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The hyperbolic Desargues theorem in the Poincaré model of hyperbolic geometry

Dorin Andrica and Cătălin Barbu

To the memory of Professor Mircea-Eugen Craioveanu (1942-2012)

Abstract. In this note, we present the hyperbolic Desargues theorem in the Poincaré disc of hyperbolic geometry.

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1. Introduction

Hyperbolic Geometry appeared in the first half of the 19^{th} century as an attempt to understand Euclid's axiomatic basis of Geometry. It is also known as a type of non-Euclidean Geometry, being in many respects similar to Euclidean Geometry. Hyperbolic Geometry includes similar concepts as distance and angle. Both these geometries have many results in common but many are different.

There are known many models for Hyperbolic Geometry, such as: Poincaré disc model, Poincaré half-plane, Klein model, Einstein relativistic velocity model, etc. The hyperbolic geometry is a non-Euclidean geometry. Here, in this study, we present a proof of Desargues theorem in the Poincaré disc model of hyperbolic geometry. The well-known Desargues theorem states that if the three straight lines joining the corresponding vertices of two triangles and all meet in a point, then the three intersections of pairs of corresponding sides lie on a straight line [1]. This result has a simple statement but it is of great interest. We just mention here few different proofs given by N. A. Court [2], H. Coxeter [3], C. Durell [4], H. Eves [5], C.Ogilvy [6], W. Graustein [7].

We begin with the recall of some basic geometric notions and properties in the Poincaré disc. Let D denote the unit disc in the complex z - plane, i.e.

$$D = \{z \in \mathbb{C} : |z| < 1\}.$$

The most general Möbius transformation of D is

$$z \to e^{i\theta} \frac{z_0 + z}{1 + \overline{z_0} z} = e^{i\theta}(z_0 \oplus z),$$

which induces the Möbius addition $\oplus$ in D, allowing the Möbius transformation of the disc to be viewed as a Möbius left gyro-translation

$$z \to z_0 \oplus z = \frac{z_0 + z}{1 + \overline{z_0} z}$$

followed by a rotation. Here $\theta \in \mathbb{R}$ is a real number, $z, z_0 \in D$, and $\overline{z_0}$ is the complex conjugate of z_0. Let $Aut(D, \oplus)$ be the automorphism group of the grupoid $(D, \oplus)$. If we define

$$gyr : D \times D \to Aut(D, \oplus), gyr[a, b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + a\overline{b}}{1 + \overline{a}b},$$

then is true gyro-commutative law

$$a \oplus b = gyr[a, b](b \oplus a).$$

A gyro-vector space $(G, \oplus, \otimes)$ is a gyro-commutative gyro-group $(G, \oplus)$ that obeys the following axioms:

(1) $gyr[\mathbf{u}, \mathbf{v}]\mathbf{a} \cdot gyr[\mathbf{u}, \mathbf{v}]\mathbf{b} = \mathbf{a} \cdot \mathbf{b}$ for all points $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in G$.

(2) G admits a scalar multiplication, $\otimes$, possessing the following properties. For all real numbers $r, r_1, r_2 \in \mathbb{R}$ and all points $\mathbf{a} \in G$:

(G1) $1 \otimes \mathbf{a} = \mathbf{a}$

(G2) $(r_1 + r_2) \otimes \mathbf{a} = r_1 \otimes \mathbf{a} \oplus r_2 \otimes \mathbf{a}$

(G3) $(r_1 r_2) \otimes \mathbf{a} = r_1 \otimes (r_2 \otimes \mathbf{a})$

(G4) $\frac{|r| \otimes \mathbf{a}}{\|r \otimes \mathbf{a}\|} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$

(G5) $gyr[\mathbf{u}, \mathbf{v}](r \otimes \mathbf{a}) = r \otimes gyr[\mathbf{u}, \mathbf{v}]\mathbf{a}$

(G6) $gyr[r_1 \otimes \mathbf{v}, r_1 \otimes \mathbf{v}] = \mathit{1}$

(3) Real vector space structure $(\|G\|, \oplus, \otimes)$ for the set $\|G\|$ of one-dimensional "vectors"

$$\|G\| = \{\pm \|\mathbf{a}\| : \mathbf{a} \in G\} \subset \mathbb{R}$$

with vector addition $\oplus$ and scalar multiplication $\otimes$, such that for all $r \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in G$,

(G7) $\|r \otimes \mathbf{a}\| = |r| \otimes \|\mathbf{a}\|$

(G8) $\|\mathbf{a} \oplus \mathbf{b}\| \leq \|\mathbf{a}\| \oplus \|\mathbf{b}\|$

Definition 1.1. *The hyperbolic distance function in D is defined by the equation*

$$d(a, b) = |a \ominus b| = \left| \frac{a - b}{1 - \overline{a}b} \right|.$$

Here, $a \ominus b = a \oplus (-b)$, for $a, b \in D$.

For further details we refer to the recent book of A. Ungar [8].

Theorem 1.2. (The Menelaus's Theorem for Hyperbolic Gyrotriangle) *Let ABC be a gyrotriangle in a Möbius gyrovector space $(V_s, \oplus, \otimes)$ with vertices $A, B, C \in V_s$, sides $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{V}_s$, and side gyrolengths $a, b, c \in (-s, s)$, $\mathbf{a} = \ominus B \oplus C$, $\mathbf{b} = \ominus C \oplus A$, $\mathbf{c} = \ominus A \oplus B$, $a = \|\mathbf{a}\|$, $b = \|\mathbf{b}\|$, $c = \|\mathbf{c}\|$. If l is a gyroline not through any vertex of a gyrotriangle ABC such that l meets BC in D, CA in E, and AB in F, then*

$$\frac{(AF)_\gamma}{(BF)_\gamma} \cdot \frac{(BD)_\gamma}{(CD)_\gamma} \cdot \frac{(CE)_\gamma}{(AE)_\gamma} = 1,$$

where $v_\gamma = \frac{v}{1-\frac{v^2}{s^2}}$.

(See [9])

Theorem 1.3. (Converse of Menelaus's Theorem for Hyperbolic Gyrotriangle) *If D lies on the gyroline BC, E on CA, and F on AB such that*

$$\frac{(AF)_\gamma}{(BF)_\gamma} \cdot \frac{(BD)_\gamma}{(CD)_\gamma} \cdot \frac{(CE)_\gamma}{(AE)_\gamma} = 1,$$

then D, E, and F are collinear.

(See [9])

2. Main results

In this section we prove the Desargues theorem in the Poincaré disc model of hyperbolic geometry.

Theorem 2.1. The Desargues Theorem for Hyperbolic Gyrotriangle *If ABC, $A'B'C'$ are two gyrotriangles such that the gyrolines AA', BB', CC' meet in O, and BC and $B'C'$ meet at L, CA and $C'A'$ at M, AB and $A'B'$ at N, then L, M, and N are collinear.*

Proof. If we use Menelaus's theorem in the gyrotriangle OBC, cut by the gyroline $B'C'$ (See Theorem 1.2, Figure 1), we get

$$\frac{(LC)_\gamma}{(LB)_\gamma} \cdot \frac{(B'B)_\gamma}{(B'O)_\gamma} \cdot \frac{(C'O)_\gamma}{(C'C)_\gamma} = 1. \tag{2.1}$$

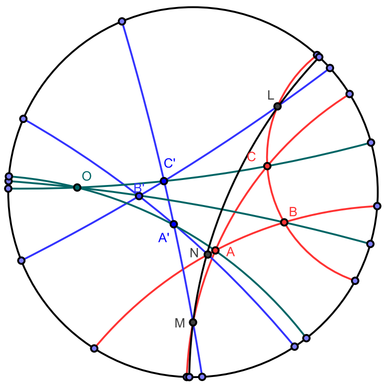

Figure 1

If we use Menelaus's theorem in the gyrotriangle OCA, cut by the gyroline $C'A'$, we get

$$\frac{(MA)_\gamma}{(MC)_\gamma} \cdot \frac{(C'C)_\gamma}{(C'O)_\gamma} \cdot \frac{(A'O)_\gamma}{(A'A)_\gamma} = 1. \tag{2.2}$$

If we use Menelaus's theorem in the gyrotriangle OAB, cut by the gyroline $A'B'$, we get

$$\frac{(NB)_\gamma}{(NA)_\gamma} \cdot \frac{(A'A)_\gamma}{(A'O)_\gamma} \cdot \frac{(B'O)_\gamma}{(B'B)_\gamma} = 1. \tag{2.3}$$

Multiplying the relations (2.1), (2.2) and (2.3), we obtain

$$\frac{(LC)_\gamma}{(LB)_\gamma} \cdot \frac{(MA)_\gamma}{(MC)_\gamma} \cdot \frac{(NB)_\gamma}{(NA)_\gamma} = 1, \tag{2.4}$$

and by Theorem 1.3 we get that the gyropoints L, M, and N are collinear. □

Naturally, one may wonder whether the converse of the Desargues theorem exists. Indeed, a partially converse theorem does exist as we show in the following theorem.

Theorem 2.2. (Converse of Desargues Theorem for Hyperbolic Gyrotriangle) *Let ABC, $A'B'C'$ are two gyrotriangles such that the gyrolines BC and $B'C'$ meet at L, CA and $C'A'$ at M, AB and $A'B'$ at N, and the gyropoints L, M, and N are collinear. If two of the gyrolines AA', BB', CC' meet, then all three are concurrent.*

Proof. Let O be a point of intersection of gyrolines AA' and BB'. Then N is the point of intersection of gyrolines $AB, A'B'$, and MN. If we use Desargues theorem for gyrotriangles $LB'B$ and MAA' we obtain that the points of intersection of the gyrolines AA' and BB', LB and MA, MA' and LB' respectively, are collinear. So, the gyropoints O, C, and C' are collinear, the conclusion follows. □

Many of the theorems of Euclidean geometry are relatively similar form in the Poincaré model of hyperbolic geometry, Desargues theorem is an example in this respect. In the Euclidean limit of large s, $s \to \infty$, v_γ reduces to v, so Desargues theorem for hyperbolic triangle reduces to the Desargues theorem of Euclidean geometry.

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Dorin Andrica
Babeş-Bolyai University
Faculty of Mathematics and Computer Sciences
1, Kogălniceanu Street
400084 Cluj-Napoca, Romania
e-mail: dandrica@yahoo.com

Cătălin Barbu
Vasile Alecsandri National College
600011 Bacău, Romania
e-mail: kafka_mate@yahoo.com