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On operads in terms of finite pointed sets

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Abstract. We prove that the definition of operads in terms of finite pointed sets is equivalent to the classical definition.

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1. Introduction

Operads are algebraic structures that model various kinds of algebras such as commutative, associative, Lie, Poisson, etc. They were introduced by J. P. May in [9] as a tool to study the algebraic structures inherent in iterated loop spaces. May's work was continued by J. M. Boardman and R. M. Vogt on homotopy invariant algebraic structures in topological spaces, where operads played a central role. Starting from the nineties, operads had their renaissance, due to the works of M. Kontsevich on graph homology, of Ginzburg an Kapranov on generalized Koszul duality, and of P. Deligne on the structure of Hochschild cohomology among others (see [6, 4]).

In the literature there are mainly two equivalent definitions of operads that are used: the first one is the classical definition of May ([9]), and the second is the "$\circ_i$-definition", that also appears in the reference book of M. Markl, S. Shnider and J. Stasheff ([8]). It is folklore that these two definitions are equivalent (and an outline of the proof can be found in [8]).

If one examines any of these two definitions of operads, one can see that the role of the natural numbers is to keep track of the arity of the abstract operations as well as to label the inputs of these operations. This approach has certain disadvantages which become apparent when we compose two abstract operations. For example, to get the labels of the resulting operation right, one has to adjust the labels of the composed operations accordingly. This adjustment gives rise to not wanted technicalities in many cases, for instance when proving that something is an operad: one will need to use block permutations to prove equivariance and associativity for example.

A possible remedy to this problem can be given by labeling our operations in P with finite sets, and when a composition occurs just take the disjoint union of the reoccurring labels for the new operation. This approach has been used in the past for example by V. Hinich and A. Vaintrob in [5]. They credit P. Deligne and J. S. Milne for the formalism (see [2]). The "finite pointed sets" approach to operads was used also by P. van der Laan in his thesis [7]. None of these sources prove that the finite pointed set approach to operads is equivalent to the classical one.

The aim of this paper is to prove that the definition of operads in terms of finite pointed sets is equivalent to the classical definitions.

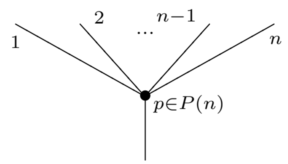
The paper is organised as follows. In Section 2 we describe operads intuitively. The goal here is to have a picture about operads in general, hence the technical details are omited (although the $\circ_i$-definition of operads in symmetrical monodal categories appears in all detail as a consequence of our constructions in Section 4). The reader interested in the technical details can find these in the work of J. P. May ([9]) and in the reference book of M. Markl, S. Shnider and J. Stasheff ([8]). In Section 3 we define operads in terms of finite pointed sets. In Section 4 we prove that this definition is equivalent to the definition in terms of the $\circ_i$ operations, hence to any of the two classical definitions, in the categorical sense.

2. An intuitive description of operads

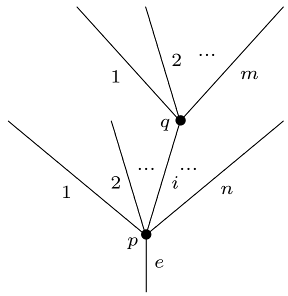
Intuitively, an operad in the classical sense consists of a "space"(vector space over a field $\Bbbk$, topological space, or more generally, an object in a symmetric monoidal category) $P(n)$ together with a right action of the symmetric group Σ_n on $P(n)$ for every $n \in \mathbb{N}$, an identity element $\mathrm{id} \in P(1)$ and composition maps

$$\circ_i \colon P(n) \otimes P(m) \longrightarrow P(n+m-1), \quad i = 1, 2, \ldots, n$$

for all n and m. The nature of the axioms this data has to satisfy comes from the intuition that the space $P(n)$ is thought of as a space of operations with n inputs and one output:



The action of the groups Σ_n permutes the inputs and the composition $p \circ_i q$ of two operations gives a new operation by using the output of q as the i-th input of p. This operation can be visualised as grafting the tree for q on the i-th leaf of the tree for p:



The unit id $\in P(1)$ can be thought of as an operation which takes one input and gives it back as output.

The axioms that the operad P has to satisfy are the formal consequences of the above intuition. In fact, the intuition can be made to a rigourous example of an operad: if the underlying category is the category of vector spaces over a field $\Bbbk$ and if V is such a vector space, define

$$\mathrm{End}_V(n) := \mathcal{V}ect_{\Bbbk}(\underbrace{V \otimes \cdots \otimes V}_{n \text{ times}}, V)$$

and follow the description above to define the rest of the structure. This operad is called the *endomorphism operad on* V. It has a prominent role in the theory of operads not only because it models the abstract definition of operads, but also because it can "realize" on the space V the algebraic structure encoded in an operad P. To be more precise, note that any map of operads $\alpha\colon P \longrightarrow \mathrm{End}_V$ takes an "abstract" n-ary operation of $P(n)$ to a "concrete" n-ary operation $V \otimes \cdots \otimes V \longrightarrow V$ and the various compatibility conditions for α impose algebraic relations between these concrete operations on the End_V side. For particular operads in $\mathcal{V}ect_{\Bbbk}$ one can describe in this way various kinds of $\Bbbk$-algebras (e.g. associative, commutative, Lie, Poisson, Leibnitz, etc). This provides a justification for the following terminology: in the literature a vector space V together with an operad map $\alpha\colon P \longrightarrow \mathrm{End}_V$ is called a *P-algebra*.

A rigourous definition of an operad that follows the intuition given above can be found in [8], although the original definition (the one by J.P. May in [9]) differs from this approach. May's definition collects the $\circ_i$ composition maps for a given $P(n)$, $i = 1, 2, \ldots, n$ under one big composition map

$$P(n) \otimes \big(P(m_1) \otimes P(m_2) \otimes \cdots \otimes P(m_n)\big) \xrightarrow{\ \gamma\ } P(m_1 + m_2 + \cdots + m_n)\ .$$

An $n+1$-tuple of operations $(p, q_1, q_2, \ldots, q_n)$ is sent by γ to a new operation which we usually write as $p(q_1, q_2, \ldots, q_n)$ and visualise as n trees corresponding to the operations q_i, grafted upon the leaves of the tree corresponding to the operation p. The equivalence of the two definitions follows from the existence of the unit-operation id $\in P(1)$, and a proof of this can be found in [8]. For example, the operation

$$\circ_i\colon P(n) \otimes P(m) \longrightarrow P(n+m-1)$$

can be obtained from

$$\gamma\colon P(n)\otimes\big(P(1)\otimes\cdots P(1)\otimes \underset{i\text{-th}}{P(m)}\otimes P(1)\otimes\cdots\otimes P(1)\big)\longrightarrow P(m+n-1).$$

3. Operads in terms of finite pointed sets

Denote by $\mathcal{F}in_*$ the category of finite pointed sets (X,x_0) and basepoint-preserving bijections. To any ordered pair $((X,x_0),(Y,y_0))\in\mathcal{F}in_*\times\mathcal{F}in_*$ and $x\in X$, $x\neq x_0$ we render $(X\sqcup_x Y,x_0)\in\mathcal{F}in_*$, defined as

$$X\sqcup_x Y = X\sqcup Y\setminus\{x,y_0\}.$$

The following properties of the $\sqcup_x$ operations are going to be important for the definition of operads:

Associativity. If $(X,x_0),(Y,y_0),(Z,z_0)\in\mathcal{F}in_*$ and $x,x'\in X$, $y\in Y$ such that $x_0\neq x\neq x'\neq x_0$ and $y\neq y_0$ then

$$(X\sqcup_x Y)\sqcup_y Z = X\sqcup_x(Y\sqcup_y Z),$$
$$(X\sqcup_x Y)\sqcup_{x'} Z = (X\sqcup_{x'} Z)\sqcup_x Y.$$

Equivariance. If $\sigma\colon(X,x_0)\longrightarrow(X',x_0')$ and $\tau\colon(Y,y_0)\longrightarrow(Y',y_0')$ are maps in $\mathcal{F}in_*$ and $x\in X$, $x\neq x_0$ then σ and τ induce a map

$$\sigma\circ_x\tau\colon(X\sqcup_x Y,x_0)\longrightarrow(X'\sqcup_{\sigma(x)}Y',x_0')$$

in $\mathcal{F}in_*$, defined as

$$\sigma\circ_x\tau|_{X\setminus\{x\}}=\sigma|_{X\setminus\{x\}},$$
$$\sigma\circ_x\tau|_{Y\setminus\{y_0\}}=\tau|_{Y\setminus\{y_0\}}.$$

Unit. For any pointed set with two elements $(U,u_0)=(\{u,u_0\},u_0)$ and any other pointed set (X,x_0) together with an element $x\in X\setminus\{x_0\}$ there are maps

$$e_{ux_0}\colon(X,x_0)\longrightarrow(U\sqcup_u X,u_0)\quad\text{and}\quad e_{ux}\colon(X,x_0)\longrightarrow(X\sqcup_x U,x_0),$$

where e_{ux_0} sends x_0 to u_0 and is the identity elsewhere, and e_{ux} sends x to u and is the identity elsewhere.

Let $(\mathcal{E},\otimes,I,a,l,r,s)$ be a symmetric monoidal category.

Definition 3.1. *A contravariant functor $P\colon\mathcal{F}in_*^{\mathrm{op}}\longrightarrow\mathcal{E}$ is called a collection or a $\mathcal{F}in_*$-module in $\mathcal{E}$.*

If P is a collection in $\mathcal{E}$ then for any map $\sigma\colon(X,x_0)\longrightarrow(X',x_0')$ in $\mathcal{F}in_*$ the induced map $P(\sigma)\colon P(X',x_0')\longrightarrow P(X,x_0)$ can be considered as acting on the right on $P(X',x_0')$. We will write instead of $P(\sigma)$ just σ.

Definition 3.2. *An operad in $\mathcal{E}$ is a collection $P\colon\mathcal{F}in_*^{\mathrm{op}}\longrightarrow\mathcal{E}$ with structure maps*

$$\circ_x\colon P(X,x_0)\otimes P(Y,y_0)\longrightarrow P(X\sqcup_x Y,x_0)$$

for any (X,x_0), $(Y,y_0)\in\mathcal{F}in_*$ and $x\in X$, $x\neq x_0$, which satisfy the following three conditions:

Associativity. *For any* (X, x_0), (Y, y_0) *and* $(Z, z_0) \in \mathcal{F}in_*$, *and any* $x, x' \in X$, $y \in Y$ *such that* $x_0 \neq x \neq x' \neq x_0$ *and* $y \neq y_0$ *the following diagrams commute:*

$$\begin{array}{ccc}
P(X,x_0)\otimes P(Y,y_0)\otimes P(Z,z_0) & \xrightarrow{\circ_x\otimes \mathrm{id}} & P(X\sqcup_x Y,x_0)\otimes P(Z,z_0)\\
\downarrow{\scriptstyle \mathrm{id}\otimes\circ_y} & & \downarrow{\scriptstyle \circ_y}\\
P(X,x_0)\otimes P(Y\sqcup_y Z,y_0) & \xrightarrow{\circ_x} & P(X\sqcup_x Y\sqcup_y Z,x_0)
\end{array}$$

$$\begin{array}{ccc}
P(X,x_0)\otimes P(Y,y_0)\otimes P(Z,z_0) & \xrightarrow{\circ_x\otimes \mathrm{id}} & P(X\sqcup_x Y,x_0)\otimes P(Z,z_0)\\
\downarrow{\scriptstyle \mathrm{id}\otimes s} & & \\
P(X,x_0)\otimes P(Z,z_0)\otimes P(Y,y_0) & & \downarrow{\scriptstyle \circ_{x'}}\\
\downarrow{\scriptstyle \circ_{x'}\otimes\mathrm{id}} & & \\
P(X\sqcup_{x'} Z,x_0)\otimes P(Y,y_0) & \xrightarrow{\circ_x} & P(X\sqcup_x Y\sqcup_{x'} Z,x_0),
\end{array}$$

where $s\colon P(Y, y_0) \otimes P(Z, z_0) \longrightarrow P(Z, z_0) \otimes P(Y, y_0)$ *is the symmetry of* $\mathcal{E}$.

Equivariance. *For any* $\sigma\colon (X, x_0) \longrightarrow (X', x'_0)$, $\tau\colon (Y, y_0) \longrightarrow (Y', y'_0)$ *maps in* $\mathcal{F}in_*$ *and* $x \in X$, $x \neq x_0$ *the following diagram commutes:*

$$\begin{array}{ccc}
P(X',x'_0)\otimes P(Y',y'_0) & \xrightarrow{\circ_{\sigma(x)}} & P(X'\sqcup_{\sigma(x)} Y',x'_0)\\
\downarrow{\scriptstyle \sigma\otimes\tau} & & \downarrow{\scriptstyle \sigma\circ_x\tau}\\
P(X,x_0)\otimes P(Y,y_0) & \xrightarrow{\circ_x} & P(X\sqcup_x Y,x_0).
\end{array}$$

Unit. *For any set with two elements* $(U, u_0) = (\{u, u_0\}, u_0) \in \mathcal{F}in_*$ *there is a map* $\eta_{(U,u_0)}\colon I \longrightarrow P(U, u_0)$, *for which the compositions*

$$I\otimes P(X,x_0)\xrightarrow{\eta_U\otimes\mathrm{id}}P(U,u_0)\otimes P(X,x_0)\xrightarrow{\circ_u}P(U\sqcup_u X,u_0)\xrightarrow{e_{ux_0}}P(X,x_0),$$

$$P(X,x_0)\otimes I\xrightarrow{\mathrm{id}\otimes\eta_U}P(X,x_0)\otimes P(U,u_0)\xrightarrow{\circ_x}P(X\sqcup_x U,x_0)\xrightarrow{e_{ux}}P(X,x_0)$$

are the left and right identities in the monoidal category $\mathcal{E}$ *for any* $(X, x_0) \in \mathcal{F}in_*$.

The following diagram commutes for any two-point sets (X, x_0) *and* (X', x'_0):

$$\begin{array}{ccc}
I & \xrightarrow{\eta_X} & P(X,x_0)\\
\| & & \downarrow{\scriptstyle \alpha}\\
I & \xrightarrow{\eta_{X'}} & P(X',x'_0)
\end{array}$$

where $\alpha\colon (X', x'_0) \longrightarrow (X, x_0)$ *is the obvious (unique) map.*

Definition 3.3. *Let* P *and* Q *be operads in* $\mathcal{E}$. *A morphism of operads* $\mu\colon P \longrightarrow Q$ *is an equivariant natural transformation from* P *to* Q *which is compatible with all the operations* $\circ_x$ *and unit maps* η_U. *Explicitly, such a* μ *is a family of maps*

$\mu_{(X,x_0)}\colon P(X,x_0) \longrightarrow Q(X,x_0)$, *such that the following diagrams commute for any possible choice of* x, σ *and* η*:*

$$\begin{array}{ccc} P(X',x_0') & \xrightarrow{\mu_{X'}} & Q(X',x_0') \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma} \\ P(X,x_0) & \xrightarrow{\mu_X} & Q(X,x_0) \end{array} \qquad \begin{array}{ccc} P(X,x_0)\otimes P(Y,y_0) & \xrightarrow{\circ_x} & P(X\sqcup_x Y,x_0) \\ {\scriptstyle\mu_X\otimes\mu_Y}\downarrow & & \downarrow{\scriptstyle\mu_{X\sqcup_x Y}} \\ Q(X,x_0)\otimes Q(Y,y_0) & \xrightarrow{\circ_x} & Q(X\sqcup_x Y,x_0) \end{array}$$

$$\begin{array}{ccc} I & \xrightarrow{\eta_U} & P(U,u_0) \\ \| & & \downarrow{\scriptstyle\mu_U} \\ I & \xrightarrow{\eta_U} & Q(U,u_0) \end{array}$$

With these maps, operads in $\mathcal{E}$ form the category of $\mathcal{F}in_*$-operads $\mathcal{O}p_{\mathcal{F}in_*}$.

4. Equivalence with the classical definition

In this Section we are going to prove that the category of $\mathcal{F}in_*$-operads arising from our definition is equivalent to the classical one, given in terms of the $\circ_i$ operations (see [8], pp. 46) which in turn is equivalent to the original definition of May [9].

In the following we are going to denote the pointed set $(\{0,1,\dots,n\},0)\in\mathcal{F}in_*$ by $\langle n\rangle$. Instead of $P(\langle n\rangle)$ let us write $P(n)$. If P is an operad in $\mathcal{E}$ then any composition map $\circ_x\colon P(X,x_0)\otimes P(Y,y_0)\longrightarrow P(X\sqcup_x Y,x_0)$ gives rise to a new one $\circ_i\colon P(m)\otimes P(n)\longrightarrow P(\langle m\rangle\sqcup_i\langle n\rangle)$ via the actions of some pointed bijections $\sigma\colon\langle m\rangle\longrightarrow(X,x_0)$ with $\sigma(i)=x$ and $\tau\colon\langle n\rangle\longrightarrow(Y,y_0)$, because of the equivariance condition:

$$\circ_x=(\sigma\circ_i\tau)^{-1}(\circ_i)(\sigma\otimes\tau)).$$

This property suggests to study more the structures induced by the operad axioms on the objects $P(m)$. Define the *renumbering map* $\varphi_i\colon\langle m+n-1\rangle\longrightarrow\langle m\rangle\sqcup_i\langle n\rangle$,

$$\varphi_i(k):=\begin{cases} k\in\langle m\rangle & \text{if } k<i,\\ (k-n+1)\in\langle m\rangle & \text{if } k>i+n-1,\\ (k-i+1)\in\langle n\rangle & \text{if } i\le k\le i+n-1. \end{cases} \tag{4.1}$$

We can infer that the composition of φ_i with $\circ_i$ defines a new operation, denoted by $\bullet_i$ which is written only in terms of the sets $\langle m\rangle$:

$$\bullet_i:=\varphi_i\circ_i\colon P(m)\otimes P(n)\longrightarrow P(m+n-1).$$

In the following we look at the axioms – induced by the associativity, equivariance and unit axioms for P – that this new operations satisfy.

Associativity. Let $\circ_i\colon P(m)\otimes P(n)\longrightarrow P(\langle m\rangle\sqcup_u\langle n\rangle)$ and $\circ_j\colon P(n)\otimes P(p)\longrightarrow P(\langle n\rangle\sqcup_j\langle p\rangle)$ be two operations. To avoid confusion we write $j_{\langle n\rangle}$ instead of j, when

it is necessary to indicate the set from which j is taken.
The squares of the following diagram commute:

$$\begin{array}{ccccc}
P(m)\otimes P(n)\otimes P(p) & \xrightarrow{\circ_i\otimes\mathrm{id}} & P(\langle m\rangle \sqcup_i \langle n\rangle)\otimes P(p) & \xrightarrow{\varphi_i\otimes\mathrm{id}} & P(m+n-1)\otimes P(p)\\
{\scriptstyle \mathrm{id}\otimes\circ_j}\downarrow & & \downarrow{\scriptstyle \circ_{j_{\langle n\rangle}}} & & \downarrow{\scriptstyle k}\\
P(m)\otimes P(\langle n\rangle \sqcup_j \langle p\rangle) & \xrightarrow{\circ_i} & P(\langle m\rangle \sqcup_i \langle n\rangle \sqcup_{j_{\langle n\rangle}} \langle p\rangle) & \xrightarrow{\varphi_i\circ_k\mathrm{id}} & P(\langle m+n-1\rangle \sqcup_k \langle p\rangle)\\
{\scriptstyle \mathrm{id}\otimes\varphi_j}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\circ_i\varphi_j} & & \downarrow{\scriptstyle \varphi_k}\\
P(m)\otimes P(n+p-1) & \xrightarrow{\circ_i} & P(\langle m\rangle \sqcup_i \langle n+p-1\rangle) & \xrightarrow{\varphi_i} & P(m+n+p-2),
\end{array}$$

where $k=\varphi_i^{-1}(j_{\langle n\rangle})$. Indeed, the commutativity of the first square is just an associativity condition of the operad P, the second and third squares are equivariance conditions of P. The commutativity of the last square follows from a straightforward computation. With the use of the $\bullet_i$ operations the bordering square of the diagram above can be written as

$$\begin{array}{ccc}
P(m)\otimes P(n)\otimes P(p) & \xrightarrow{\bullet_i\otimes\mathrm{id}} & P(m+n-1)\otimes P(p)\\
{\scriptstyle \mathrm{id}\otimes\bullet_j}\downarrow & & \downarrow{\scriptstyle \bullet_{j+i-1}}\\
P(m)\otimes P(n+p-1) & \xrightarrow{\bullet_i} & P(m+n+p-2).
\end{array}$$

We proceed similarly for operations $\circ_i\colon P(m)\otimes P(n)\longrightarrow P(\langle m\rangle\sqcup_i\langle n\rangle)$ and $\circ_j\colon P(m)\otimes P(p)\longrightarrow P(\langle m\rangle\sqcup_j\langle p\rangle)$, where $i\neq j$. In this case we use the second axiom for associativity of the operad P. We obtain the diagram

$$\begin{array}{ccccc}
P(m)\otimes P(n)\otimes P(n) & \xrightarrow{\circ_i\otimes\mathrm{id}} & P(\langle m\rangle \sqcup_i \langle n\rangle)\otimes P(p) & \xrightarrow{\varphi_i\otimes\mathrm{id}} & P(m+n-1)\otimes P(p)\\
{\scriptstyle \mathrm{id}\otimes s}\downarrow & & & & \\
P(m)\otimes P(p)\otimes P(n) & & \downarrow{\scriptstyle \circ_{j_{\langle m\rangle}}} & & \downarrow{\scriptstyle \circ_k}\\
{\scriptstyle \circ_j\otimes\mathrm{id}}\downarrow & & & & \\
P(\langle m\rangle \sqcup_j \langle p\rangle)\otimes P(n) & \xrightarrow{\circ_{i_{\langle m\rangle}}} & P(\langle m\rangle \sqcup_{i_{\langle m\rangle}} \langle n\rangle \sqcup_{j_{\langle m\rangle}} \langle p\rangle) & \xrightarrow{\varphi_i\circ_k\mathrm{id}} & P(\langle m+n-1\rangle \sqcup_k \langle p\rangle)\\
{\scriptstyle \varphi_j\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \varphi_j\circ_l\mathrm{id}} & & \downarrow{\scriptstyle \varphi_k}\\
P(m+p-1)\otimes P(n) & \xrightarrow{\circ_l} & P(\langle m+p-1\rangle \sqcup_l \langle n\rangle) & \xrightarrow{\varphi_l} & P(m+n+p-2),
\end{array}$$

where $l=\varphi_j^{-1}(i_{\langle m\rangle})$ and $k=\varphi_i^{-1}(j_{\langle m\rangle})$. Again, only the commutativity of the last square must be checked, because the other squares are commutative from the associativity and equivariance properties of P. If we write the bordering square with the operations $\bullet_i$, then we have

$$\begin{CD}
P(m)\otimes P(n)\otimes P(p) @>{\bullet_i\otimes\mathrm{id}}>> P(m+n-1)\otimes P(p)\\
@V{\mathrm{id}\otimes s}VV @.\\
P(m)\otimes P(p)\otimes P(n) @. @VV{\bullet_{j+n-1}}V\\
@V{\bullet_j\otimes\mathrm{id}}VV @.\\
P(m+p-1)\otimes P(n) @>{\bullet_i}>> P(m+n+p-2)
\end{CD}$$

if $i < j$: in this case $l = i$ and $k = j + n - 1$;

$$\begin{CD}
P(m)\otimes P(n)\otimes P(p) @>{\bullet_i\otimes\mathrm{id}}>> P(m+n-1)\otimes P(p)\\
@V{\mathrm{id}\otimes s}VV @.\\
P(m)\otimes P(p)\otimes (n) @. @VV{\bullet_{j}}V\\
@V{\bullet_j\otimes\mathrm{id}}VV @.\\
P(m+p-1)\otimes P(n) @>{\bullet_{i+p-1}}>> P(m+n+p-2)
\end{CD}$$

if $i > j$: in this case $l = i + p - 1$ and $k = j$.
The obtained three commutative diagrams are the associativity axioms for the $\bullet_i$ operations. After a suitable renumbering, they can be expressed in the following equations:

$$\bullet_j(\bullet_i\otimes\mathrm{id}) = \begin{cases} \bullet_i(\mathrm{id}\otimes\bullet_{j-i+1}), & \text{if } 1\le i\le j\le n\le m+n-1;\\ \bullet_i(\bullet_{j+n-1}\otimes\mathrm{id})(\mathrm{id}\otimes s), & \text{if } n\le i+n-1<j\le m+n-1;\\ \bullet_{i+p-1}(\bullet_j\otimes\mathrm{id})(\mathrm{id}\otimes s), & \text{if } 1\le j<i\le m. \end{cases} \tag{4.2}$$

Equivariance. Let $\sigma\colon \langle m\rangle \longrightarrow \langle m\rangle$, $\tau\colon \langle n\rangle \longrightarrow \langle n\rangle$ be two maps in $\mathcal{F}in_*$. The equivariance property of P induces the commutative diagram

$$\begin{CD}
P(m)\otimes P(n) @>{\circ_i}>> P(\langle m\rangle \sqcup_i \langle n\rangle) @>{\varphi_i}>> P(m+n-1)\\
@V{\sigma\otimes\tau}VV @V{\sigma\circ_k\tau}VV @VV{\sigma\bullet_k\tau}V\\
P(m)\otimes P(n) @>{\circ_k}>> P(\langle m\rangle \sqcup_k \langle n\rangle) @>{\varphi_k}>> P(m+n-1),
\end{CD}$$

where $\sigma(k) = i$ and $\sigma \bullet_k \tau\colon \langle m+n-1\rangle \longrightarrow \langle m+n-1\rangle$,

$$\sigma \bullet_k \tau = (\varphi_i)^{-1}(\sigma \circ_k \tau)(\varphi_k).$$

A straightforward computation shows that

$$\sigma \bullet_k \tau = \sigma_{(1,\dots,1,n,1,\dots,1)} \circ (\mathrm{id}\times\cdots\times\mathrm{id}\times\tau\times\mathrm{id}\times\cdots\times\mathrm{id}), \tag{4.3}$$

where on the right hand side of the equation, n and τ occur at the k^{th} position. We infer that the equivariance property induces the commutativity of the diagram

$$\begin{array}{ccc} P(m)\otimes P(n) & \xrightarrow{\bullet_{\sigma(k)}} & P(m+n-1) \\ {\scriptstyle \sigma\otimes\tau}\downarrow & & \downarrow{\scriptstyle \sigma\bullet_k\tau} \\ P(m)\otimes P(n) & \xrightarrow{\bullet_k} & P(m+n-1) \end{array}$$

or

$$(\sigma \bullet_k \tau)\bullet_{\sigma(k)} = (\bullet_k)(\sigma\otimes\tau). \tag{4.4}$$

Unit. Let us take in the unit condition for an operad P the two-element pointed set $(X, x_0) = \langle 1\rangle$. It follows that for any $n \in \mathbb{N}^*$ we have

$$e_{xy_0} = \varphi_1 = \mathrm{id}\colon \langle n\rangle \longrightarrow \langle 1\rangle \sqcup_1 \langle n\rangle,$$

$$e_{xy} = \varphi_i\colon \langle n\rangle \longrightarrow \langle n\rangle \sqcup_i \langle 1\rangle,$$

hence the unit conditions for the $\bullet_i$ operations say that the following compositions must be the corresponding left and right identities in $\mathcal{E}$:

$$I\otimes P(n) \xrightarrow{\eta\otimes\mathrm{id}} P(1)\otimes P(n) \xrightarrow{\bullet_1} P(n); \tag{4.5}$$

$$P(n)\otimes I \xrightarrow{\mathrm{id}\otimes\eta} P(n)\otimes P(1) \xrightarrow{\bullet_i} P(n). \tag{4.6}$$

These properties imply the following definition:

Definition 4.1. *Let Σ denote the symmetric groupoid (i.e. the category whose objects are the finite sets $[n] = \{1, 2, \ldots, n\}$ for every $n \in \mathbb{N}^*$ and the maps are permutations $\sigma\colon [n] \longrightarrow [n]$). A Σ-operad in a symmetric monoidal category $\mathcal{E}$ is a contravariant functor $P\colon \Sigma^{\mathrm{op}} \longrightarrow \mathcal{E}$ with operations*

$$\bullet_i\colon P(m)\otimes P(n) \longrightarrow P(m+n-1)$$

for every $1 \le i \le m$ (here we denote $P([m])$ by $P(m)$), which satisfy the conditions given in equations (4.2), (4.4), (4.5) and (4.6).

This definition agrees with Markl, Shinder and Stasheff's definition of an operad in [8], and it is equivalent to the definition given by May [9]. *Morphisms of* Σ*-operads* are defined as that of operads: they are collections of maps $\mu_m\colon P(m) \longrightarrow Q(m)$, for which the following diagrams commute:

$$\begin{array}{ccc} P(m) & \xrightarrow{\mu_m} & Q(m) \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma} \\ P(m) & \xrightarrow{\mu_m} & Q(m) \end{array} \qquad \begin{array}{ccc} P(m)\otimes P(n) & \xrightarrow{\bullet_i} & P(m+n-1) \\ {\scriptstyle\mu_m\otimes\mu_n}\downarrow & & \downarrow{\scriptstyle\mu_{m+n-1}} \\ Q(m)\otimes Q(n) & \xrightarrow{\bullet_i} & Q(m+n-1) \end{array} \qquad \begin{array}{ccc} I & \xrightarrow{\eta} & P(1) \\ \| & & \downarrow{\scriptstyle\mu_1} \\ I & \xrightarrow{\eta} & Q(1) \end{array}$$

It follows that we have a category of Σ-operads in $\mathcal{E}$, which we denote by $\mathcal{O}p_\Sigma$.

We turn to prove that $\mathcal{O}p_\Sigma$ and $\mathcal{O}p_{\mathcal{F}in_*}$ are equivalent categories. For this, first observe that the usual restriction and extension functors $R\colon \mathcal{F}in_* \longrightarrow \Sigma$ and $E\colon \Sigma \longrightarrow \mathcal{F}in_*$ are equivalences and even $RE = \mathrm{id}$ is satisfied. Denote the induced functors on the categories of $\mathcal{E}$-collections by $R^\#\colon \mathcal{E}^{\Sigma^{\mathrm{op}}} \longrightarrow \mathcal{E}^{\mathcal{F}in_*^{\mathrm{op}}}$ and

$E^{\#}\colon \mathcal{E}^{\mathcal{F}in_*^{\mathrm{op}}} \longrightarrow \mathcal{E}^{\Sigma^{\mathrm{op}}}$. By a slight abuse of notation, we will not distinguish between the finite set $[n]$ and the finite pointed set $(\langle n\rangle, 0)$ in what follows.

Lemma 4.2. *$P\colon \mathcal{F}in_*^{op} \longrightarrow \mathcal{E}$ defines a $\mathcal{F}in_*$-operad if and only if $E^{\#}(P)\colon \Sigma^{op} \longrightarrow \mathcal{E}$ defines a Σ-operad.*

Proof. If P is a $\mathcal{F}in_*$-operad, then (by the abuse of notation mentioned above) $E^{\#}(P)(n) = P(n)$ and $E^{\#}(P)(\sigma) = P(\sigma)$ for any $n \in \mathbb{N}^*$ and $\sigma \in \Sigma_n$. The construction of the $\bullet_i$ operations as above gives a Σ-operad structure to $E^{\#}(P)$.
Conversely, suppose that $E^{\#}(P)$ is a Σ-operad. Then we have operations

$$\bullet_i\colon P(m) \otimes P(n) \longrightarrow P(m+n-1)$$

which satisfy the respective associativity, equivariance and unit conditions.
First, define the operations

$$\circ_i\colon P(m) \otimes P(n) \longrightarrow P(\langle m\rangle \sqcup_i \langle n\rangle)$$

with the composition: $\circ_i := \varphi_i^{-1}\bullet_i$ where the maps φ_i are defined by (4.1). It follows from the Σ-equivariance condition that the diagram

$$\begin{array}{ccc} P(m)\otimes P(n) & \xrightarrow{\circ_{\sigma(i)}} & P(\langle m\rangle \sqcup_i \langle n\rangle) \\ {\scriptstyle \sigma\otimes\tau}\big\downarrow & & \big\downarrow{\scriptstyle \sigma\circ_i\tau} \\ P(m)\otimes P(n) & \xrightarrow{\circ_i} & P(\langle m\rangle \sqcup_i \langle n\rangle) \end{array} \tag{4.7}$$

also commutes.
Second, define the operations $\circ_x\colon P(X)\otimes P(Y) \longrightarrow P(X \sqcup_x Y)$ by requiring the diagram

$$\begin{array}{ccc} P(X)\otimes P(Y) & \xrightarrow{\circ_x} & P(X \sqcup_x Y) \\ {\scriptstyle \sigma\otimes\tau}\big\downarrow & & \big\downarrow{\scriptstyle \sigma\circ_i\tau} \\ P(m)\otimes P(n) & \xrightarrow{\circ_i} & P(\langle m\rangle \sqcup_i \langle n\rangle) \end{array}$$

to be commutative. Here $\sigma\colon \langle m\rangle \longrightarrow X$, $\tau\colon \langle n\rangle \longrightarrow Y$ are chosen maps in $\mathcal{F}in_*$ with the property that $\sigma(i) = x$. The operations $\circ_x$ do not depend on the choice of σ and τ, because of the commutative square (4.7). Indeed, if σ' and τ' define an operation $(\circ_x)' \neq \circ_x$ by

$$\begin{array}{ccc} P(X)\otimes P(Y) & \xrightarrow{(\circ_x)'} & P(X \sqcup_x Y) \\ {\scriptstyle \sigma'\otimes\tau'}\big\downarrow & & \big\downarrow{\scriptstyle \sigma'\circ_i\tau'} \\ P(m)\otimes P(n) & \xrightarrow{\circ_i} & P(\langle m\rangle \sqcup_i \langle n\rangle) \end{array}$$

then patching together the last two diagrams follows that the diagram (4.7) is not commutative with the maps $\sigma'\sigma^{-1}$ and $\tau'\tau^{-1}$, which is contradiction.
Thus the operations $\circ_x$ are well defined. The axioms for the $\mathcal{F}in_*$-operad are easily verified: we just have to do the diagram-chasing with $\bullet_i$ and $\circ_x$ backwards. □

Lemma 4.3. *$\mu\colon P \longrightarrow Q$ is a map of $\mathcal{F}in_*$-operads if and only if $E^{\#}(\mu)$ is a map of Σ-operads.*

Proof. A straightforward check, using the maps φ_i and that $E^{\#}(\mu)_n = \mu_{\langle n\rangle}$. □

Theorem 4.4. *The categories $\mathcal{O}p_{\mathcal{F}in_*}$ and $\mathcal{O}p_{\Sigma}$ are equivalent.*

Proof. For any Σ-operad Q we have $E^{\#}R^{\#}(Q) = Q$. We infer by Lemma 4.2 that $R^{\#}(Q)$ is an operad. This and Lemma 4.2 again show that $E^{\#}$ is an essentially surjective functor when viewed between the operad-categories.
On the other hand, because $E^{\#}$ is fully faithful, Lemma 4.3 implies that $E^{\#}$ is also fully faithful between the operad-categories. Hence $\mathcal{O}p_{\mathcal{F}in_*}$ and $\mathcal{O}p_{\Sigma}$ are equivalent. □

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