

# Exact discrete Morse functions on surfaces

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*To the memory of Professor Mircea-Eugen Craioveanu (1942-2012)*

**Abstract.** In this paper, we review some basic facts about discrete Morse theory, we introduce the Morse-Smale characteristic for a finite simplicial complex, and we construct  $\mathbb{Z}_2$ -exact discrete Morse functions on the torus with two holes  $\mathbf{T}_2$  and on the dunce hat  $\mathbf{DH}$ .

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Let  $K$  be a finite simplicial complex. A function  $f : K \rightarrow \mathbb{R}$  is a *discrete Morse function* if for every simplex  $\alpha^{(p)} \in K$  we have simultaneously :

$$\#\{\beta^{(p+1)} > \alpha^{(p)} \mid f(\beta) \leq f(\alpha)\} \leq 1, \quad \#\{\gamma^{(p+1)} < \alpha^{(p)} \mid f(\gamma) \geq f(\alpha)\} \leq 1,$$

where  $\#A$  denotes the cardinality of the set  $A$ .

Note that a discrete Morse function is not a continuous function on the complex  $K$  since we did not consider any topology on  $K$ . Rather, it is an assignment of a single number to each simplex.

The other main ingredient in discrete Morse theory is the notion of a critical simplex of discrete function. A  $p$ -dimensional simplex  $\alpha^{(p)}$  is *critical* if the following relations hold simultaneously :

$$\#\{\beta^{(p+1)} > \alpha^{(p)} \mid f(\beta) \leq f(\alpha)\} = 0, \quad \#\{\gamma^{(p+1)} < \alpha^{(p)} \mid f(\gamma) \geq f(\alpha)\} = 0.$$

The study of the discrete version of the Morse theory was initiated by R. Forman [9], [10].

If  $K$  is a  $m$ -dimensional simplicial complex with a discrete Morse function, then let  $\mu_j = \mu_j(f)$  denote the number of critical simplices of dimension  $j$  of the function  $f$ . For any field  $F$ , let  $\beta_j = \dim H_j(K, F)$  be the  $j$ -th Betti number with respect to  $F$ ,  $j = 0, 1, \dots, m$ .

Let  $K$  be a  $m$ -dimensional simplicial complex with a discrete Morse function  $f$ . Then the following relations also hold in the discrete context.

(1) The weak discrete Morse's inequalities.

(i)  $\mu_j \geq \beta_j, j = 0, 1, \dots, m;$

(ii)  $\mu_0 - \mu_1 + \mu_2 - \dots + (-1)^m \mu_m = \beta_0 - \beta_1 + \beta_2 - \dots + (-1)^m \beta_m = \chi(K)$ .

The last relation is called Euler's relation.

(2) Also, the strong discrete Morse's inequalities are valid in this context, that is for each  $j = 0, 1, \dots, m - 1$ , we have

$$\mu_j - \mu_{j-1} + \dots + (-1)^j \mu_0 \geq \beta_j - \beta_{j-1} + \dots + (-1)^j \beta_0.$$

Let  $K$  be a  $m$ -dimensional simplicial complex containing exactly  $c_j$  simplices of dimension  $j$ , for each  $j = 0, 1, \dots, m$ . Let  $C_j(K, \mathbb{Z})$  denote the space  $\mathbb{Z}^{c_j}$ . More precisely,  $C_j(K, \mathbb{Z})$  denotes the free Abelian group generated by the  $j$ -simplices of  $K$ , each endowed with an orientation. Then for each  $j$ , there are boundary maps  $\partial_j : C_j(K, \mathbb{Z}) \rightarrow C_{j-1}(K, \mathbb{Z})$ , such that  $\partial_{j-1} \circ \partial_j = 0$ .

The resulting differential complex

$$0 \longrightarrow C_m(K, \mathbb{Z}) \xrightarrow{\partial_m} C_{m-1}(K, \mathbb{Z}) \xrightarrow{\partial_{m-1}} \dots \xrightarrow{\partial_1} C_0(K, \mathbb{Z}) \longrightarrow 0$$

calculates the singular homology of  $K$ . That is, if we define the quotient space

$$H_j(C, \partial) = \text{Ker}(\partial_j) / \text{Im}(\partial_{j+1}),$$

then for each  $j$  we have the isomorphism

$$H_j(C, \partial) \cong H_j(K, \mathbb{Z}),$$

where  $H_j(K, \mathbb{Z})$  denotes the singular homology of  $K$ .

The discrete Morse theory is the main tool in studying some geometric properties of finite simplicial complexes. In this respect we refer to the papers of D.Andrica and I.C.Lazăr [3]-[6], K.Crowley [8], and I.C. Lazăr [11], [12].

### 1. The discrete Morse-Smale characteristic

Consider  $K^m$  to be a  $m$ -dimensional finite simplicial complex.

The discrete Morse-Smale characteristic of  $K$  was considered in paper [13], and it is a natural extension of the well-known Morse-Smale characteristic of a manifold (see the monograph [2]).

Let  $\Omega(K)$  be the set of all discrete Morse functions defined on  $K$ . It is clear that  $\Omega(K)$  is nonempty, because, for instance, the trivial example of discrete Morse function defined by  $f(\sigma) = \dim \sigma$ ,  $\sigma \in K$ .

For  $f \in \Omega(K)$ , let  $\mu_j(f)$  be the number of  $j$ -dimensional critical simplices of  $f$ ,  $j = 0, 1, \dots, m$ .

Let  $\mu(f)$  be the number defined as follows:

$$\mu(f) = \sum_{j=0}^m \mu_j(f),$$

i.e.  $\mu(f)$  is the total number of critical simplices of  $f$ . The number

$$\gamma(K) = \min\{\mu(f) : f \in \Omega(K)\}$$

is called the *discrete Morse Smale characteristic* of the simplicial complex  $K$ . So, the discrete Morse-Smale characteristic represents the minimal number of critical simplices for all discrete Morse functions defined on  $K$ .

In analogous way, one can define the numbers  $\gamma_j(K), j = 0, 1, \dots, m$ , by

$$\gamma_j(K) = \min\{\mu_j(f) : f \in \Omega(K)\},$$

that is the minimal numbers of critical of  $j$ -dimensional simplices, for all discrete Morse functions defined on  $K$ .

The effective computation of these numbers associated to a finite simplicial complex is an extremely complicated problem in combinatorial topology. A finite algorithm for the determination of these numbers for any simplicial complex is not yet known.

## 2. Exact discrete Morse functions and $F$ -perfect Morse functions

Consider the finite  $m$ -dimensional simplicial complex  $K$ . For  $j = 0, 1, \dots, m$ , let  $H_j(K, F)$  be the singular homology groups with the coefficients in the field  $F$ , and let  $\beta_j(K, F) = \text{rank}H_j(K, F) = \dim_F H_j(K, F)$  be the Betti numbers with respect to  $F$ .

For every  $f \in \Omega(K)$ , we have the discrete weak Morse inequalities :

$$\mu_j(f) \geq \beta_j(K, F), \quad j = 0, 1, \dots, m.$$

The discrete Morse function  $f \in \Omega(K)$  is called *exact* (or *minimal*) if  $\mu_j(f) = \gamma_j(K)$ , for all  $j = 0, 1, \dots, m$ . So, an exact discrete Morse function has a minimal number of critical simplices in each dimension.

The discrete Morse function  $f \in \Omega(K)$  is called  *$F$ -perfect* if  $\mu_j(f) = \beta_j(K, F)$ ,  $j = 0, 1, \dots, m$ .

Using the discrete weak Morse inequalities and the definition of the discrete Morse-Smale characteristic, we obtain the inequalities:

$$\mu(f) \geq \min\{\mu_j(f) : f \in \Omega(K)\} = \gamma_j(K) \geq \beta_j(K, F).$$

**Theorem.** *The simplicial complex  $K$  has  $F$ -perfect discrete Morse functions if and only if  $\gamma(K) = \beta(K, F)$ , where*

$$\beta(K, F) = \sum_{j=0}^m \beta_j(K, F)$$

*is the total Betti number of  $K$  with respect to the field  $F$ .*

*Proof.* To prove the direct implication, let  $f \in \Omega(K)$  be a fixed  $F$ -perfect discrete Morse function. Using the weak Morse inequalities, it follows:

$$\mu(f) = \sum_{j=0}^m \mu_j(f) \geq \sum_{j=0}^m \beta_j(K, F) = \beta(K, F),$$

hence  $\mu(f) \geq \beta(K, F)$ . Using the definition of the discrete Morse-Smale characteristic of  $K$ , we get

$$\gamma(K) = \min\{\mu(f) : f \in \Omega(K)\} \geq \beta(K, F).$$

Because  $f$  is a discrete  $F$ -perfect Morse function on  $K$ , we have  $\mu(f) = \beta(K, F)$ . On the other hand, clearly we have the inequality

$$\gamma(K) = \min\{\mu(g) : g \in \Omega(K)\} \leq \beta(K, F).$$

Therefore, we get  $\gamma(K) \leq \beta(K, F)$  and the desired relation follows.

For the converse implication, let  $f \in \Omega(K)$  be a discrete Morse function. From the relations

$$\mu(f) = \sum_{j=0}^m \mu_j(f) \text{ and } \beta(K, F) = \sum_{j=0}^m \beta_j(K, F),$$

using the hypothesis  $\gamma(K) = \beta(K, F)$ , it follows

$$\sum_{j=0}^m [\mu_j(f) - \beta_j(K, F)] = 0.$$

From the discrete weak Morse inequalities, we get  $\mu_j(f) - \beta_j(K, F) \geq 0, \quad j = 0, 1, \dots, m$ . All in all, the following relations hold  $\mu_j(f) = \beta_j(K, F), \quad j = 0, 1, \dots, m$ . Therefore,  $f$  is a discrete  $F$ -perfect Morse function.  $\square$

If  $K$  is a simplicial complex of dimension  $m$ , one knows that  $C_j(K, \mathbb{Z}), \quad j = 0, 1, \dots, m$ , is a finitely generated free Abelian group generated by the  $j$ -simplices in  $K$ . Since subgroups and quotient groups of finitely generated groups are again finitely generated, it follows that the homology group  $H_j(K, \mathbb{Z})$  is finitely generated. Therefore, by the fundamental theorem about such groups, we can write  $H_j(K, \mathbb{Z}) \simeq A_j \oplus B_j$ , where  $A_j$  is a free group and  $B_j$  is the torsion subgroup of  $H_j(K, \mathbb{Z})$ .

Therefore, the singular homology groups  $H_j(K, \mathbb{Z}), \quad j = 0, 1, \dots, m$ , are finitely generated. For every  $j = 0, 1, \dots, m$ , we can write

$$H_j(K, \mathbb{Z}) \simeq (\mathbb{Z} \oplus \dots \oplus \mathbb{Z}) \oplus (\mathbb{Z}_{n_{j_1}} \oplus \dots \oplus \mathbb{Z}_{n_{j_{\beta(j)}}}),$$

where  $\mathbb{Z}$  is taken  $\beta_j$  times in the free group,  $j = 0, 1, \dots, m$ . Here  $\beta_j$  represents the Betti numbers of  $K$  with respect to the group  $(\mathbb{Z}, +)$ , that is we have  $\beta_j(K, \mathbb{Z}) = \text{rank}H_j(K, \mathbb{Z}), \quad j = 0, 1, \dots, m$ .

### 3. A $\mathbb{Z}_2$ -exact Morse function on the two holes torus $\mathbf{T}_2$

The torus with two holes  $\mathbf{T}_2$  is the connected sum of two copies of torus  $\mathbb{T}^2$ , that is  $\mathbf{T}_2 = \mathbb{T}^2 \# \mathbb{T}^2$ . In this section we consider the triangulation of  $\mathbb{T}^2$  represented in Figure 1. The singular homology of the torus with two holes  $\mathbf{T}_2$  is given by

$$H_0(\mathbf{T}_2) = \mathbb{Z}, \quad H_1(\mathbf{T}_2) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}, \quad H_2(\mathbf{T}_2) = \mathbb{Z}.$$

Then, using the universal coefficients formula, we easily obtain

$$H_0(\mathbf{T}_2, \mathbb{Z}_2) \simeq \mathbb{Z}_2, \quad H_1(\mathbf{T}_2, \mathbb{Z}_2) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2, \quad H_2(\mathbf{T}_2, \mathbb{Z}_2) \simeq \mathbb{Z}_2.$$

This implies that the  $\mathbb{Z}_2$ -Betti numbers of  $\mathbf{T}_2$  are given by

$$\beta_0(\mathbf{T}_2, \mathbb{Z}_2) = 1, \quad \beta_1(\mathbf{T}_2, \mathbb{Z}_2) = 4, \quad \beta_2(\mathbf{T}_2, \mathbb{Z}_2) = 1,$$

hence the  $\mathbb{Z}_2$ - total Betti number of  $\mathbf{T}_2$  is

$$\beta(\mathbf{T}_2, \mathbb{Z}_2) = \sum_{j=0}^2 \beta_j(\mathbf{T}_2, \mathbb{Z}_2) = 1 + 4 + 1 = 6.$$

We obtain  $\gamma(\mathbf{T}_2) = \beta(\mathbf{T}_2, \mathbb{Z}_2) = 6$ , and this relation implies that we can define on the simplicial complex given by the triangulation of torus with two holes  $\mathbf{T}_2$  in Figure 1,

a discrete Morse function with exactly six critical simplices. This function is  $\mathbb{Z}_2$ -exact and is defined in Figure 1, where the critical simplices are encircled.

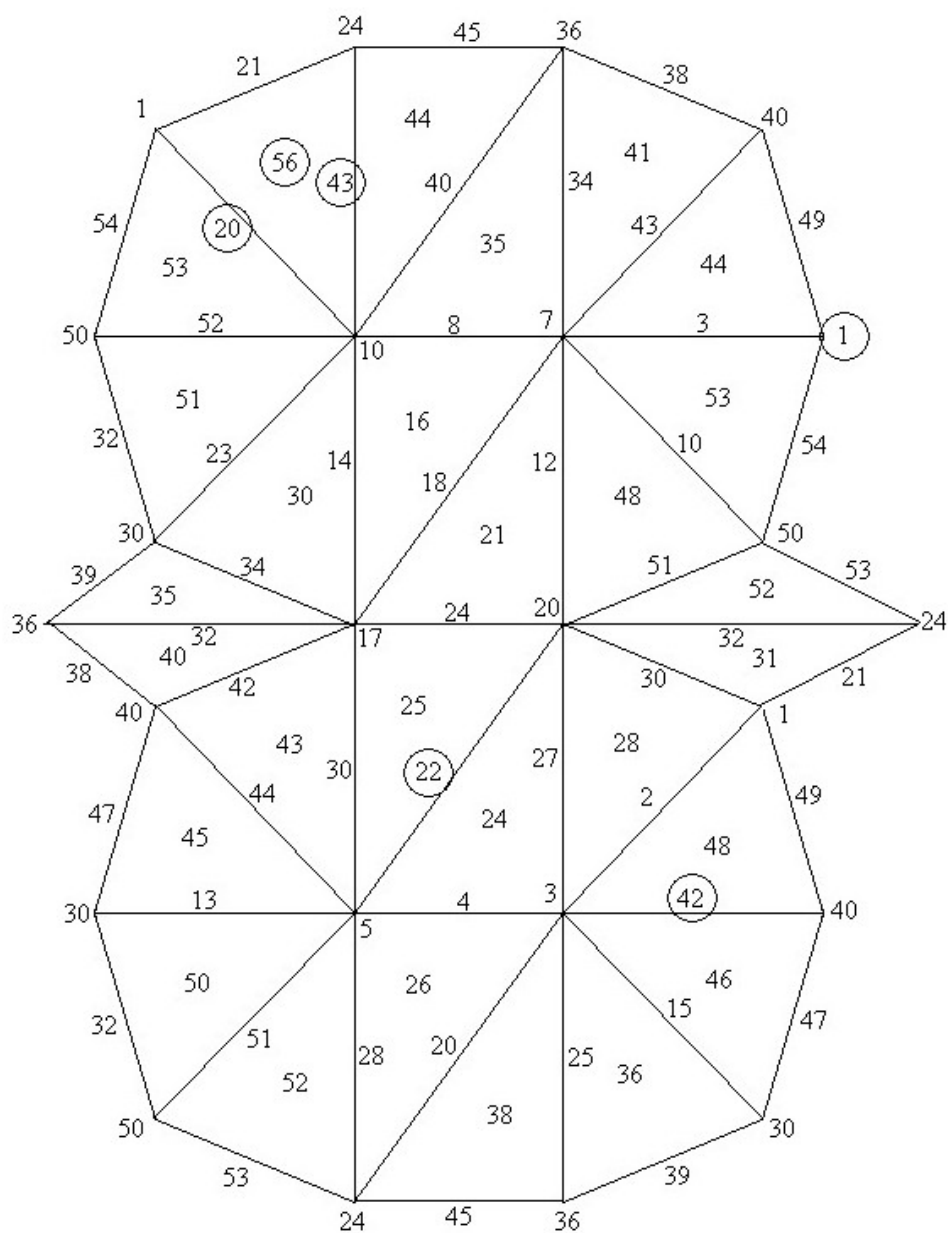


Figure 1.

### 4. A $\mathbb{Z}_2$ -exact Morse function on the dunce hat $\mathbf{DH}$

In topology, the *dunce hat*  $\mathbf{DH}$  is a compact topological space formed by taking a solid triangle and gluing all three sides together, with the orientation of one side reversed. Simply gluing two sides oriented in the same direction would yield a cone much like the layman's dunce cap, but the gluing of the third side results in identifying the base of the cap with a line joining the base to the point.

The space  $\mathbf{DH}$  is contractible, but not collapsible. Contractibility can be easily seen by noting that the dunce hat embeds in the 3-ball and the 3-ball deformation retracts onto the dunce hat. Alternatively, note that the dunce hat is the CW-complex obtained by gluing the boundary of a 2-cell onto the circle. The gluing map is homotopic to the identity map on the circle and so the complex is homotopy equivalent to the disc. By contrast, it is not collapsible because it does not have a free face.

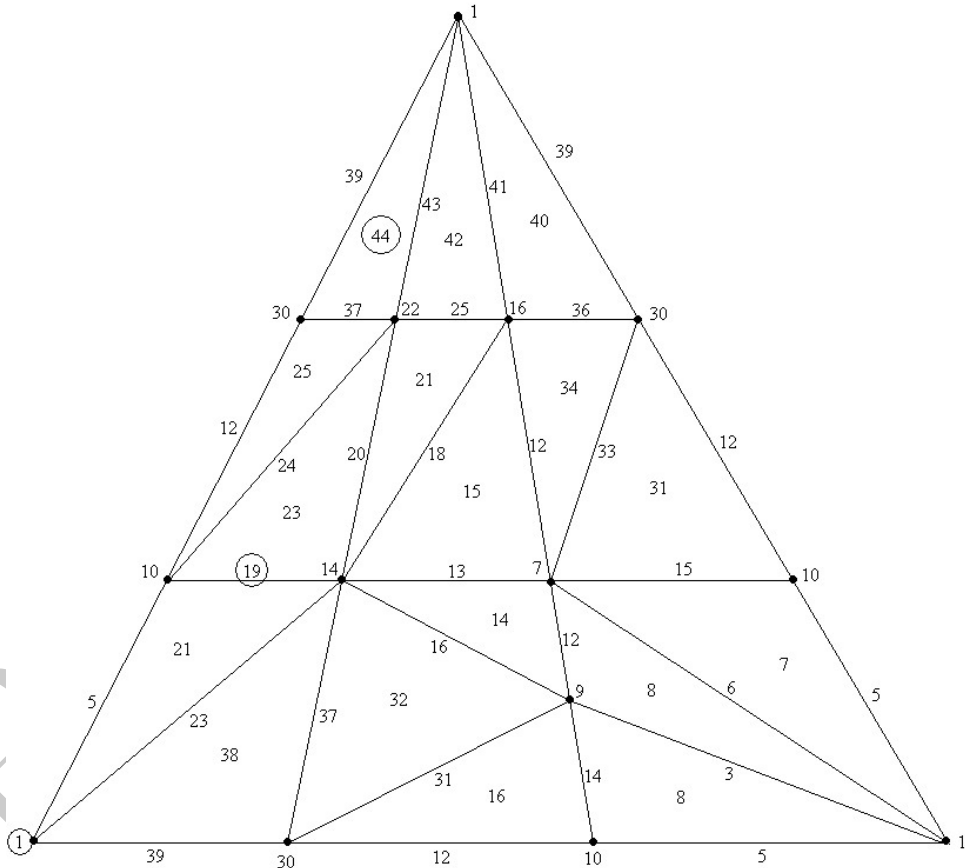


Figure 2.

The name is due to E. C. Zeeman [15], who observed that any contractible 2-complex (such as the dunce hat) after taking the Cartesian product with the closed unit interval seemed to be collapsible. This observation became known as the Zeeman conjecture and was shown by Zeeman to imply the Poincaré conjecture.

We consider the triangulation of the dunce hat **DH** which is shown in Figure 2.

The singular homology of the dunce hat **DH** is

$$H_0(\mathbf{DH}) = \mathbb{Z}, \quad H_1(\mathbf{DH}) = \mathbb{Z}, \quad H_2(\mathbf{DH}) = \mathbb{Z}.$$

Then, using again the universal coefficients formula, we obtain

$$H_0(\mathbf{DH}, \mathbb{Z}_2) \simeq \mathbb{Z}_2, \quad H_1(\mathbf{DH}, \mathbb{Z}_2) \simeq \mathbb{Z}_2, \quad H_2(\mathbf{DH}, \mathbb{Z}_2) \simeq \mathbb{Z}_2.$$

This implies that the  $\mathbb{Z}_2$ -Betti numbers of **DH** are given by

$$\beta_0(\mathbf{DH}, \mathbb{Z}_2) = 1, \quad \beta_1(\mathbf{DH}, \mathbb{Z}_2) = 1, \quad \beta_2(\mathbf{DH}, \mathbb{Z}_2) = 1,$$

and the total Betti number is

$$\beta(\mathbf{DH}, \mathbb{Z}_2) = \sum_{j=0}^2 \beta_j(\mathbf{DH}, \mathbb{Z}_2) = 1 + 1 + 1 = 3.$$

We obtain  $\gamma(\mathbf{DH}) = \beta(\mathbf{DH}, \mathbb{Z}_2) = 3$ , and this property implies that we can define on the simplicial complex given by the triangulation of **DH** in Figure 2 a discrete Morse function with exactly three critical simplices. This function is  $\mathbb{Z}_2$ -exact and is defined in Figure 2, the critical simplices are encircled.

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