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Barycentric and trilinear coordinates of some remarkable points of a hyperbolic triangle

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To the memory of Professor Mircea-Eugen Craioveanu (1942-2012)

Abstract. In this paper we establish the barycentric and trilinear equations of the altitudes and perpendicular bisectors of a hyperbolic triangle and we compute the barycentric and trilinear coordinates of the orthocenter and circumcenter. We, also, indicate necessary and sufficient conditions for these two points to be ordinary points.

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1. Introduction

The purpose of this paper is to give some methods to compute the barycentric and trilinear coordinates for some important points in the hyperbolic triangle. For this, we will use the *Cayley-Klein model*, also called the *projective model*.

In the following, we shall consider the hyperbolic triangle ABC in which the *trilinear coordinates* are defined in a natural way, as the hyperbolic distances from an arbitrary point M in the plane of the triangle to the sides of the triangle. The *barycentric coordinates* are obtained from trilinear coordinates, multiplying the values by the hyperbolic sines of the hyperbolic lengths of the sides of the triangle.

The definition of these coordinates can be given, also, by specifying a particular choice of the polarity that defines the Absolute. This is reflected in the definition of the polarity matrices $[c_{\mu\nu}]$ and $[C_{\mu\nu}]$. We remind these matrices for both of the coordinates systems:

For the trilinear coordinates system:

$$[c_{\mu\nu}] = \frac{1}{\Gamma}\begin{pmatrix} \sin^2 a & \sin A \sin B \cosh c & \sin A \sin C \cosh b \\ \sin A \sin B \cosh c & \sin^2 B & \sin B \sin C \cosh A \\ \sin A \sin C \cosh b & \sin B \sin C \cos a & \sin^2 C \end{pmatrix}$$

and

$$[C_{\mu\nu}] = \begin{pmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{pmatrix}$$

For barycentric coordinates system:

$$[c_{\mu\nu}] = \begin{pmatrix} 1 & \cosh c & \cosh b \\ \cosh c & 1 & \cosh a \\ \cosh b & \cosh a & 1 \end{pmatrix}$$

and

$$[C_{\mu\nu}] = \frac{1}{\Gamma}\begin{pmatrix} -\sinh^2 a & \sinh a \sinh b \cos C & \sinh a \sinh c \cos B \\ \sinh a \sinh b \cos C & -\sinh^2 b & \sinh b \sinh c \cos A \\ \sinh a \sinh c \cos B & \sinh b \sinh c \cos A & -\sinh^2 c \end{pmatrix}.$$

We can pass from point coordinates to line coordinates (and the other way around) by using the relations:

$$\begin{cases} x_\mu = c_{\mu\nu} \cdot \xi_\nu \\ \xi_\mu = C_{\mu\nu} x_\mu. \end{cases}$$

where x_μ are the point coordinates and ξ_ν are the line coordinates.

See [1, 2] for details.

Remark 1.1. A different approach to barycentric coordinates, using the Poincaré disk model, was taken by A. Ungar (see [4]).

2. Barycentric and trilinear equation of the altitudes. Coordinates of the Orthocenter

In the following we will present the coordinates in barycentric coordinates. Having the result in barycentric coordinates system, the reader can easily obtain at any time the coordinates in trilinear coordinate system by dividing each component with the hyperbolic sinus of corresponding side length of the triangle.

Having ABC a hyperbolic triangle, we denote by A' the orthogonal projection of the vertex A on the side BC, B' the orthogonal projection of the vertex B on the side AC and C' the orthogonal projection of the vertex C on the side AB.

For start, we want to obtain the equation of the line AA'. We know that in barycentic coordinates, A is defined by $(1, 0, 0)$.

A general equation of a line, both in barycentric and trilinear coordinates, is of the form:

$$\alpha_0 X_0 + \alpha_1 X_1 + \alpha_2 X_2 = 0. \tag{2.1}$$

Because AA' passes through A, this means that AA' is of the form: $\alpha_1 x_1 + \alpha_2 x_2 = 0$, or, to put it another way, the *line* coordinates of AA', denoted by ξ are:

$$\xi = [0, \alpha_1, \alpha_2]. \tag{2.2}$$

We also know that the side BC, denoted by η has the line coordinates

$$\eta = [1, 0, 0] \tag{2.3}$$

As $AA' \perp BC$, we have the relation

$$[\xi,\eta]=0. \tag{2.4}$$

By definition,

$$\begin{aligned}[\xi,\eta]=\ &C_{00}\cdot\xi_0\nu_0+C_{01}\cdot\xi_0\nu_1+C_{02}\cdot\xi_0\nu_2+\\ &C_{10}\cdot\xi_1\nu_0+C_{11}\cdot\xi_1\nu_1+C_{12}\cdot\xi_1\nu_2+\\ &C_{20}\cdot\xi_2\nu_0+C_{21}\cdot\xi_2\nu_1+C_{22}\cdot\xi_2\nu_2.\end{aligned}$$

From 2.2 and 2.3 we have:

$$\begin{aligned}[\xi,\eta]=\ &C_{00}\cdot 0\cdot 1+C_{01}\cdot 0\cdot 1+C_{02}\cdot 0\cdot 0+\\ &C_{10}\cdot\alpha_1\cdot 1+C_{11}\cdot\alpha_1\cdot 0+C_{12}\cdot\alpha_1\cdot 1+\\ &C_{20}\cdot\alpha_2\cdot 1+C_{21}\cdot\alpha_2\cdot 0+C_{22}\cdot\alpha_2\cdot 0=\\ &=C_{10}\cdot\alpha_1+C_{20}\cdot\alpha_2.\end{aligned}$$

If we use the matrix $[C_{\mu\nu}]$ for barycentric coordinate, we obtain:

$$[\xi,\eta]=\sinh a\cdot\sinh b\cdot\cos C\cdot\alpha_1+\sinh a\cdot\sinh c\cdot\cos B\cdot\alpha_2.$$

By using the condition (2.4), we have:

$$\sinh a\cdot\sinh b\cdot\cos C\cdot\alpha_1+\sinh a\cdot\sinh c\cdot\cos B\cdot\alpha_2=0$$

thus, the relation between α_1 and α_2 is:

$$\alpha_2=-\alpha_1\frac{\sinh b\cdot\cos C}{\sinh c\cdot\cos B}.$$

In conclusion we have the following coordinates for ξ

$$\xi=\left[0,1,-\frac{\sinh b\cdot\cos C}{\sinh c\cdot\cos B}\right] \tag{2.5}$$

or, more simplified:

$$\xi=[0,\sinh c\cdot\cos B,-\sinh b\cdot\cos C].$$

Thus the equation of the altitude AA' is:

$$AA' : X_1\sinh c\cos B-X_2\sinh b\cdot\cos C=0.$$

By performing the same computation for the other two altitudes, we obtain the following

Theorem 2.1. *The equations of the altitudes of the hyperbolic triangle ABC, written in the barycentric coordinates determined by the triangle, are:*

$$\begin{aligned}&AA' : X_1\sinh c\cos B-X_2\sinh b\cdot\cos C=0,\\ &BB' : X_0\sinh c\cdot\cos A-X_2\sinh a\cdot\cos C=0,\\ &CC' : X_0\sinh b\cdot\cos A-X_1\sinh a\cdot\cos B=0.\end{aligned} \tag{2.6}$$

By solving the system (2.6), we obtain:

Consequence 2.2. *The barycentric coordinates of the orthocenter of the hyperbolic triangle ABC are given by:*

$$H\left(\frac{1}{\sinh b \sinh c \cos A}, \frac{1}{\sinh c \sinh a \cos B}, \frac{1}{\sinh a \sinh b \cos C}\right). \tag{2.7}$$

The orthocenter is a real point (i.e. the altitudes do *intersect), iff we have $(H,H) > 0$, i.e. iff*

$$\begin{aligned}
&\left(\sinh^2 a \cos^2 B + 2\sinh a \sinh b \cosh c \cos A \cos B + \sinh^2 b \cos^2 A\right)\cos^2 C+\\
&+\left(2\sinh a \cosh b \sinh c \cos A \cos^2 B + 2\cosh a \sinh b \sinh c \cos^2 A \cos B\right)\cos C+\\
&+\sinh^2 c \cos^2 A \cos^2 B > 0
\end{aligned} \tag{2.8}$$

If we want to use trilinear coordinates, instead, we simple apply a coordinate change to the equations from the Theorem 2.1 and the Consequence 2.2 and we get:

Theorem 2.3. *The equations of the altitudes of the hyperbolic triangle ABC, written in the trilinear coordinates determined by the triangle, are:*

$$\begin{aligned}
AA' &: x_1 \cos B - x_2 \cos C = 0,\\
BB' &: x_0 \cos A - x_2 \cos C = 0,\\
CC' &: x_0 \cos A - x_1 \cos B = 0
\end{aligned} \tag{2.9}$$

Corollary 2.4. *The trilinear coordinates of the orthocenter of the hyperbolic triangle ABC are given by:*

$$H\left(\frac{1}{\cos A}, \frac{1}{\cos B}, \frac{1}{\cos C}\right). \tag{2.10}$$

The orthocenter is an ordinary point iff $(H,H) > 0$, i.e. iff

$$\begin{aligned}
&\cos^2 A \cos^2 B \sin^2 C+\\
&+\left(2\cosh a \cos^2 A \cos B \sin B + 2\cosh b \cos A \sin A \cos^2 B\right)\cos C \sin C+\\
&\left(\cos^2 A \sin^2 B + 2\cosh c \cos A \sin A \cos B \sin B + \sin^2 A \cos^2 B\right)\cos^2 C > 0.
\end{aligned} \tag{2.11}$$

It can be proved that the equations (2.8) and (2.11) are equivalent.

3. Barycentric and trilinear equation of the perpendicular bisectors. Coordinates of the Circumcenter

In order to obtain the line coordinates of the perpendicular bisectors, we use the already known coordinates of the midpoints of the sides of the triangle (see [1]).

If we consider A'' to be the midpoint of BC, B'' – the midpoint of AC and C'' – the midpoint of AB, we have their coordinates in trilinear coordinates:

$$A''\left(0, \sinh\frac{a}{2}\sin C, \sinh\frac{a}{2}\sin B\right);$$
$$B''\left(\sinh\frac{a}{2}\sin C, 0, \sinh\frac{b}{2}\sin A\right);$$
$$C''\left(\sinh\frac{c}{2}\sin B, \sinh\frac{c}{2}\sin A, 0\right);$$

or

$$\begin{aligned} &A''\left(0, \sin C, \sin B\right); \\ &B''\left(\sin C, 0, \sin A\right); \\ &C''\left(\sin B, \sin A, 0\right). \end{aligned} \tag{3.1}$$

From these, we can easily obtain the barycentric coordinates (see [1]).

We denote by ξ the line perpendicular to BC at the point A''. Then the general equation of ξ (in trilinear coordinates) is:

$$\xi : \alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2.$$

We also know that the equation of BC is $\eta : x_0 = 0$. So we have:

$$\xi = [\alpha_0, \alpha_1.\alpha_2],$$
$$\eta = [1, 0, 0].$$

Because we know that A'' is on ξ we have:

$$\alpha_1 \sinh\frac{a}{2}\sin C + \alpha_2 \sinh\frac{a}{2}\sin B = 0$$
$$\alpha_2 = -\alpha_1 \cdot \frac{\sin C}{\sin B}$$

If we replace in the quation of ξ we get:

$$\xi : \alpha_0 \sin B x_0 + \alpha_1 \sin B x_1 - \alpha_1 \sin C x_2 = 0;$$

We know that $\xi \perp \eta$, which implies that $[\xi, \eta] = 0$.

Using the relation (2.4), we obtain:

$$\begin{aligned} [\xi, \eta] &= C_{00}\xi_0\eta_0 + C_{10}\xi_1\eta_1 + C_{20}\xi_2\eta_0 \\ &= C_{00}\alpha_0 \sin B + C_{10}\alpha_1 \sin B - C_{20}\alpha_1 \sin C \\ &= -\alpha_0 \sin B + \cos C \sin B \alpha_1 - \cos B \sin C \alpha_1 \\ &= 0 \end{aligned}$$

Thus

$$\alpha_0 = \alpha_1 \frac{\sin C \cos C - \cos B \sin C}{\sin B} = \alpha_1 \frac{\sin(B - C)}{\sin B}$$

If we replace in the general equation for AA'', we have the form for the perpendicular bisector AA'':

$$\xi_{AA''} = [\sin B \cos C - \cos B \sin C, \sin B, -\sin C]. \tag{3.2}$$

After similar computations, we obtain the theorem:

Theorem 3.1. *The trilinear line coordinates of the perpendicular bisectors of the sides of the hyperbolic triangle ABC are*

$$\begin{aligned} \xi_{AA''} &= [\sin B\cos C - \cos B\sin C, \sin B, -\sin C], \\ \xi_{BB''} &= [\sin A, \sin C\cos A - \cos C\sin A, -\sin C], \\ \xi_{CC''} &= [\sin A, -\sin B, \cos B\sin A - \cos A\sin B]. \end{aligned} \tag{3.3}$$

or

$$\begin{aligned} \xi_{AA''} &= [\sin(B-C), \sin B, -\sin C], \\ \xi_{BB''} &= [\sin A, \sin(C-A), -\sin C], \\ \xi_{CC''} &= [\sin A, -\sin B, \sin(A-B)]. \end{aligned} \tag{3.4}$$

After solving the system of equations of the perpendicular bisectors, we get

Corollary 3.2. *The trilinear coordinates of the circumcenter of the hyperbolic triangle ABC are*

$$\begin{aligned} &O\left(\sin B - \sin(C-A), \sin(C-B) + \sin A,\right. \\ &\left.\sin(C-A)\sin(C-B) + \sin A\sin B\right). \end{aligned} \tag{3.5}$$

O is an ordinary point iff $(O,O) > 0$, i.e. iff

$$\begin{aligned} &\big(\sin^2 C\sin^2(C-A) + 2\cosh a\sin B\sin C\sin(C-A) + \sin^2 B\big)\cdot \\ &\cdot\sin^2(C-B) + \big(-2\cosh b\sin A\sin C\sin^2(C-A) + \\ &\big(2\sin A\sin B\sin^2 C + (2\cosh b + \cosh a)\sin A\sin B\sin C - \\ &-2\cosh c\sin A\sin B\big)\sin(C-A) + 2\cosh a\sin A\sin^2 B\sin C + \\ &+(2\cosh c + 2)\sin A\sin^2 B\big)\sin(C-B) + \sin^2 A\sin^2(C-A) + \\ &+\big((-2\cosh c - 2)\sin^2 A\sin B - 2\cosh b\sin^2 A\sin B\sin C\big)\sin(C-A) + \\ &+\sin^2 A\sin^2 B\sin^2 C + (2\cosh b + 2\cosh a)\sin^2 A\sin^2 B\sin C + \\ &+(2\cosh c + 2)\sin^2 A\sin^2 B > 0. \end{aligned} \tag{3.6}$$

If we pass to the barycentric coordinates, we get immediately, from the Theorem (3.1):

Theorem 3.3. *The barycentric line coordinates of the perpendicular bisectors of the sides of the hyperbolic triangle ABC are*

$$\begin{aligned} \xi_{AA''} &= [\sinh b\sinh c\sin(B-C), \sinh a\sinh c\sin B, -\sinh a\sinh b\sin C], \\ \xi_{BB''} &= [\sinh b\sinh c\sin A, \sinh a\sinh c\sin(C-A), -\sinh a\sinh b\sin C], \\ \xi_{CC''} &= [\sinh b\sinh c\sin A, -\sinh a\sinh c\sin B, \sinh a\sinh b\sin(A-B)]. \end{aligned} \tag{3.7}$$

Also, the consequence (3.2) gives rise to the consequence

Consequence 3.4. *The barycentric coordinates of the circumcenter of the hyperbolic triangle ABC are*

$$\begin{aligned} &O\left(\sinh a(\sin B - \sin(C-A)), \sinh b(\sin(C-B) + \sin A),\right. \\ &\left.\sinh c(\sin(C-A)\sin(C-B) + \sin A\sin B)\right). \end{aligned} \tag{3.8}$$

The point O is ordinary iff $(O,O)>0$, i.e. iff

$$
\begin{aligned}
&\big(\sinh^2 c \sin^2(C-A) + 2\cosh a \sinh b \sinh c \sin(C-A) + \sinh^2 b\big)\cdot\\
&\cdot \sin^2(C-B) + \big(-2\sinh a \cosh b \sinh c \sin^2(C-A) +\\
&+ \big(\big(2\sinh^2 c \sin A + 2\sinh a \cosh b \sinh c\big)\sin B +\\
&+2\cosh a \sinh b \sinh c \sin A - 2\sinh a \sinh b \cosh c)\sin(C-A) +\\
&+ \ (2\cosh a \sinh b \sinh c \sin A + 2\sinh a \sinh b \cosh c)\sin B +\\
&+2\sinh^2 b \sin A\big)\sin(C-B) + \sinh^2 a \sin^2(C-A) +\\
&+ \big(\big(-2\sinh a \cosh b \sinh c \sin A - 2\ \sinh^2 a\big)\sin B -\\
&-2\sinh a \sinh b \cosh c \sin A)\sin(C-A) + \big(\sinh^2 c \sin^2 A +\\
&+2\sinh a \cosh b \sinh c \sin A + \sinh^2 a\big)\sin^2 B +\\
&+ \big(2\cosh a \sinh b \sinh c \sin^2 A + 2\sinh a \sinh b \cosh c \sin A\big)\sin B +\\
&+ \sinh^2 b \sin^2 A > 0.
\end{aligned}
\tag{3.9}
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