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On some asymptotic properties of the Rössler dynamical system

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To the memory of Professor Mircea-Eugen Craioveanu (1942-2012)

Abstract. In this paper we present a method to stabilize asymptotically the Lyapunov stable equilibrium states of the Rössler dynamical system.

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1. Introduction

The Rössler dynamical system [12] has been widely investigated over the last years, mainly from the chaotic dynamics perspective. In this work we are concerned with the analysis of the conservative properties of this system. Among the studied topics related to the conservative properties of the Rössler dynamical system, one can mention various types of integrability, namely Darboux integrability ([8], [13]), formal and analytic integrability [7], the description of the global dynamics in the Poincaré sphere [6] and a dynamical analysis from the Hamiltonian point of view [14].

The aim of this work is to analyze further the Rössler dynamical system from the stability theory point of view. More exactly, we present a method to associate to each Lyapunov stable equilibrium state of the Rössler system, a special type of dissipative system in such a way that each Lyapunov stable equilibrium state of the Rössler system generates a one dimensional attracting neighborhood for the dissipative system.

The structure of the paper is as follows. In the second section of this work, we recall from [14] the geometric framework adopted in our study, namely a Hamiltonian realization of the Rössler system. In the third section of the paper we recall from [14] the main results regarding the Lyapunov stability analysis of the equilibrium states of the Rössler system. In the fourth section, we recall the definition of a metriplectic system and construct explicitly a metriplectic perturbation associated to the Rössler system. The metriplectic perturbation of the Rössler system, prove to have all the equilibrium states of the Rössler system. The last part of the paper contains the main results, namely it describes explicitly the method of associating, to each Lyapunov stable equilibrium state of the Rössler system, a special type of metriplectic system, in such a way that each Lyapunov stable equilibrium of the unperturbed system generates a one dimensional attracting neighborhood for the dissipative system.

For details on Hamiltonian dynamics, and respectively metriplectic dissipative systems, see, e.g. [1], [2], [11], [3], [4], [9].

2. Setting of the problem from the Poisson geometry point of view

As the purpose of this paper is to study a special type of perturbations of a Hamiltonian system from the Poisson dynamics and geometry point of view, the first step in this approach is to prepare the geometric framework of the problem. The results from this chapter are from [14].

The Rössler system we consider for our study, is governed by the equations:

$$\begin{cases} \dot{x} = -y - z \\ \dot{y} = x \\ \dot{z} = xz. \end{cases} \tag{2.1}$$

Note that in the article [13] it is proved that the above system it is the only case when the Rössler system it is completely integrable.

Let us recall now some results from [14] concerning the geometric framework of the problem. The following proposition from [14] provides a Hamiltonian formulation of the Rössler system on an appropriate Poisson manifold.

Theorem 2.1. *The dynamics* (2.1) *admit the following Hamilton-Poisson realization:*

$$(\mathbb{R}^3, \nu\Pi_C, H) \tag{2.2}$$

where

$$\Pi_C(x,y,z) = \begin{bmatrix} 0 & e^{-y} & ze^{-y} \\ -e^{-y} & 0 & 0 \\ -ze^{-y} & 0 & 0 \end{bmatrix}$$

is the Poisson structure generated by the smooth function $C(x,y,z) := ze^{-y}$, the rescaling ν is given by $\nu(x,y,z) = -e^{y}$, and the Hamiltonian $H \in C^\infty(\mathbb{R}^3, \mathbb{R})$ is given by $H(x,y,z) := \frac{1}{2}(x^2+y^2)+z$.

Note that, by Poisson structure generated by the smooth function C, we mean the Poisson structure generated by the Poisson bracket

$$\{f,g\}_C := \nabla C \cdot (\nabla f \times \nabla g),$$

for any smooth functions $f, g \in C^\infty(\mathbb{R}^3, \mathbb{R})$.

Next remark from [14] provides a class of first integrals for all the Hamiltonian dynamical systems modeled on the Poisson manifold $(\mathbb{R}^3, \nu\Pi_C)$.

Remark 2.2. By definition we have that the center of the Poisson algebra $(C^\infty(\mathbb{R}^3,\mathbb{R}), \{\cdot,\cdot\}_C)$ is generated by the Casimir invariant $C(x,y,z) = ze^{-y}$.

3. Equilibrium states and Lyapunov stability

In this short section, we recall some results from [14] regarding the Lyapunov stability of the equilibrium states of the Rössler system (2.2). As our main purpose is to perturb the Rössler system in such a way that, each Lyapunov stable equilibrium of the unperturbed system, turnes to an asymptotically stable equilibrium for the perturbed system, we do not consider here the unstable equilibrium states of the unperturbed system.

Note that the set of equilibrium states of the Rössler system is given by

$$\mathcal{E} := \{(0, -M, M) : M \in \mathbb{R}\}.$$

Let us recall from [14], the following theorem describing the stability properties of the equilibrium states of the Rössler system.

Theorem 3.1. *Let $e_M = (0, -M, M) \in \mathcal{E}$ be an arbitrary equilibrium state of the Rössler system* (2.1)*. The equilibrium $e_M \in \mathcal{E}$ is Lyapunov stable for $M > -1$ and unstable for $M \leq -1$.*

Proof. See [14]. □

4. Metriplectic perturbations of the system (2.2)

The purpose of this section is to associate to the Rössler system (2.2), a class of metriplectic systems (parameterized by a smooth real function $\varphi \in C^{\infty}(\mathbb{R}, \mathbb{R})$) in such a way that the equilibrium states of the Hamilton-Poisson system (2.2) are also equilibrium states for all the associated metriplectic systems. By metriplectic system we mean a dynamical system consisting of a compatible pair consisting of a conservative system (modeled by a Hamiltonian system), together with a dissipative (nonconservative) system (modeled by a gradient system with respect to a symmetric tensor G). For details regarding the properties of metriplectic systems, see e.g. [10], [3].

Let us give first the definition of a general metriplectic perturbation of a Hamiltonian system on the Poisson manifold $(\mathbb{R}^3, \nu\Pi_C)$.

Definition 4.1. *A metriplectic perturbation of a Hamiltonian system on $(\mathbb{R}^3, \nu\Pi_C)$ is a dynamical system of the type:*

$$\dot{u} = \nu(u)\Pi_C(u) \cdot \nabla H(u) + G(u) \cdot \nabla(\varphi \circ C)(u), \ u^T = (x, y, z) \in \mathbb{R}^3,$$

where $\nu, H, C \in C^{\infty}(\mathbb{R}^3, \mathbb{R})$, $C(x, y, z) = ze^{-y}$, $\nu(x, y, z) = -e^{y}$, G is a symmetric covariant tensor, and $\varphi \in C^{\infty}(\mathbb{R}, \mathbb{R})$, such that the following compatibility conditions hold:

(i) $G \cdot \nabla H = \bar{0}$,
(ii) $(\nabla(\varphi \circ C))^T \cdot G \cdot \nabla(\varphi \circ C) \leq 0$.

Let us now construct a metriplectic perturbation of the Rössler system (2.2). In order to do that, we associate to the Hamiltonian $H \in C^{\infty}(\mathbb{R}^3, \mathbb{R})$, $H(x, y, z) = \frac{1}{2}(x^2 + y^2) + z$, of the system (2.2), a second order covariant symmetric tensor, given by

$G=\nabla H\otimes\nabla H-\|\nabla H\|^2\,\mathrm{Id}$, in order to get a candidate for a metriplectic perturbation of the system (2.2).

Note that, in coordinates:

$$G(x,y,z)=\begin{bmatrix} -y^2-1 & xy & x \\ xy & -x^2-1 & y \\ x & y & -x^2-y^2 \end{bmatrix}.$$

Next proposition gives a family of metriplectic perturbations of the Rössler system, parameterized by a smooth real function $\varphi\in C^\infty(\mathbb{R},\mathbb{R})$.

Proposition 4.2. *The system:*

$$\dot{u}=\nu(u)\Pi_C(u)\cdot\nabla H(u)+G\cdot\nabla(\varphi\circ C)(u),\ \ u^T=(x,y,z), \tag{4.1}$$

is a metriplectic perturbation of the Rössler system, where $\nu,H,C\in C^\infty(\mathbb{R}^3,\mathbb{R})$ are given by $\nu(x,y,z)=-e^y$, $H(x,y,z)=\dfrac{1}{2}(x^2+y^2)+z$, and respectively $C(x,y,z)=ze^{-y}$.

Proof. In order to obtain the conclusion, we need to check the condition (i) and respectively (ii) from the above definition. The condition (i) follows by straightforward computations. To verify the condition (ii), note that:

$$\begin{aligned}(\nabla(\varphi\circ C)(x,y,z))^T\cdot G(x,y,z)\cdot\nabla(\varphi\circ C)(x,y,z)&=\\ &=-\left[\varphi'\left(ze^{-y}\right)\right]^2\cdot e^{-2y}\left[x^2+x^2z^2+(y+z)^2\right]\\ &\leq 0.\end{aligned}$$

□

Before analyzing the equilibrium states of the metriplectic system, let us write the system in coordinates.

Remark 4.3. The metriplectic system (4.1) is given in coordinates by:

$$\begin{cases}\dot{x}=-y-z+\varphi'\left(ze^{-y}\right)\cdot xe^{-y}(1-yz),\\ \dot{y}=x+\varphi'\left(ze^{-y}\right)\cdot e^{-y}(y+z+x^2z),\\ \dot{z}=xz-\varphi'\left(ze^{-y}\right)\cdot e^{-y}(x^2+y^2+yz).\end{cases} \tag{4.2}$$

Next remark gives a relation between the equilibrium states of the Hamilton-Poisson system (2.2) and the associated metriplectic perturbation (4.1).

Remark 4.4. All of the equilibrium states of the Rössler system (2.2) are also equilibrium states for the perturbed metriplectic system (4.1), for any smooth real function $\varphi\in C^\infty(\mathbb{R},\mathbb{R})$.

5. Asymptotically stabilizing the metriplectically perturbed system

The aim of this section is to discuss the asymptotic stability of some special equilibrium states of the metriplectic system (4.1). In the previous section we obtained that for any smooth real function $\varphi\in C^\infty(\mathbb{R},\mathbb{R})$, all the equilibrium states of the Rössler system (2.1) are also equilibrium states of his metriplectic perturbation (4.1). The aim of this section is to make use of this important property in order to metriplectically perturb the Rössler system in such a way that each Lyapunov stable equilibrium of the unperturbed system, generates a one-dimensional attracting neighborhood for the associated metriplectically perturbed system.

Before stating the main results of this paper, let us recall the principle of LaSalle [5].

Theorem 5.1. *Let $x_0 \in \mathbb{R}^n$ be an equilibrium state of the dynamical system $\dot{x} = f(x)$, where $f \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$, and let U be a compact neighborhood around x_0. Suppose there exists $L : U \to \mathbb{R}$ a C^1 function with $L(x) > 0$ for $x \neq x_0$, $L(x_0) = 0$ and $\dot{L}(x) \leq 0$. Let $E := \{x \in U : \dot{L}(x) = 0\}$ and $M \subset E$ be the largest dynamically invariant subset of E. Then there exists $V \subset U$ a neighborhood of x_0 such that the ω-limit set $\omega(x) \subset M$ for all $x \in V$.*

Let us now state the main result of this article.

Theorem 5.2. *Let $e_M \in \mathcal{E}$ be a Lyapunov stable equilibrium state of the Rössler systems* (2.1)*, and respectively* (4.1)*. Then there exists a smooth function $\varphi_{e_M} \in C^\infty(\mathbb{R}, \mathbb{R})$, a compact neighborhood K around e_M and a neighborhood $U \subset K$ such that any solution of the metriplectic system* (4.1) *(corresponding to φ_{e_M}) starting in U approaches $K \cap \mathcal{E}$.*

Proof. Let $e_M = (0, -M, M) \in \mathcal{E}$ be a Lyapunov stable equilibrium state of the Rössler system (2.1). Recall from Theorem (3.1) that e_M is a Lyapunov stable equilibrium state for the system (2.1) if and only if $M > -1$. Recall that e_M it is also an equilibrium state for the system (4.1) for any smooth real function φ. In order to prove the theorem, we construct a Lyapunov type function that verifies the hypothesis of LaSalle's principle. Let

$$(x, y, z) \in \mathbb{R}^3 \mapsto L_{\varphi_{e_M}}(x, y, z) = \frac{1}{2}(x^2 + y^2) + z + \varphi_{e_M}\left(\, ze^{-y}\right) \in \mathbb{R}$$

be a smooth real function, where $\varphi_{e_M} \in C^\infty(\mathbb{R}, \mathbb{R})$ is given by

$$\varphi_{e_M}(t) = e^{-2M} \cdot \frac{M+2}{M+1} \cdot \frac{t^2}{2} - e^{-M} \cdot \left[\frac{M(M+2)}{M+1} + 1\right] \cdot t.$$

Using these functions, we construct a candidate for a Lyapunov type function that verifies LaSalle's principle.

Let $L_{e_M} \in C^\infty(\mathbb{R}^3, \mathbb{R})$ be the smooth function given by

$$L_{e_M}(x, y, z) = L_{\varphi_{e_M}}(x, y, z) - L_{\varphi_{e_M}}(0, -M, M).$$

Note that the condition $L_{e_M}(e_M) = 0$ is automatically satisfied, and also we have that $\mathbf{d}L_{e_M}(e_M) = 0$. Hence, to check the first condition of LaSalle's principle, i.e., $L_{e_M}(x, y, z) > L_{e_M}(e_M) = 0$, locally for $(x, y, z) \neq e_M = (0, -M, M)$, it is enough to prove that $\mathbf{d}^2 L_{e_M}(e_M)$ is positive definite.

This is true indeed, because

$$\mathbf{d}^2 L_{e_M}(e_M) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & M^2 + (M+1)^{-1} & 1 - M(M+2)(M+1)^{-1} \\ 0 & 1 - M(M+2)(M+1)^{-1} & (M+2)(M+1)^{-1} \end{bmatrix}$$

is positive definite, since $M > -1$.

To check the last condition of LaSalle's principle we compute first $\dot{L}_{e_M}$.

$$\begin{aligned}
\dot{L}_{e_M}(x,y,z) &= [\nabla H(x,y,z) + \nabla(\varphi_{e_M} \circ C)(x,y,z)]^T (\dot{x}, \dot{y}, \dot{z})^T \\
&= [\nabla H(x,y,z) + \nabla(\varphi_{e_M} \circ C)(x,y,z)]^T [\nu(x,y,z)\Pi_C(x,y,z)\nabla H(x,y,z) \\
&+ G(x,y,z)\nabla(\varphi_{e_M} \circ C)(x,y,z)] \\
&= -\left[\varphi'_{e_M}\left(ze^{-y}\right)\right]^2 \cdot e^{-2y} \cdot \left[x^2 + x^2 z^2 + (y+z)^2\right] \\
&\leq 0.
\end{aligned}$$

Using the above relation and the analytic expression of φ'_{e_M}, we get that

$$E_{e_M} := \{(x,y,z) \in \mathbb{R}^3 \mid \dot{L}_{e_M}(x,y,z) = 0\} = \mathcal{E} \cup \Sigma_M,$$

where $\Sigma_M := \left\{(x,y,z) \in \mathbb{R}^3 \mid ze^{-y} = e^M[M + (M+1)(M+2)^{-1}]\right\}$ is a symplectic leaf of the Poisson manifold $(\mathbb{R}^3, \nu\Pi_C)$, and consequently a dynamically invariant set. Hence, the largest dynamically invariant subset $\mathcal{M}_{e_M} \subseteq E_{e_M}$ coincides with E_{e_M}. Now the conclusion follows from LaSalle's principle together with the remark that $e_M = (0, -M, M) \in \mathcal{E}$. $\square$

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