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On approximation of functions of one variable in spaces with a polynomial weight

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Abstract. In this paper we give some approximation theorems for a general class of discrete type operators. We discuss the linear and nonlinear cases.

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1. Introduction

The creation of the basics of approximation theory can be attributed to the Russian mathematician Chebyshev, who formulated and examined the existence of polynomials furnishing best approximations from a particular function over 150 years ago. One of the first problems in the area was to find the polynomial that best approximated the function $f(x) = x^n$ in the interval $[-1, 1]$ in the class of algebraic polynomials having the degree $n - 1$. Solving that problem, Chebyshev defined polynomials $T_n(x) = cos(narccosx)$, which are now called Chebyshev polynomials and which have been widely used in uniform function approximation. The origin of function approximation theory are also connected with K. Weierstrass, S. N. Bernstein, L. Fejer and D. Jackson. It was at the turn of the 20th century that basic problems of continuous function approximation were formulated. The authors proved that, among other things, each continuous function on the closed and bounded interval could be approximated by an algebraic (trigonometric) polynomial with any predetermined order of accuracy. Another important issue was to efficiently obtain operators approximating a particular function with a predetermined accuracy. In addition, research was centred around estimating the rate of convergence of a series of polynomials to a particular function approximated by the polynomials.

Research on function approximation was justified by it being used in other mathematical fields (especially mathematical analysis, functional analysis and the theory of differential equations) and the progress in that field was influenced by other branches of science. I would like to mention that the Fourier series is used in physics and technology and examining many limit problems comes down to studying approximation issues. The merging of function approximation and other branches of science is now particularly visible thanks to numerical analysis and problems, e.g. Bernstein polynomials are widely used in e.g. computer graphics.

The development and directions of research on function approximation by linear operators have been defined in numerous publications and dissertations.

Our research has aimed at generalizing the aforementioned results concerning approximation of functions by positive linear operators. Such research usually needs to be carried out using more subtle proving methods, and results obtained in this way make it possible to come up with additional conclusions.

In the sections more important definitions and theorems are designated by consecutive figures. Definitions and certain properties of the polynomial weighted space and some other designations are denoted as in M. Becker [1].

Similarly to [1], let $p \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$, and let

$$w_0(x) := 1, \quad w_p(x) := (1 + x^p)^{-1}, \qquad \text{if} \qquad p \geq 1. \tag{1.1}$$

Denote by C_p, $p \in \mathbb{N}_0$, the set of all real-valued functions f, continuous on $\mathbb{R}_0 := [0, \infty)$ and such that $w_p f$ is uniformly continuous and bounded on $\mathbb{R}_0$. The norm on C_p is defined by the formula

$$\|f\|_p \equiv \|f(\cdot)\|_p := \sup_{x \in \mathbb{R}_0} w_p(x) \, |f(x)|. \tag{1.2}$$

In the paper [12] it was constructed for any real function f on the interval $\mathbb{R}_0$ a sequence of positive linear operators S_n defined by

$$S_n(f; x) = \sum_{k=0}^{\infty} a_{nk}(x; q) f\left(\frac{k+q}{n}\right), \qquad n, q \in \mathbb{N} := \{1, 2, \cdots\}, \tag{1.3}$$

where $a_{nk}(x; q) := \frac{(nx)^k}{g(nx; q)(k+q)!}$ and $g(0; q) = \frac{1}{q!}$, $g(t; q) = \frac{1}{t^q}\left(e^t - \sum_{k=0}^{q-1} \frac{t^j}{j!}\right)$. These operators possess many remarkable properties. We present a few of them. It is known [12] that for $f \in C_p$, $p \in \mathbb{N}_0$

$$\lim_{n \to \infty} S_n(f; x) = f(x), \tag{1.4}$$

uniformly on every interval $[x_1, x_2]$, $x_2 > x_1 \geq 0$. In [12] it was proved that

$$\lim_{n \to \infty} n(S_n(f; x) - f(x)) = \frac{x}{2} f''(x) \tag{1.5}$$

for all $f \in C_p^2$.

The operators (1.3) are related to the well-known Szász-Mirakyan operators

$$B_n(f; x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right),$$

$x \in \mathbb{R}_0$, $n \in \mathbb{N}$.

In many papers various modification of B_n were introduced and examined. They have been studied intensively. We refer the reader to A. Ciupa [2]- [4], L. Rempulska, A. Thiel [9]- [10]. Many publications on the topic allude to the research of V.Gupta [5] and V. Gupta, P. Maheshwari [6], V. Gupta, R. Yadav [7] and V. Gupta, D. K. Verma [8]. Their results improve other related results in the literature.

The paper pays special attention to defining various classes of operators and examining their certain approximation properties. Because of properties of examined operators, classical (and widely used) methods of proving approximation theorems were employed, in which traditional mathematical problems were subject to subtle and sometimes difficult analytical techniques.

We shall use the modulus of continuity of $f \in C_p$,

$$\omega_1(f;C_p;t) := \sup_{0\le h\le t} \|\Delta_h f(\cdot)\|_p, \qquad t \ge 0,$$

and the modulus of smoothness of $f \in C_p$

$$\omega_2(f;C_p;t) := \sup_{0\le h\le t} \|\Delta_h^2 f(\cdot)\|_p, \qquad t \ge 0,$$

where

$$\Delta_h f(x) := f(x+h) - f(x),\ \Delta_h^2 f(x) := f(x) - 2f(x+h) + f(x+2h).$$

In this paper we shall denote by $M_k(\alpha,\beta), k = 1,2,...,$ suitable positive constants depending only on indicated parameters α,β.

Similarly as in the paper [14] we introduce the following class of operators in C_p.

Definition 1.1. *We define the class of operators S_n by the formula*

$$S_n(f;F_{n,r};x) := \sum_{k=0}^{\infty} a_{nk}(x;q) F_{n,r}\left(f\left(\frac{k+q}{n}\right)\right),\ f \in C_p,\ p \in \mathbb{N}_0,\ q \in \mathbb{N}, \quad (1.6)$$

where $(F_{n,r})_1^\infty$, is a sequence of continuous functions on $\mathbb{R} := (-\infty,+\infty)$ such that $\sup_{x\in\mathbb{R}} w_r(|x|)\,|F_{n,r}(x)-x| \le \frac{M_1(r)}{b_n}$, $(b_n)_1^\infty$ is an increasing sequence of positive numbers with the property $\lim_{n\to\infty} b_n = \infty$.

2. Preliminary results

In this section we shall give some results, which we shall apply to the proofs of the main theorems.

First we give some properties of the operators S_n.

Lemma 2.1. ([12]) *Let $p \in \mathbb{N}_0$ and $q \in \mathbb{N}$ be fixed numbers. Then there exists $M_2(p,q)$ such that*

$$\|S_n(1/w_p)\|_p \le M_2(p,q). \quad (2.1)$$

Moreover for every $f \in C_p$ we have

$$\|S_n(f)\|_p \le M_2(p,q)\|f\|_p. \quad (2.2)$$

The formula (2.2) shows that $S_n(f)$ is a positive linear operators on C_p.

Now we shall give approximation theorems for S_n.

Theorem 2.2. ([12]) *Let $p \in \mathbb{N}_0$ be a fixed number. Then there exists $M_3(p,q)$ such that for every $f \in C_p$ and $n \in \mathbb{N}$ we have*

$$w_p(x)|S_n(f;x)-f(x)| \le M_3(p,q)\omega_1\left(f;C_p;\sqrt{\frac{x+1}{n}}\right),\ x \in \mathbb{R}_0. \tag{2.3}$$

Now we shall give some properties of the operators (1.6).

Lemma 2.3. *Let $(F_{n,r})_1^\infty$, $n,r \in \mathbb{N}$, be a sequence of continuous functions on $\mathbb{R}$ such that $\sup_{x\in\mathbb{R}} w_r(|x|)\,|F_{n,r}(x)-x| \le \frac{M_1(r)}{b_n}$, where $(b_n)_1^\infty$ is an increasing sequence of positive numbers and $\lim_{n\to\infty} b_n = \infty$. For every $p \in \mathbb{N}_0$ we have*

$$\|S_n(f;F_{n,r})\|_{pr} \le M_4(p,q,r,b_1),\ \ f \in C_p.$$

The above inequality shows that $S_n(f;F_{n,r})$ is well-defined on the space C_{pr}.

Proof. For $f \in C_p$ and $p,q,r \in \mathbb{N}$ we have

$$w_{pr}(x)|S_n(f;F_{n,r};x)| \le w_{pr}(x)\sum_{k=0}^{\infty} a_{nk}(x;q)\left|F_{n,r}\left(f\left(\frac{k+q}{n}\right)\right)\right|.$$

$$\le w_{pr}(x)\sum_{k=0}^{\infty} a_{nk}(x;q)$$

$$\times\left\{\left|F_{n,r}\left(f\left(\frac{k+q}{n}\right)\right)-f\left(\frac{k+q}{n}\right)\right|+\left|f\left(\frac{k+q}{n}\right)\right|\right\}$$

From (1.3) by our assumption we get

$$w_{pr}(x)|S_n(f;F_{n,r};x)| \le w_{pr}(x)\sum_{k=0}^{\infty} a_{nk}(x;q)$$

$$\times\left\{\frac{M_1(r)}{b_n}\left(1+\left|f\left(\frac{k+q}{n}\right)\right|^r\right)+\left|f\left(\frac{k+q}{n}\right)\right|\right\}$$

$$\le M_5(r,q,b_1)w_{pr}(x)\left\{1+S_n\left(|f(t)|^r;x\right)+S_n\left(|f(t)|;x\right)\right\}$$

Observe that

$$w_{pr}(x)S_n\left(|f(t)|^r;x\right) \le M_6(p,q,r)\|f\|_{pr}w_{pr}(x)S_n\left(1/w_{pr}(t);x\right) \le M_7(p,q,r). \tag{2.4}$$

From this we immediately obtain

$$\|S_n(f;F_{n,r})\|_{pr} \le M_8(p,q,r,b_1),\ \ f \in C_p,\ \ p \in \mathbb{N}.$$

The proof is similar for $p=0$. Thus the proof is completed. □

Theorem 2.4. *If the assumptions of Lemma 2.3 are satisfied then there exists $M_9(p,q,r)$ such that for every $f \in C_p$ and $p \in \mathbb{N}_0$ we have*

$$w_{pr}(x)|S_n(f;F_{n,r};x)-f(x)| \le M_9(p,q,r)\left\{b_n^{-1}+\omega_1\left(f;C_p;\sqrt{\frac{x+1}{n}}\right)\right\}. \tag{2.5}$$

Proof. By (1.6) and (1.3) we get

$$w_{pr}(x)(S_n(f;F_{n,r};x)-f(x))=w_{pr}(x)\sum_{k=0}^{\infty}a_{nk}(x;q)$$
$$\times\left\{\left(F_{n,r}\left(f\left(\frac{k+q}{n}\right)\right)-f\left(\frac{k+q}{n}\right)\right)+\left(f\left(\frac{k+q}{n}\right)-f\left(x\right)\right)\right\}$$
$$=w_{pr}(x)\sum_{k=0}^{\infty}a_{nk}(x;q)\left(F_{n,r}\left(f\left(\frac{k+q}{n}\right)\right)-f\left(\frac{k+q}{n}\right)\right)$$
$$+w_{pr}(x)(S_n(f(t);x)-f(x)).$$

From Theorem 2.2 we have

$$w_{pr}(x)|S_n(f(t);x)-f(x)|\le M_3(p,q,r)\omega_1\left(f;C_p;\sqrt{\frac{x+1}{n}}\right).$$

By our assumptions we get

$$w_{pr}(x)\sum_{k=0}^{\infty}a_{nk}(x;q)\left|F_{n,r}\left(f\left(\frac{k+q}{n}\right)\right)-f\left(\frac{k+q}{n}\right)\right|$$
$$\le\frac{M_1(r)}{b_n}w_{pr}(x)\sum_{k=0}^{\infty}a_{nk}(x;q)\left(1+\left|f\left(\frac{k+q}{n}\right)\right|^r\right)$$
$$=\frac{M_1(r)}{b_n}\left(1+w_{pr}(x)S_n\left(\left|f\left(t\right)\right|^r;x\right)\right).$$

Applying (2.4) we obtain (2.5). □

3. Main results

In this section we shall use the same method to obtain a general class of operators.

Similarly as in the paper [14] let Ω be the set of all infinite matrices $A=[a_{nk}(x)]_{n\in\mathbb{N},k\in\mathbb{N}_0}$, of functions $a_{nk}\in C_0$ having the following properties:

(a) $a_{nk}(x)\ge 0$ for $x\in\mathbb{R}_0$, $n\in\mathbb{N}$, $k\in\mathbb{N}_0$,

(b) $\sum_{k=0}^{\infty}a_{nk}(x)=1$ for $x\in\mathbb{R}_0$, $n\in\mathbb{N}$,

(c) $\sum_{k=0}^{\infty}k^pa_{nk}(x)$, $x\in\mathbb{R}_0$, $n\in\mathbb{N}$, $p\in\mathbb{N}$, is uniformly convergent on $\mathbb{R}_0$ and its sum is a function belonging to the space C_p

(d) for given $p\in\mathbb{N}$ there exists a positive constant $M_{10}(p,A)$ dependent on p and A such that the function

$$T_{n,p}(A;x):=\sum_{k=0}^{\infty}a_{nk}(x)\left(\frac{k}{n}-x\right)^p,\qquad x\in\mathbb{R}_0,\quad n\in\mathbb{N},\tag{3.1}$$

satisfies the conditions

$$\|T_{n,2p}(A;\cdot)\|_{2p}\le M_{10}(p,A)n^{-p},\qquad n\in\mathbb{N},\tag{3.2}$$

and

$$T_{n,1}(A;x)=0.\tag{3.3}$$

We introduce the following class of operators in C_p.

Definition 3.1. *Let $A \in \Omega$ and let $r \in \mathbb{N}$ be a fixed number. We define the class of operators S_n and $S_{n,p}$ by the formulas*

$$S_n(f; F_{n,r}; A; x) := \sum_{k=0}^{\infty} a_{nk}(x) F_{n,r}\left(f\left(\frac{k}{n}\right)\right), \ f \in C_p, \ p \in \mathbb{N}_0, \tag{3.4}$$

$x \in \mathbb{R}_0$, where $(F_{n,r})_1^{\infty}$, is a sequence of continuous functions on $\mathbb{R} := (-\infty, +\infty)$ such that $\sup_{x\in\mathbb{R}} w_r(|x|)\,|F_{n,r}(x) - x| \leq \frac{M_1(r)}{b_n}$, $(b_n)_1^{\infty}$ is an increasing sequence of positive numbers with the property $\lim_{n\to\infty} b_n = \infty$.

In this section we shall give some results, which we shall apply to the proofs of the main theorems.

Definition 3.2. *Let the matrix $A \in \Omega$ and let C_p for a given space with $p \in \mathbb{N}_0$. For $f \in C_p$ we define the operators*

$$K_n(f; A; x) := \sum_{k=0}^{\infty} a_{nk}(x) f\left(\frac{k}{n}\right), \qquad n \in \mathbb{N}, \quad x \in \mathbb{R}_0. \tag{3.5}$$

First we shall give some properties of the operators K_n.

Lemma 3.3. *Let $A \in \Omega$ and $p \in \mathbb{N}_0$. Then there exists $M_{11}(p, A)$ such that*

$$\|K_n(1/w_p; A)\|_p \leq M_{11}(p, A). \tag{3.6}$$

Moreover for every $f \in C_p$ we have

$$\|K_n(f; A)\|_p \leq M_{11}(p, A)\|f\|_p. \tag{3.7}$$

*The formulas (*3.6*) and (*3.7*) show that $K_n(f; A)$ is a positive linear operators on C_p.*

Proof. If $p = 0$, then by (1.1), (1.2) and the property (b) we have $\|K_n(1/w_0)\|_0 = 1$.

Let $p \in N$. By (3.5), 1.1), (3.1), the property (b) and the Hölder inequality we get

$$\begin{aligned}
w_p(x)K_n(1/w_p(t); A; x) &= w_p(x)(1 + K_n(t^p; A; x)) \\
&= w_p(x)(1 + K_n(2^{p-1}(|t - x|^p + x^p); A; x)) \\
&= w_p(x)(1 + 2^{p-1}x^p + 2^{p-1}K_n(|t - x|^p; A; x)) \\
&\leq M_{12}(p) + 2^{p-1}(w_p^2(x)T_{n,2p}(A; x))^{1/2} \\
&\leq M_{12}(p) + 2^{p-1}(w_{2p}(x)T_{n,2p}(A; x))^{1/2}
\end{aligned}$$

From this and by (3.2) we can write

$$\|K_n(1/w_p; A)\|_p \leq M_{12}(p)(1 + \|T_{n,2p}(A)\|_{2p}^{1/2}) \leq M_{11}(p, A).$$

The formulas (3.5) and (1.1) yield

$$\|K_n(f; A)\|_p \leq \|f\|_p \|K_n(1/w_p; A)\|_p$$

for $f \in C_p$. Applying (3.6) we obtain (3.7). □

Now we shall give approximation theorems for K_n.

Theorem 3.4. *Let $p \in \mathbb{N}_0$ be a fixed number. Then there exists a positive constant $M_{13}(p, A)$ such that for every $f \in C_p^2$ we have*

$$w_p(x)|K_n(f;A;x) - f(x)| \le M_{13}(p, A)\frac{\|f''\|_p(1+x)^2}{n}, \qquad n \in \mathbb{N},\ x \in \mathbb{R}_0. \tag{3.8}$$

Proof. For a fixed $x \in \mathbb{R}_0$ and $f \in C_p^2$ we have

$$f(t) = f(x) + f'(x)(t-x) + \int_x^t \int_x^s f''(u)duds, \quad t \in \mathbb{R}_0,$$

which yields

$$f(t) = f(x) + f'(x)(t-x) + \int_x^t (t-u)f''(u)du, \quad t \in \mathbb{R}_0.$$

From this and by (3.5) we deduce that

$$K_n(f(t);A;x) = f(x) + f'(x)K_n(t-x;A;x) + K_n\left(\int_x^t (t-u)f''(u)du;A;x\right) \tag{3.9}$$

for $n \in \mathbb{N}$. By (1.1) and (1.2) we can write

$$\left|\int_x^t (t-u)f''(u)du\right| \le \|f''\|_p\left(\frac{1}{w_p(t)} + \frac{1}{w_p(x)}\right)(t-x)^2.$$

Applying the above inequality, the Hölder inequality (1.1), (3.1), (3.3) and (3.5), we derive from (3.9)

$$\begin{aligned} &w_p(x)\,|K_n(f;A;x) - f(x)| \\ &\le \|f''\|_p\left\{w_p(x)K_n\left(\frac{(t-x)^2}{w_p(t)};A;x\right) + T_{n,2}(A;x)\right\} \le \\ &\le M_{14}(p,A)\|f''\|_p(T_{n,4}(A;x))^{1/2}\{(w_p^2(x)K_n\left(1/w_p^2(t);A;x\right))^{1/2} + 1\} \\ &\le M_{15}(p,A)\|f''\|_p(T_{n,4}(A;x))^{1/2}\{(w_{2p}(x)K_n\left(1/w_{2p}(t);A;x\right))^{1/2} + 1\} \end{aligned}$$

for $n \in \mathbb{N}$. Using (1.1), (1.2), (3.2) and (3.6), we obtain the desired estimate (3.8). □

Theorem 3.5. *Let $p \in \mathbb{N}_0$ be a fixed number. Then there exists $M_{16}(p, A)$ such that for every $f \in C_p$ and $n \in \mathbb{N}$ we have*

$$w_p(x)|K_n(f;A;x) - f(x)| \le M_{16}(p,A)\omega_2\left(f;C_p;\frac{x+1}{n^{1/2}}\right), \ x \in \mathbb{R}_0. \tag{3.10}$$

Proof. Let $x \in \mathbb{R}_0$. Similarly as in [1] we apply the Stieklov function of $f \in C_p$

$$f_h(x) := \frac{4}{h^2}\int_0^{\frac{h}{2}}\int_0^{\frac{h}{2}}[f(x+s+t) - f(x+2(s+t))]dsdt \tag{3.11}$$

for $x \in \mathbb{R}_0$, $h > 0$. From (3.11) we get

$$f_h'(x) = \frac{1}{h^2}\int_0^{\frac{h}{2}}[8\Delta_{h/2}f(x+s) - 2\Delta_h f(x+2s)]ds,$$

$$f_h''(x) = \frac{1}{h^2}\left[8\Delta_{h/2}^2 f(x) - \Delta_h^2 f(x)\right].$$

Consequently

$$\|f_h - f\|_p \le \omega_2 \left(f, C_p; h, \right), \tag{3.12}$$

$$\|f_h''\|_p \le 9h^{-2}\omega_2 \left(f, C_p; h\right), \tag{3.13}$$

for $h > 0$. We see that $f_h \in C_p^2$ if $f \in C_p$. Hence, for $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$, we can write

$$w_p(x) \left|K_n(f; A; x) - f(x)\right| \le w_p(x) \left\{\left|K_n\left(f - f_h; A; x\right)\right| \right.$$
$$\left. + \left|K_n\left(f_h; A; x\right) - f_h(x)\right| + \left|f_h(x) - f(x)\right|\right\} := Z_1 + Z_2 + Z_3.$$

By (3.7) and (3.12) we have

$$Z_1 \le M_{17}(p; A) \left\|f - f_h\right\|_p \le M_{17}(p; A)\omega_2 \left(f, C_p; h\right), \quad Z_3 \le \omega_2 \left(f, C_p; h\right).$$

Applying Theorem 3.4 and (3.13), we get

$$Z_2 \le M_{18}(p, A) \frac{\|f_h''\|_p (1+x)^2}{n} \le$$
$$\le M_{18}(p, A) \frac{9(1+x)^2}{h^2 n} \omega_2 \left(f, C_p; h\right).$$

Combining these and setting $h = \frac{1+x}{n^{1/2}}$, for fixed $n \in \mathbb{N}$, we obtain the inequality (3.10). □

Now we shall give some properties of the operators (3.4).

Lemma 3.6. *Let $(F_{n,r})_1^\infty$, $n, r \in \mathbb{N}$, be a sequence of continuous functions on $\mathbb{R}$ such that $\sup_{x\in\mathbb{R}} w_r(|x|) \left|F_{n,r}(x) - x\right| \le \frac{M_{19}(r)}{b_n}$, where $(b_n)_1^\infty$ is an increasing sequence of positive numbers with the property $\lim_{n\to\infty} b_n = \infty$. For every $A \in \Omega$ and $p \in \mathbb{N}_0$ we have*

$$\|S_n(f; F_{n,r}; A)\|_{pr} \le M_{20}(p, r, A, b_1), \;\; f \in C_p.$$

The above inequality show that $S_n(f; F_{n,r}; A)$ is well-defined on the space C_{pr}.

Proof. For $f \in C_p$ and $p, r \in \mathbb{N}$ we have

$$w_{pr}(x)|S_n(f; F_{n,r}; A; x)| \le w_{pr}(x) \sum_{k=0}^{\infty} a_{nk}(x) \left|F_{n,r}\left(f\left(\frac{k}{n}\right)\right)\right|.$$

$$\le w_{pr}(x) \sum_{k=0}^{\infty} a_{nk}(x) \left\{\left|F_{n,r}\left(f\left(\frac{k}{n}\right)\right) - f\left(\frac{k}{n}\right)\right| + \left|f\left(\frac{k}{n}\right)\right|\right\}$$

From (3.5) by our assumption we get

$$w_{pr}(x)|S_n(f; F_{n,r}; A; x)|$$
$$\le w_{pr}(x) \sum_{k=0}^{\infty} a_{nk}(x) \left\{\frac{M_1(r)}{b_n}\left(1 + \left|f\left(\frac{k}{n}\right)\right|^r\right) + \left|f\left(\frac{k}{n}\right)\right|\right\}$$
$$\le M_{21}(r, A, b_1) w_{pr}(x) \left\{1 + K_n\left(\left|f(t)\right|^r; A; x\right) + K_n\left(\left|f(t)\right|; A; x\right)\right\}$$

Observe that

$$w_{pr}(x) K_n\left(\left|f(t)\right|^r; A; x\right)$$
$$\le M_{21}(p, r, A) \|f\|_{pr} w_{pr}(x) K_n\left(1/w_{pr}(t); A; x\right) \le M_{22}(p, r, A) \tag{3.14}$$

From this we immediately obtain

$$\|S_n(f; F_{n,r}; A)\|_{pr} \le M_{23}(p, r, A, b_1),\ f \in C_p,\ p \in \mathbb{N}.$$

The proof is similar for $p = 0$. Thus the proof is completed. □

Theorem 3.7. *If assumptions of Lemma 3.6 are satisfied then there exists $M_{24}(p, r, A)$ such that for every $f \in C_p$ and $p \in \mathbb{N}_0$ we have*

$$w_{pr}(x)|S_n(f; F_{n,r}; A; x) - f(x)| \le M_{24}(p, r, A) \left\{ b_n^{-1} + \omega_2 \left(f; C_p; \frac{x+1}{n^{1/2}} \right) \right\}. \tag{3.15}$$

Proof. By (3.4) and (3.5) we get

$$\begin{aligned} &w_{pr}(x)(S_n(f; F_{n,r}; A; x) - f(x)) \\ &= w_{pr}(x) \sum_{k=0}^{\infty} a_{nk}(x) \left\{ \left(F_{n,r}\left(f\left(\frac{k}{n}\right)\right) - f\left(\frac{k}{n}\right) \right) + \left(f\left(\frac{k}{n}\right) - f(x) \right) \right\} \\ &= w_{pr}(x) \sum_{k=0}^{\infty} a_{nk}(x) \left(F_{n,r}\left(f\left(\frac{k}{n}\right)\right) - f\left(\frac{k}{n}\right) \right) \\ &\quad + w_{pr}(x)(K_n(f(t); A; x) - f(x)). \end{aligned}$$

From Theorem 3.5 we have

$$w_{pr}(x)|K_n(f(t); A; x) - f(x)| \le M_{25}(p, r, A)\omega_2 \left(f; C_p; \frac{x+1}{n^{1/2}} \right).$$

By our assumptions we get

$$\begin{aligned} &w_{pr}(x) \sum_{k=0}^{\infty} a_{nk}(x) \left| F_{n,r}\left(f\left(\frac{k}{n}\right)\right) - f\left(\frac{k}{n}\right) \right| \\ &\le \frac{M_1(r)}{b_n} w_{pr}(x) \sum_{k=0}^{\infty} a_{nk}(x) \left(1 + \left| f\left(\frac{k}{n}\right) \right|^r \right) \\ &= \frac{M_1(r)}{b_n} \left(1 + w_{pr}(x) K_n\left(|f(t)|^r ; A; x \right) \right). \end{aligned}$$

Applying (3.14) we obtain (3.15). □

Now we shall give one example of operators of the $S_n(f; F_{n,r}; A)$ type. The Baskakov operators

$$V_n(f; x) := \sum_{k=0}^{\infty} \binom{n-1+k}{k} x^k (1+x)^{-n-k} f\left(\frac{k}{n}\right), \qquad x \in \mathbb{R}_0,\ n \in \mathbb{N},$$

for $f \in C_p$, are generated by the matrix $A^* = [a^*_{nk}(x)]_{n\in\mathbb{N},\ k\in\mathbb{N}_0}$ with

$$a^*_{nk}(x) := \binom{n-1+k}{k} x^k (1+x)^{-n-k}, \qquad x \in \mathbb{R}_0,$$

i.e. $V_n(f; x) = K_n(f; A^*; x)$. If $F_{n,r}(x) = x$ for $n \in \mathbb{N}$ and $x \in \mathbb{R}$, then the operators $S_n(f; F_{n,r}; A^*)$ and $V_n(f)$ are identical.

It is worth remarking that the introduced definitions also cover the case of nonlinear operators. To the best of the author's knowledge, there are not many publications on this topic. Another benefit from the definitions that we have proposed is the ability to use the research method to modify other positive linear operators known in literature. We would like to stress that the approximation theorems found in this paper covered results presented in many other papers.

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References

[1] Becker, M., *Global approximation theorems for Szász - Mirakyan and Baskakov operators in polynomial weight spaces*, Indiana Univ. Math. J., **27**(1978), no. 1, 127-142.

[2] Ciupa, A., *Approximation by a generalized Szász type operator*, J. Comput. Anal. Appl., **5**(2003), no. 4, 413-424.

[3] Ciupa, A. *On the approximation by Jakimovski-Leviatan operators*, Automat. Comput. Appl. Math. (ACAM), **16**(2007), no. 2, 9-15.

[4] Ciupa, A., *Approximation properties of a modified Jakimovski-Leviatan operator*, Automat. Comput. Appl. Math. (ACAM), **17**(2008), no. 3, 401-408.

[5] Gupta, V., *An estimate on the convergence of Baskakov-Bézier operators*, J. Math. Anal. Appl., **312**(2005), no. 1, 280-288.

[6] Gupta, V., Maheshwari, P., *On Baskakov-Szász type operators*, Kyungpook Math. J., **43**(2003), no. 3, 315-325.

[7] Gupta, V., Yadav, R., *Rate of convergence for generalized Baskakov operators*, Arab. J. Math. Sci., doi:10.1016/j.ajmsc.2011.08.001.

[8] Gupta, V., Verma, D.K., *Approximation by complex Favard-Szśz-Mirakjan-Stancu operators in compact disks*, Mathematical Sciences, 2012, 6:25 doi:10.1186/2251-7456-6-2.

[9] Rempulska, L., Thiel, A., *Approximation of functions by certain nonlinear integral operators*, Lith. Math. J., **48**(2008), no. 4, 451-462.

[10] Rempulska, L., Thiel, A., *Approximation properties of certain nonlinear summation operators*, Int. J. Pure Appl. Math., **44**(2008), no. 1, 63-74.

[11] Rempulska, L., Walczak, Z., *The strong approximation of differentiable functions by operators of SzászMirakyan and Baskakov type*, Rend. Sem. Mat. Univ. Pol. Torino, **63**(2005), no. 2, 187-196.

[12] Walczak, Z., *Approximation properties of certain linear positive operators in polynomial weighted spaces of functions of one and two variables*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat., **15**(2004), 52-65.

[13] Walczak, Z., *Convergence of Szász-Mirakyan type operators*, Rev. Anal. Numer. Theor. Approx., **36**(2007), no. 1, 107-113.

[14] Walczak, Z., *Approximation theorems for a general class of truncated operators*, Appl. Math. Comput., **217**(2010), 2142-2148.

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