Stud. Univ. Babeş-Bolyai Math. 58(2013), No. 3, 393–400

The equiform differential geometry of curves in 4-dimensional galilean space $\mathbb{G}_4$

M. Evren Aydin and Mahmut Ergüt

Abstract. In this paper, we establish equiform differential geometry of curves in 4-dimensional Galilean space $\mathbb{G}_4$. We obtain the angle between the equiform Frenet vectors and their derivatives in $\mathbb{G}_4$. Also, we characterize generalized helices with respect to their equiform curvatures.

Mathematics Subject Classification (2010): 53A35.

Keywords: Equiform geometry, generalized helices.

1. Introduction

Differential geometry of the Galilean space $\mathbb{G}_3$ has been largely developed in O. Röschel's paper [12]. The Frenet formulas of a curve in 4-dimensional Galilean space $\mathbb{G}_4$ are given by [13]. The helices in $\mathbb{G}_3$ are characterized by [8]. The equiform differential geometry of isotropic spaces and Galilean-pseudo Galilean spaces are represented by [9, 4, 5]. In this paper, we construct equiform differential geometry of curves in $\mathbb{G}_4$.

The Galilean space is three dimensional complex projective space, $\mathbb{P}_3$, in which absolute figure $\{w, f, I_1, I_2\}$ consist of a real plane w (absolute plane), a real line $f \subset w$ (absolute line) and two complex conjugate points, I_1, $I_2 \in f$ (absolute points) [7].

The equiform geometry of Cayley - Klein space is defined by requesting that similarity group of the space preserves angles between planes and lines, respectively. Cayley-Klein geometries are studied for many years. However, they recently have become interesting again since their importance for other fields, like soliton theory [11], have been rediscovered. The positive aspect of this paper is the equiform Frenet formulas and equiform curvatures of $\mathbb{G}_3$ to generalize these of $\mathbb{G}_4$.

2. Preliminaries

Four-dimensional Galilean geometry can be described as the study of properties of four-dimensional space with coordinates that are invariant under general Galilean transformations

$$\begin{aligned} x' &= (\cos\beta\cos\alpha - \cos\gamma\sin\beta\sin\alpha)\,x + (\sin\beta\cos\alpha - \cos\gamma\sin\beta\sin\alpha)\,y \\ &\quad + (\sin\gamma\sin\alpha)\,z + (v\cos\delta_1)\,t + a, \\ y' &= -(\cos\beta\sin\alpha + \cos\gamma\sin\beta\cos\alpha)\,x + (-\sin\beta\sin\alpha - \cos\gamma\cos\beta\cos\alpha)\,y \\ &\quad + (\sin\gamma\cos\alpha)\,z + (v\cos\delta_2)\,t + b, \\ z' &= (\sin\gamma\sin\beta)\,x - (\sin\gamma\cos\beta)\,y + (\cos\gamma)\,z + (v\cos\delta_3)\,t + c, \\ t' &= t + d, \end{aligned}$$

where $\cos^2\delta_1 + \cos^2\delta_2 + \cos^2\delta_3 = 1$.
Given two vectors $\overrightarrow{\alpha}$ and $\overrightarrow{\beta}$ with the coordinates $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and $(\beta_1, \beta_2, \beta_3, \beta_4)$, respectively, then the Galilean scalar product g between the vectors is defined as follows

$$g\left(\overrightarrow{\alpha}, \overrightarrow{\beta}\right) = \begin{cases} \alpha_1\beta_1, & \text{if} \quad \alpha_1 \neq 0 \quad \text{or} \quad \beta_1 \neq 0, \\ \alpha_2\beta_2 + \alpha_3\beta_3 + \alpha_4\beta_4, & \text{if} \quad \alpha_1 = 0 \quad \text{and} \quad \beta_1 = 0. \end{cases} \tag{2.1}$$

For the vectors $\overrightarrow{\alpha}$, $\overrightarrow{\beta}$, $\overrightarrow{\gamma}$ with the coordinates $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, $(\beta_1, \beta_2, \beta_3, \beta_4)$, $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$, the cross product of $\mathbb{G}_4$ given by

$$\overrightarrow{\alpha} \times_{\mathbb{G}} \overrightarrow{\beta} \times_{\mathbb{G}} \overrightarrow{\gamma} = \begin{vmatrix} 0 & \overrightarrow{e}_2 & \overrightarrow{e}_3 & \overrightarrow{e}_4 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \end{vmatrix}, \tag{2.2}$$

where $\overrightarrow{e}_i$ are the standard basis vectors.

Let $C : I \subset \mathbb{R} \longrightarrow \mathbb{G}_4$ be a curve, parametrized by the invariant parameter $s = x$, is given in the coordinate form

$$C(s) = (s, c_1(s), c_2(s), c_3(s)),$$

the Frenet vector fields of the curve C defined by

$$\begin{aligned} V_1 &= (1, \dot{c}_1, \dot{c}_2, \dot{c}_3), \\ V_2 &= \frac{1}{k_1}(0, \ddot{c}_1, \ddot{c}_2, \ddot{c}_3), \\ V_3 &= \frac{1}{k_2}\left(0, \frac{d\left(\frac{1}{k_1}\ddot{c}_1\right)}{ds}, \frac{d\left(\frac{1}{k_1}\ddot{c}_2\right)}{ds}, \frac{d\left(\frac{1}{k_1}\ddot{c}_3\right)}{ds}\right), \\ V_4 &= V_1 \times_{\mathbb{G}} V_2 \times_{\mathbb{G}} V_3, \end{aligned} \tag{2.3}$$

where k_1, k_2, k_3 are the first, second and third curvature functions, respectively, defined by

$$
\begin{aligned}
k_1 &= \left((\ddot{c}_1)^2 + (\ddot{c}_2)^2 + (\ddot{c}_3)^2\right)^{\frac{1}{2}}, \\
k_2 &= \left[g\left(\dot{V}_2, \dot{V}_2\right)\right]^{\frac{1}{2}}, \\
k_3 &= g\left(\dot{V}_3, V_4\right),
\end{aligned}
\tag{2.4}
$$

where the derivative with respect to s denote by a dot. Thus, the Frenet equations of $\mathbb{G}_4$ given by as follows ([13])

$$
\begin{aligned}
\dot{V}_1 &= k_1 V_2, \\
\dot{V}_2 &= k_2 V_3, \\
\dot{V}_3 &= -k_2 V_2 + k_3 V_4, \\
\dot{V}_4 &= -k_3 V_3.
\end{aligned}
$$

3. Frenet formulas in equiform geometry of $\mathbb{G}_4$

Let $C : I \subset \mathbb{R} \longrightarrow \mathbb{G}_4$ be a curve parametrized by arclength s. The equiform parameter of the curve $C(s)$ defined by

$$\sigma = \int \frac{ds}{\rho}, \tag{3.1}$$

where $\rho = \frac{1}{k_1}$ is radius of curvature of the curve. Considering the equation (3.1), it is written that

$$\frac{ds}{d\sigma} = \rho. \tag{3.2}$$

Suppose that h is a homothety with the center in the origin and the coefficient λ. If we take $\tilde{C} = h(C)$, then it can easily be seen that

$$\tilde{s} = \lambda s \quad \text{and} \quad \tilde{\rho} = \lambda \rho, \tag{3.3}$$

where $\tilde{s}$ is the arc-length parameter of $\tilde{C}$ and $\tilde{\rho}$ the radius of curvature of this curve. Hence σ is an equiform invariant parameter of C.

Remark 3.1. Denote by k_1, k_2, k_3 the curvature functions of the curve C. Then, the curvatures k_1, k_2, k_3 are not invariants of the homothety group, because from (2.4), it follows that

$$\tilde{k}_1 = \frac{1}{\lambda} k_1, \ \tilde{k}_2 = \frac{1}{\lambda} k_2, \ \tilde{k}_3 = \frac{1}{\lambda} k_3.$$

Now, if we get

$$\mathbb{V}_1 = \frac{dC}{d\sigma}, \tag{3.4}$$

then using (2.1), we have

$$\mathbb{V}_1 = \rho V_1. \tag{3.5}$$

Also, we define the vectors $\mathbb{V}_2, \mathbb{V}_3, \mathbb{V}_4$ by

$$\mathbb{V}_2 = \rho V_2, \quad \mathbb{V}_3 = \rho V_3, \quad \mathbb{V}_4 = \rho V_4 \tag{3.6}$$

Thus, $\{\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3, \mathbb{V}_4\}$ is an equiform invariant tetrahedron of the curve C.

Now, we will find the derivatives of these vectors with respect to σ using by (3.2), (3.4) and (3.6). For this purpose, it can be written that

$$\mathbb{V}_1' = \frac{d}{d\sigma}(\mathbb{V}_1) = \dot{\rho}\mathbb{V}_1 + \mathbb{V}_2.$$

Similarly, we obtain

$$\begin{aligned} \mathbb{V}_2' &= \frac{d\mathbb{V}_2}{d\sigma} = \dot{\rho}\mathbb{V}_2 + \frac{k_2}{k_1}\mathbb{V}_3, \\ \mathbb{V}_3' &= \frac{d\mathbb{V}_3}{d\sigma} = -\frac{k_2}{k_1}\mathbb{V}_2 + \dot{\rho}\mathbb{V}_3 + \frac{k_3}{k_1}\mathbb{V}_4, \\ \mathbb{V}_4' &= \frac{d\mathbb{V}_4}{d\sigma} = -\frac{k_3}{k_1}\mathbb{V}_3 + \dot{\rho}\mathbb{V}_4, \end{aligned}$$

where the derivatives of the vectors $\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3, \mathbb{V}_4$ with respect to σ are denoted by a dash $(')$.

Definition 3.2. The function $\mathbb{K}_i : I \longrightarrow \mathbb{R}$ $(i = 1, 2, 3)$ is defined by

$$\mathbb{K}_1 = \dot{\rho}, \ \mathbb{K}_2 = \frac{k_2}{k_1}, \ \mathbb{K}_3 = \frac{k_3}{k_1} \tag{3.7}$$

is called $i.th$ equiform curvature of the curve C. It is easy to prove that $\mathbb{K}_i$ is differential invariant of the group of equiform transformations.

Thus the formulas analogous to famous the Frenet formulas in the equiform geometry of the Galilean $4-$space $\mathbb{G}_4$ have the following form:

$$\begin{aligned} \mathbb{V}_1' &= \mathbb{K}_1\mathbb{V}_1 + \mathbb{V}_2, \\ \mathbb{V}_2' &= \mathbb{K}_1\mathbb{V}_2 + \mathbb{K}_2\mathbb{V}_3, \\ \mathbb{V}_3' &= -\mathbb{K}_2\mathbb{V}_2 + \mathbb{K}_1\mathbb{V}_3 + \mathbb{K}_3\mathbb{V}_4, \\ \mathbb{V}_4' &= -\mathbb{K}_3\mathbb{V}_3 + \mathbb{K}_1\mathbb{V}_4, \end{aligned} \tag{3.8}$$

where the functions $\mathbb{K}_1$, $\mathbb{K}_2$, $\mathbb{K}_3$ is the equiform curvatures of this curve.

These formulas can be written in matrix form as follows:

$$\begin{bmatrix} \mathbb{V}_1' \\ \mathbb{V}_2' \\ \mathbb{V}_3' \\ \mathbb{V}_4' \end{bmatrix} = \begin{bmatrix} \mathbb{K}_1 & 1 & 0 & 0 \\ 0 & \mathbb{K}_1 & \mathbb{K}_2 & 0 \\ 0 & -\mathbb{K}_2 & \mathbb{K}_1 & \mathbb{K}_3 \\ 0 & 0 & -\mathbb{K}_3 & \mathbb{K}_1 \end{bmatrix} \begin{bmatrix} \mathbb{V}_1 \\ \mathbb{V}_2 \\ \mathbb{V}_3 \\ \mathbb{V}_4 \end{bmatrix}$$

Because of the equiform Frenet formulas (3.8), the below equalities regarding equiform curvatures can be given

$$\mathbb{K}_i = \begin{cases} \dfrac{1}{\rho^2} g\left(\mathbb{V}_j', \mathbb{V}_j\right), & (j = 1, 2, 3, 4), \quad \text{for } i = 1, \\ \dfrac{1}{\rho^2} g\left(\mathbb{V}_i', \mathbb{V}_{i+1}\right) = -\dfrac{1}{\rho^2} g\left(\mathbb{V}_i, \mathbb{V}_{i+1}'\right), & \text{for } i = 2, 3, \end{cases} \tag{3.9}$$

where $\rho = \frac{1}{k_1}$ is radius of curvature of C.

Theorem 3.3. *Let $C : I \subset \mathbb{R} \longrightarrow \mathbb{G}_4$ be a curve parametrized by arclength s, $\{\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3, \mathbb{V}_4\}$ be the equiform invariant tetrahedron and the function $\mathbb{K}_i : I \longrightarrow \mathbb{R}$ $(i = 1, 2, 3)$ be i.th equiform curvature of the curve C. Then for $1 \leq i \leq 4$, the angle between the vectors $\mathbb{V}_i$ and $\mathbb{V}_i'$ is given as follows*

$$\measuredangle(\mathbb{V}_i, \mathbb{V}_i') = \begin{cases} \rho\sqrt{\mathbb{K}_1^2 - 2\mathbb{K}_1 + 2} & \text{for } i = 1, \\ \arccos\left(\frac{\mathbb{K}_1}{\sqrt{\mathbb{K}_1^2+\mathbb{K}_2^2}}\right), & \text{for } i = 2, \\ \arccos\left(\frac{\mathbb{K}_1}{\sqrt{\mathbb{K}_1^2+\mathbb{K}_2^2+\mathbb{K}_3^2}}\right), & \text{for } i = 3, \\ \arccos\left(\frac{\mathbb{K}_1}{\sqrt{\mathbb{K}_1^2+\mathbb{K}_3^2}}\right), & \text{for } i = 4 \end{cases} \tag{3.10}$$

Proof. For $i = 1$, let θ_1 be the angle between the vectors $\mathbb{V}_1$ and $\mathbb{V}_1'$. Since these vectors are non-isotropic, it is obtained as follows

$$\begin{aligned} \theta_1 &= \left[g(\mathbb{V}_1 - \mathbb{V}_1', \mathbb{V}_1 - \mathbb{V}_1')\right]^{\frac{1}{2}} \\ &= \rho\sqrt{\mathbb{K}_1^2 - 2\mathbb{K}_1 + 2}. \end{aligned}$$

For $i = 2$, denote by θ_2, the angle between the vectors $\mathbb{V}_2$ and $\mathbb{V}_2'$. The vectors $\mathbb{V}_2$ and $\mathbb{V}_2'$ are isotropic and we have

$$\begin{aligned} \cos\theta_2 &= \frac{g(\mathbb{V}_2, \mathbb{V}_2')}{\left[g(\mathbb{V}_2, \mathbb{V}_2)\right]^{\frac{1}{2}}\left[g(\mathbb{V}_2', \mathbb{V}_2')\right]^{\frac{1}{2}}} \\ &= \frac{\mathbb{K}_1}{\sqrt{\mathbb{K}_1^2 + \mathbb{K}_2^2}}. \end{aligned}$$

The others are obtained in a similar way.

4. The characterizations of the curves

The equiform curvatures $\mathbb{K}_i$ $(i = 1, 2, 3)$ in $\mathbb{G}_4$ have important geometric interpretation. For example,

(i) The equiform curvatures of a curve have following form

$$\mathbb{K}_2 = const., \ \mathbb{K}_3 = const., \tag{4.1}$$

if and only if the curve is generalized helix. Here, we do not set condition on $\mathbb{K}_1$.

(ii) If (4.1) holds and $\mathbb{K}_1$ is identically zero, then the curve is a $W-$curve.

Now, we present a few characterizations regarding a curve in $\mathbb{G}_4$ with respect to the its equiform curvatures.

Theorem 4.1. *Let C be a curve in $\mathbb{G}_4$ with the equiform invariant tetrahedron $\{\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3, \mathbb{V}_4\}$ and with equiform curvatures $\mathbb{K}_1 \neq 0$. Then C has $\mathbb{K}_2 \equiv 0$ if and only if C lies fully in a 2-dimensional subspace of $\mathbb{G}_4$.*

Theorem 4.2. *Let C be a curve in $\mathbb{G}_4$ with the equiform invariant tetrahedron $\{\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3, \mathbb{V}_4\}$ and with equiform curvatures $\mathbb{K}_1, \mathbb{K}_2 \neq 0$. Then C has $\mathbb{K}_3 \equiv 0$ if and only if C lies fully in a hyperplane of $\mathbb{G}_4$.*

Proof. If C has $\mathbb{K}_3 \equiv 0$, then from (3.8), we have

$$\begin{aligned} C' &= \mathbb{V}_1, \\ C'' &= \mathbb{K}_1\mathbb{V}_1 + \mathbb{V}_2, \\ C''' &= \left(\rho\dot{\mathbb{K}}_1 + \mathbb{K}_1^2\right)\mathbb{V}_1 + 2\mathbb{K}_1\mathbb{V}_2 + \mathbb{K}_2\mathbb{V}_3, \\ C^{(4)} &= \left(\frac{d\left(\rho\dot{\mathbb{K}}_1 + \mathbb{K}_1^2\right)}{d\sigma} + \left(\rho\dot{\mathbb{K}}_1 + \mathbb{K}_1^2\right)\mathbb{K}_1\right)\mathbb{V}_1 \\ &\quad + \left(\rho\dot{\mathbb{K}}_1 + 3\mathbb{K}_1^2 + 2\rho\dot{\mathbb{K}}_1 - \mathbb{K}_2^2\right)\mathbb{V}_2 \\ &\quad + \left(3\mathbb{K}_1\mathbb{K}_2 + \rho\dot{\mathbb{K}}_2\right)\mathbb{V}_3. \end{aligned}$$

Hence, by using Mclauren expansion for C, given by

$$C(\sigma) = C(0) + C'(0)\sigma + C''(0)\frac{\sigma^2}{2!} + C'''(0)\frac{\sigma^3}{3!} +, ...,$$

we obtain that C lies fully in a hyperplane of $\mathbb{G}_4$ by spanned

$$\{C'(0), C''(0), C'''(0)\}.$$

Conversely, we suppose that C lies fully in a hyperplane Γ of $\mathbb{G}_4$. Then, there exist the points $p, q \in \mathbb{G}_4$ such that C satisfies the equation of Γ given by

$$g(C(\sigma) - p, q) = 0, \tag{4.2}$$

where $q \in \Gamma^{\perp}$. Differentiating (4.2) with respect to σ, we can write

$$g(C', q) = g(C'', q) = g(C''', q) = 0.$$

Since

$$C' = \mathbb{V}_1 \text{ and } C'' = \mathbb{K}_1\mathbb{V}_1 + \mathbb{V}_2,$$

it follows that

$$g(\mathbb{V}_1, q) = g(\mathbb{V}_2, q) = 0. \tag{4.3}$$

Similarly, we have

$$g(\mathbb{V}_3, q) = 0. \tag{4.4}$$

Again, differentiating (4.4)

$$\begin{aligned} 0 &= g(-\mathbb{K}_2\mathbb{V}_2 + \mathbb{K}_1\mathbb{V}_3 + \mathbb{K}_3\mathbb{V}_4, q) \\ 0 &= \mathbb{K}_3 g(\mathbb{V}_4, q), \end{aligned}$$

because $\mathbb{V}_4$ is the only vector perpendicular to $\{\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3\}$, we obtain

$$\mathbb{K}_3 = 0,$$

this completes the proof. □

Last, we give a characterization for a generalized helix in $\mathbb{G}_4$ with respect to the curvatures in equiform geometry.

Theorem 4.3. *Let C be a curve with equiform invariant vector $\mathbb{V}_3$ in the equiform geometry of $\mathbb{G}_4$ is a generalized helix if and only if*

$$\mathbb{V}_3'' + \varphi_1\mathbb{V}_3 = \varphi_2\mathbb{V}_2 + \varphi_3\mathbb{V}_4, \tag{4.5}$$

where $\varphi_1 = \mathbb{K}_2^2 + \mathbb{K}_3^2 - \mathbb{K}_1^2 - \rho\dot{\mathbb{K}}_1$, $\varphi_2 = -2\mathbb{K}_1\mathbb{K}_2$ *and* $\varphi_3 = 2\mathbb{K}_1\mathbb{K}_3$.

Proof. Suppose that the curve C is a generalized helix. Thus, we have

$$\mathbb{K}_2 = const. \text{ and } \mathbb{K}_3 = const. \tag{4.6}$$

From (3.8) and (4.6), it is easy to prove that the equation (4.5) is satisfied.

Conversely, we assume that the equation (4.5) holds. Then from (3.8), it follows that

$$\mathbb{V}_3' = -\mathbb{K}_2\mathbb{V}_2 + \mathbb{K}_1\mathbb{V}_3 + \mathbb{K}_3\mathbb{V}_4, \tag{4.7}$$

and differentiating (4.7) with respect to σ

$$\begin{aligned} \mathbb{V}_3'' &= \left(-\rho\dot{\mathbb{K}}_2 - 2\mathbb{K}_1\mathbb{K}_2\right)\mathbb{V}_2 \\ &\quad + \left(\rho\dot{\mathbb{K}}_1 + \mathbb{K}_1^2 - \mathbb{K}_2^2 - \mathbb{K}_3^2\right)\mathbb{V}_3 \\ &\quad + \left(\rho\dot{\mathbb{K}}_3 + 2\mathbb{K}_1\mathbb{K}_3\right)\mathbb{V}_4, \end{aligned}$$

so, we obtain

$$\dot{\mathbb{K}}_2 = 0 \quad \text{and} \quad \dot{\mathbb{K}}_3 = 0$$

which completes the proof. □

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M. Evren Aydin and Mahmut Ergüt
Firat University, Department of Mathematics
23119, Elazig, Turkey
e-mail: aydnevren@gmail.com