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About a class of rational TC-Bézier curves with two shape parameters

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To the memory of Professor Mircea-Eugen Craioveanu (1942-2012)

Abstract. In this paper, we will study some properties concerning the cubic rational trigonometric Bézier curve attached at a class of cubic trigonometric Bézier curves with two shape parameters (for short TC-Bézier curves) introduced in paper [6].

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1. Introduction

In the following lines, we will present some well known results about Bézier curves.

A Bézier curve is defined using the classical Bernstein polynomials, in the following way:

$$P(t) = \sum_{i=0}^{n} B_{i,n}(t) p_i \tag{1.1}$$

where p_i with $i = \overline{0,n}$, represent the control points attached to Bézier curve and

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

with $t \in [0,1]$ represent the Bernstein polynomials.

A cubic Bézier curve can be obtained for $n = 3$ and have the following form:

$$P(t) = \binom{3}{0}(1-t)^3 p_0 + \binom{3}{1} t(1-t)^2 p_1 + \binom{3}{2} t^2(1-t) p_2 + \binom{3}{3} t^3 p_3$$

or, for short:

$$P(t) = (1-t)^3 p_0 + 3t(1-t)^2 p_1 + 3t^2(1-t) p_2 + t^3 p_3$$

A rational Bézier curve is given by:

$$x(t) = \frac{w_0 p_0 B_{0,n}(t) + ... + w_n p_n B_{n,n}(t)}{w_0 B_{0,n}(t) + ... + w_n B_{n,n}(t)} \tag{1.2}$$

Here, w_i with $i = \overline{0,n}$ represent the weights of the control points p_i. We can rewrite (1.2) in the following way:

$$x(t) = \frac{\sum_{i=0}^{n} w_i p_i B_{i,n}(t)}{\sum_{i=0}^{n} w_i B_{i,n}(t)} \tag{1.3}$$

The authors of paper [6], H. Liu, L. Li and Z. Daming, have replaced the classical Bernstein base of the cubic Bézier curve with a new one which has 2 parameters λ and μ.

The trigonometric base choosed by the authors of paper [6] for the cubic TC Bézier curve, is:

$$\begin{cases} B_{0,3}(t) = 1 - (1+\lambda)\sin t + \lambda \sin^2 t \\ B_{1,3}(t) = (1+\lambda)\sin t - (1+\lambda)\sin^2 t \\ B_{2,3}(t) = (1+\mu)\cos t - (1+\mu)\cos^2 t \\ B_{3,3}(t) = 1 - (1+\mu)\cos t + \mu \cos^2 t \end{cases} \tag{1.4}$$

where $t \in \left[0, \frac{\pi}{2}\right]$ and $\lambda, \mu \in [-1, 1]$.

Other results concerning classical and trigonometric Bézier curves are obtained in the following papers:[1], [2], [3], [4], [5], [7] and [8].
Next, we will present some important results obtained in paper [6].

Theorem 1.1. ([6]) *The basis functions (1.4) have the following properties:*
(1) Nonnegativity and partition of unity: $B_{i,3}(t) \geq 0$, $i \in \{0, 1, 2, 3\}$.
(2) Monotonicity: For a given parameter t, $B_{0,3}(t)$ *and* $B_{3,3}(t)$ *are monotonically decreasing for the shape parameters* λ *and* μ*; respectively;* $B_{1,3}(t)$ *and* $B_{2,3}(t)$ *are monotonically increasing for the shape parameters* λ *and* μ*; respectively;*
(3) Symmetry: $B_{i,3}(t; \lambda, \mu) = B_{3-i,3}(\frac{\pi}{2} - t; \lambda, \mu)$ *for* $i = \overline{0,3}$.

Definition 1.2. ([6]) *Given points* p_i, *(*$i = \overline{0,3}$*) in* $\mathbb{R}^2$, $\mathbb{R}^3$, *then*

$$r(t) = \sum_{i=0}^{3} p_i B_{i,3}(t) \tag{1.5}$$

$t \in \left[0, \frac{\pi}{2}\right]$; $\lambda, \mu \in [0, 1]$, *is called a cubic trigonometric Bézier curve with two shape parameters, i.e. the TC-Bézier curve for short.*

Theorem 1.3. *([6]) (partial enounce) The cubic TC-Bézier curves (1.5) have the following properties:*
(1) Terminal properties:

$$\begin{cases} r(0) = p_0 \\ r\left(\frac{\pi}{2}\right) = p_3; \end{cases} \quad \begin{cases} r'(0) = (1+\lambda)(p_1 - p_0) \\ r'\left(\frac{\pi}{2}\right) = (1+\mu)(p_3 - p_2); \end{cases}$$

$$\begin{cases} r''(0) = 2\lambda p_0 - 2(1+\lambda)p_1 + (1+\mu)p_2 + (1-\mu)p_3 \\ r''\left(\frac{\pi}{2}\right) = (1-\lambda)p_0 + (1+\lambda)p_1 - 2(1+\mu)p_2 + 2\mu p_3; \end{cases}$$

(2) Symmetry: p_0, p_1, p_2, p_3 and p_3, p_2, p_1, p_0 define the same TC-Bézier curve in different parametrizations.
(3) Convex hull property: The entire TC-Bézier segment must lie inside its control polygon spanned by p_0, p_1, p_2, p_3.

For more details on TC-Bézier curves, please see [6].

2. Main results

Using the TC-Bézier curve presented before in this paper, we can introduce the cubic rational TC-Bézier curves, as follows:

$$r(t) = \frac{\sum_{i=0}^{3} w_i p_i B_{i,3}(t)}{\sum_{i=0}^{3} w_i B_{i,3}(t)} \tag{2.1}$$

with $\lambda, \mu \in [-1, 1]$ and w_i are the weights of the control points p_i with $i = \overline{0,3}$ and $B_{i,3}(t)$ represent the trigonometric basis introduced in (1.4).

We can rewrite (2.1) in the following way:

$$r(t) = \frac{(1-(1+\lambda)\sin t+\lambda\sin^2 t)w_0p_0+(1+\lambda)(\sin t-\sin^2 t)w_1p_1+(1+\mu)(\cos t-\cos^2 t)w_2p_2+(1-(1+\mu)\cos t+\mu\cos^2 t)w_3p_3}{(1-(1+\lambda)\sin t+\lambda\sin^2 t)w_0+(1+\lambda)(\sin t-\sin^2 t)w_1+(1+\mu)(\cos t-\cos^2 t)w_2+(1-(1+\mu)\cos t+\mu\cos^2 t)w_3}$$

where $\lambda, \mu \in [-1, 1]$, $t \in \left[0, \frac{\pi}{2}\right]$.

Theorem 2.1. *The curvature in $t = 0$ for the rational TC-Bézier curve (2.1), is:*

$$K(0) = \left(\frac{1+\mu}{1+\lambda}\right)\frac{w_0}{w_1^2}\left(w_3\frac{\|\overline{p_0p_1}\times\overline{p_0p_3}\|}{\|\overline{p_0p_1}\|^3} - w_2\frac{\|\overline{p_0p_1}\times\overline{p_0p_2}\|}{\|\overline{p_0p_1}\|^3}\right)$$

Proof. We start with r(t) defined in (2.1). After tedious computations for $\overline{r'(t)}$ and $\overline{r''(t)}$, one obtains for $t = 0$, the following result:

$$\overline{r'(0)} = -\frac{w_1}{w_0}(p_0\lambda - p_1\lambda + p_0 - p_1) = -(\lambda+1)\frac{w_1}{w_0}\overline{p_0p_1}$$

Then, we obtain:

$$\overline{r''(0)} = -\frac{1}{w_0^2}[w_0w_2(1+\mu)\overline{p_0p_2} -$$
$$-2\left(w_1^2 + 2w_1^2\lambda + w_1^2\lambda^2 - w_0w_1\lambda - w_0w_1\lambda^2\right)\overline{p_0p_1} - w_0w_3(1+\mu)\overline{p_0p_3}]$$

From the curvature definition, for $t = 0$, we know that:

$$K(0) = \frac{\left\|\overline{r'(0)}\times\overline{r''(0)}\right\|}{\left\|\overline{r'(0)}\right\|^3}$$

Now, we compute:

$$\overline{r'(0)} \times \overline{r''(0)} = -\lambda^2 \frac{w_1}{w_0^3} \left[w_0 w_2 (1+\mu)(\overline{p_0 p_1} \times \overline{p_0 p_2} - w_0 w_3 (1+\mu)(\overline{p_0 p_1} \times \overline{p_0 p_3}) \right] =$$

$$\lambda^2 \frac{w_1}{w_0^2} \left[w_3 (\overline{p_0 p_1} \times \overline{p_0 p_3}) - w_2 (\overline{p_0 p_1} \times \overline{p_0 p_2}) \right]$$

Also, one obtains:

$$\left\| \overline{r'(0)} \right\|^3 = \lambda^3 \frac{w_1^3}{w_0^3} \left\| \overline{p_0 p_1} \right\|^3$$

Finally, we get:

$$K(0) = \frac{\lambda^2 \frac{w_0 w_1}{w_0^3} \left[w_3 (1+\mu) \left\| \overline{p_0 p_1} \times \overline{p_0 p_3} \right\| - w_2 (1+\mu) \left\| \overline{p_0 p_1} \times \overline{p_0 p_2} \right\| \right]}{\lambda^3 \frac{w_1^3}{w_0^3} \left\| \overline{p_0 p_1} \right\|^3}$$

$$= \left(\frac{1+\mu}{1+\lambda} \right) \frac{w_0}{w_1^2} \left(w_3 \frac{\left\| \overline{p_0 p_1} \times \overline{p_0 p_3} \right\|}{\left\| \overline{p_0 p_1} \right\|^3} - w_2 \frac{\left\| \overline{p_0 p_1} \times \overline{p_0 p_2} \right\|}{\left\| \overline{p_0 p_1} \right\|^3} \right)$$

and this complete the proof. □

Remark 2.2. For the particular case, when we have the same weights $w_0 = w_1$, one obtains one of the well known results from Theorem 1.3, which was:

$$r'(0) = (1+\lambda)(p_1 - p_0).$$

Next, we will reparametrizate the TC-Bézier rational curve and we take $t = \arcsin(u)$ with $t \in [0,1] \subset \left[0, \frac{\pi}{2}\right]$.
After reparametrization, we get:

$$r(t) = \frac{(1-(1+\lambda)u+\lambda u^2)w_0 p_0 + (1+\lambda)(u-u^2)w_1 p_1 + (1+\mu)(\sqrt{1-u^2}-1+u^2)w_2 p_2 + (1-(1+\mu)\sqrt{1-u^2}+\mu(1-u^2))w_3 p_3}{(1-(1+\lambda)u+\lambda u^2)w_0 + (1+\lambda)(u-u^2)w_1 + (1+\mu)(\sqrt{1-u^2}-1+u^2)w_2 + (1-(1+\mu)\sqrt{1-u^2}+\mu(1-u^2))w_3} \tag{2.2}$$

Remark 2.3. For $\lambda = \mu = 1$, in the above expression (2.2), one obtains the following TC-Bézier rational curve:

$$r(t) = \frac{(1-u)^2 w_0 p_0 + 2(u-u^2)w_1 p_1 + 2(\sqrt{1-u^2}-1+u^2)w_2 p_2 + (1-\sqrt{1-u^2})^2 w_3 p_3}{(1-u)^2 w_0 + 2(u-u^2))w_1 + 2(\sqrt{1-u^2}-1+u^2)w_2 + (1-\sqrt{1-u^2})^2 w_3} \tag{2.3}$$

Theorem 2.4. *The hodograph of the TC-Bézier rational curve (2.3), for $u = 0$, is*

$$2\frac{w_1}{w_0}(p_1 - p_0).$$

Proof. We start with the above expression of the TC-Bézier rational curve (2.3), and we compute:

$$\overline{r'(u)} = \frac{-2(1-u)w_0 p_0 + (2-4u)w_1 p_1 + \left(-\frac{2u}{\sqrt{(1-u^2)}} + 4u\right) w_2 p_2 + \frac{2(1-\sqrt{1-u^2})w_3 p_3 u}{\sqrt{1-u^2}}}{(1-u)^2 w_0 + (2u-2u^2)w_1 + (2\sqrt{1-u^2}+2u^2-2)w_2 + (1-\sqrt{1-u^2})^2 w_3}$$

$$- \frac{(1-u)^2 w_0 p_0 + (2u-2u^2)w_1 p_1 + (2\sqrt{1-u^2}+2u^2-2)w_2 p_2 + (1-\sqrt{1-u^2})^2 w_3 p_3}{(1-u)^2 w_0 + (2u-2u^2)w_1 + (2\sqrt{1-u^2}+2u^2-2)w_2 + (1-\sqrt{1-u^2})^2 w_3} \cdot$$

$$\cdot \left(-2(1-u)w_0 p_0 + (2-4u)w_1 p_1 + \left(-\frac{2u}{\sqrt{(1-u^2)}} + 4u\right) w_2 p_2 + \frac{2(1-\sqrt{1-u^2})w_3 p_3 u}{\sqrt{1-u^2}} \right)$$

Replacing in the above expression $u = 0$, we get:

$$2\frac{w_1}{w_0}(p_1 - p_0). \tag{2.4}$$

and this end the proof of the theorem. $\square$

Remark 2.5. The hodograph of the classical rational Bézier curve for $u = 0$ is

$$3\frac{w_1}{w_0}(p_1 - p_0),$$

and this is a closed result obtained by us in (2.4).

Conclusion. In this paper we proved that the two shape parameters of one TC-Bézier rational curve have a key role when we compute the curvature of the curve. The computation of the torsion for this class of TC-Bézier rational curve is not an easy task. In a future paper we will try to continue our investigations on TC-Bézier rational curves.

References

[1] Farin, G.E., *Curves and surfaces for computer-aided geometric design: a practical code*, Academic Press Inc., 1996

[2] Han, X., *Quadratic trigonometric polynomial curves with a shape parameter*, Computer Aided Geometric Design, **19**(7)(2002), 479-502.

[3] Han, X., *Cubic Trigonometric Polynomial Curves with a Shape parameter*, Computer Aided Geometric Design, **21**(6)(2004), 535-548.

[4] Han, X., *C2 Quadratic trigonometric polynomial curves with local bias*, Journal of Computational and Applied Math., **180**(1)(2005), 161-172.

[5] Han, X., Ma, Y., Huang, L., *The cubic trigonometric Bézier curve with two shape parameters*, Applied Math. Letters, **22**(2)(2009)(2), 226-231.

[6] Liu, H., Li, L., Zhang, D., *Study on a class of TC-Bézier curve with shape parameters*, Journal of Information and Computational Science, **8**(7)(2011), 1217-1223.

[7] Sarfraz, M., Khan, M., *Automatic outline capture of arabic fonts*, Journal of Information Sciences, **140**(3)(2002), 269-281.

[8] Uzma, B., Abbas, M., Awang, M., Ali, J., *The quadratic trigonometric Bézier curve with single shape parameter*, Journal of Basic and Applied Scientific Research, **2**(3)(2012), 2541-2546.

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