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Analysis of a viscoelastic unilateral and frictional contact problem with adhesion

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Abstract. We consider a mathematical model which describes the quasistatic frictional contact between a viscoelastic body with long memory and a foundation. The contact is modelled with a normal compliance condition in such a way that the penetration is limited and restricted to unilateral constraint and associated to the nonlocal friction law with adhesion, where the coefficient of friction is solution-independent. The bonding field is described by a first order differential equation. We derive a variational formulation written as the coupling between a variational inequality and a differential equation. The existence and uniqueness result of the weak solution under a smallness assumption on the coefficient of friction is established. The proof is based on arguments of time-dependent variational inequalities, differential equations and Banach fixed point theorem.

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1. Introduction

Contact problems involving deformable bodies are quite frequent in the industry as well as in daily life and play an important role in structural and mechanical systems. Contact processes involve complicated surface phenomena, and are modelled by highly nonlinear initial boundary value problems. Taking into account various contact conditions associated with more and more complex behavior laws lead to the introduction of new and non standard models, expressed by the aid of evolution variational inequalities. An early attempt to study frictional contact problems within the framework of variational inequalities was made in [10]. The mathematical, mechanical and numerical state of the art can be found in [23]. In this reference we find a detailed analysis and numerical studies of the adhesive contact problems. Recently a new book [25] introduces to the reader the theory of variational inequalities with emphasis on the study of contact mechanics and, more specifically, on antiplane frictional contact problems. Also, recently existence results were established in [1, 5, 6, 8, 11, 20, 26, 29, 30, 31] in the study of unilateral and frictional contact problems with or without adhesion. In [31] a quasistatic viscoelastic unilateral contact problem with adhesion and friction was studied and an existence and uniqueness result was proved for a coefficient of friction sufficiently small. Also in [7] a dynamic contact problem with nonlocal friction and adhesion between two viscoelastic bodies of Kelvin-Voigt type was studied. An existence result was proved without condition on the coefficient of friction. Here as in [16] we study a mathematical model which describes a frictional and adhesive contact problem between a viscoelastic body with long memory and a foundation. The contact is modelled with a normal compliance condition associated to unilateral constraint and the nonlocal friction law with adhesion. Recall that models for dynamic or quasistatic processes of frictionless adhesive contact between a deformable body and a foundation have been studied in [2, 3, 4, 5, 7, 8, 12, 19, 21, 23, 24, 27, 28]. Following [13, 14] we use the bonding field as an additional state variable β, defined on the contact surface of the boundary. The variable satisfies the restrictions $0 \leq \beta \leq 1$. At a point on the boundary contact surface, when $\beta = 1$ the adhesion is complete and all the bonds are active; when $\beta = 0$ all the bonds are inactive, severed, and there is no adhesion; when $0 < \beta < 1$ the adhesion is partial and only a fraction β of the bonds is active. However, according to [17], the method presented here considers a compliance model in which the compliance term does not represent necessarily a compact perturbation of the original problem without contact. This leads us to study such models, where a strictly limited penetration is permitted with the limit procedure to the Signorini contact problem. In this work as in [31] we derive a variational formulation of the mechanical problem written as the coupling between a variational inequality and a differential equation. We prove the existence of a unique weak solution if the coefficient of friction is sufficiently small, and obtain a partial regularity result for the solution.

The paper is structured as follows. In section 2 we present some notations and preliminaries. In section 3 we state the mechanical problem and give a variational formulation. In section 4 we establish the proof of our main existence and uniqueness result, Theorem 4.1.

2. Notations and preliminaries

Everywhere in this paper we denote by S^d the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$ while $|.|$ represents the Euclidean norm on $\mathbb{R}^d$ and S^d. Thus, for every $u, v \in \mathbb{R}^d$, $u.v = u_i v_i$, $|v| = (v.v)^{\frac{1}{2}}$, and for every $\sigma, \tau \in S^d$, $\sigma.\tau = \sigma_{ij}\tau_{ij}$, $|\tau| = (\tau.\tau)^{\frac{1}{2}}$. Here and below, the indices i and j run between 1 and d and the summation convention over repeated indices is adopted. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary Γ and let ν denote the unit outer normal on Γ. We shall use the notation:

$$H = \left(L^2(\Omega)\right)^d, \; H_1 = \left(H^1(\Omega)\right)^d, \; Q = \left\{\sigma = (\sigma_{ij}) : \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\right\},$$

$$Q_1 = \{\sigma \in Q : div\sigma \in H\}.$$

Note that H and Q are real Hilbert spaces endowed with the respective canonical inner products

$$(u,v)_H = \int_\Omega u_i v_i dx, \quad \langle \sigma, \tau \rangle_Q = \int_\Omega \sigma_{ij}\tau_{ij} dx.$$

The strain tensor is

$$\varepsilon(u) = (\varepsilon_{ij}(u)) = \frac{1}{2}(u_{i,j} + u_{j,i});$$

$div\sigma = (\sigma_{ij,j})$ is the divergence of σ. For every $v \in H_1$ we also use the notation v for the trace of v on Γ and we denote by v_ν and v_τ the normal and tangential components of v on the boundary Γ, given by

$$v_\nu = v.\nu, \quad v_\tau = v - v_\nu \nu.$$

We define, similarly, by σ_ν and σ_τ the normal and the tangential traces of a function $\sigma \in Q_1$, and when σ is a regular function then

$$\sigma_\nu = (\sigma\nu).\nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu \nu,$$

and the following Green's formula holds:

$$\langle \sigma, \varepsilon(v) \rangle_Q + (div\sigma, v)_H = \int_\Gamma \sigma\nu.vda \quad \forall v \in H_1,$$

where da is the surface measure element. Let $T > 0$. For every real Hilbert space X we employ the usual notation for the spaces $L^p(0,T;X)$, $1 \le p \le \infty$, and $W^{1,\infty}(0,T;X)$. Recall that the norm on the space $W^{1,\infty}(0,T;X)$ is given by

$$\|u\|_{W^{1,\infty}(0,T;X)} = \|u\|_{L^\infty(0,T;X)} + \|\dot{u}\|_{L^\infty(0,T;X)},$$

where $\dot{u}$ denotes the first derivative of u with respect to time. Finally, we denote by $C([0,T];X)$ the space of continuous functions from $[0,T]$ to X, with the norm

$$\|x\|_{C([0,T];X)} = \max_{t\in[0,T]} \|x(t)\|_X.$$

Moreover, for a real number r, we use r_+ to represent its positive part, that is $r_+ = \max\{r, 0\}$.

3. Problem statement and variational formulation

We consider the following physical setting. A viscoelastic body with long memory occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ with a regular boundary Γ that is partitioned into three disjoint measurable parts $\Gamma_1, \Gamma_2, \Gamma_3$ such that $meas(\Gamma_1) > 0$. The body is acted upon by a volume force of density φ_1 on Ω and a surface traction of density φ_2 on Γ_2. The body is in unilateral contact with adhesion following the nonlocal friction law with a foundation, over the potential contact surface Γ_3.
Thus, the classical formulation of the mechanical problem is written as follows.

Problem P_1. Find a displacement $u : \Omega \times [0,T] \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times [0,T] \to \mathbf{S}^d$ and a bonding field $\beta : \Gamma_3 \times [0,T] \to [0,1]$ such that for all $t \in [0,T]$,

$$\sigma(t) = F\varepsilon(u(t)) + \int_0^t \mathcal{F}(t-s)\varepsilon(u(s))ds \quad \text{in } \Omega, \tag{3.1}$$

$$div\sigma(t) + \varphi_1(t) = 0 \text{ in } \Omega, \tag{3.2}$$

$$u(t) = 0 \quad \text{on } \Gamma_1, \tag{3.3}$$

$$\sigma(t)\nu = \varphi_2(t) \quad \text{on } \Gamma_2, \tag{3.4}$$

$$\left.\begin{array}{l} u_\nu(t) \le g,\ \sigma_\nu(t) + p(u_\nu(t)) - c_\nu\beta^2(t)R_\nu(u_\nu(t)) \le 0 \\ \big(\sigma_\nu(t) + p(u_\nu(t)) - c_\nu\beta^2(t)R_\nu(u_\nu(t))\big)(u_\nu(t) - g) = 0 \end{array}\right\} \text{ on } \Gamma_3, \tag{3.5}$$

$$\left.\begin{array}{l} \left|\sigma_\tau(t) + c_\tau\beta^2(t)R_\tau(u_\tau(t))\right| \le \mu\left|R\sigma_\nu(u(t))\right| \\ \left|\sigma_\tau(t) + c_\tau\beta^2(t)R_\tau(u_\tau(t))\right| < \mu\left|R\sigma_\nu(u(t))\right| \Rightarrow u_\tau(t) = 0 \\ \left|\sigma_\tau(t) + c_\tau\beta^2(t)R_\tau(u_\tau(t))\right| = \mu\left|R\sigma_\nu(u(t))\right| \Rightarrow \\ \exists\lambda \ge 0 \text{ s.t. } u_\tau(t) = -\lambda\left(\sigma_\tau(t) + c_\tau\beta^2(t)R_\tau(u_\tau(t))\right) \end{array}\right\} \text{ on } \Gamma_3, \tag{3.6}$$

$$\dot{\beta}(t) = -\left[\beta(t)\left(c_\nu(R_\nu(u_\nu(t)))^2 + c_\tau\left|R_\tau(u_\tau(t))\right|^2\right) - \varepsilon_a\right]_+ \quad \text{on } \Gamma_3, \tag{3.7}$$

$$\beta(0) = \beta_0 \quad \text{on } \Gamma_3. \tag{3.8}$$

Equation (3.1) represents the viscoelastic constitutive law with long memory of the material; F is the elasticity operator and $\int_0^t \mathcal{F}(t-s)\varepsilon(u(s))\,ds$ is the memory term in which $\mathcal{F}$ denotes the tensor of relaxation; the stress $\sigma(t)$ at current instant t depends on the whole history of strains up to this moment of time. Equation (3.2) represents the equilibrium equation while (3.3) and (3.4) are the displacement and traction boundary conditions, respectively, in which $\sigma\nu$ represents the Cauchy stress vector. The conditions (3.5) represent the unilateral contact with adhesion in which c_ν is a given adhesion coefficient which may dependent on $x \in \Gamma_3$ and R_ν, R_τ are truncation operators defined by

$$R_\nu(s) = \begin{cases} L & \text{if } s < -L \\ -s & \text{if } -L \le s \le 0 \\ 0 & \text{if } s > 0 \end{cases}, \quad R_\tau(v) = \begin{cases} v & \text{if } |v| \le L, \\ L\dfrac{v}{|v|} & \text{if } |v| > L. \end{cases}$$

Here $L > 0$ is the characteristic length of the bond, beyond which the latter has no additional traction (see [23]) and p is a normal compliance function which satisfies the assumption (3.19); g denotes the maximum value of the penetration which satisfies $g \ge 0$. When $u_\nu < 0$ i.e. when there is separation between the body and the foundation then the condition (3.5) combined with hypothese (3.19) and definition of R_ν shows that $\sigma_\nu = c_\nu\beta^2 R_\nu(u_\nu)$ and does not exeed the value $L\|c_\nu\|_{L^\infty(\Gamma_3)}$. When $g > 0$, the body may interpenetrate into the foundation, but the penetration is limited that is $u_\nu \le g$. In this case of penetration (i.e. $u_\nu \ge 0$), when $0 \le u_\nu < g$ then $-\sigma_\nu = p(u_\nu)$ which means that the reaction of the foundation is uniquely determined by the normal displacement and $\sigma_\nu \le 0$. Since p is an increasing function then the reaction is increasing with the penetration. When $u_\nu = g$ then $-\sigma_\nu \ge p(g)$ and σ_ν is not uniquely determined. When $g > 0$ and $p = 0$, conditions (3.5) become the Signorini's contact conditions with a gap and adhesion

$$u_\nu \le g,\ \sigma_\nu - c_\nu\beta^2 R_\nu(u_\nu) \le 0,\ (\sigma_\nu - c_\nu\beta^2 R_\nu(u_\nu))(u_\nu - g) = 0.$$

When $g = 0$, the conditions (3.5) combined with hypothese (3.19) lead to the Signorini contact conditions with adhesion, with zero gap, given by

$$u_\nu \leq 0,\ \sigma_\nu - c_\nu \beta^2 R_\nu (u_\nu) \leq 0,\ (\sigma_\nu - c_\nu \beta^2 R_\nu (u_\nu)) u_\nu = 0.$$

These contact conditions were used in [26, 29]. It follows from (3.5) that there is no penetration between the body and the foundation, since $u_\nu \leq 0$ during the process. Also, note that when the bonding field vanishes, then the contact conditions (3.5) become the classical Signorini contact conditions with zero gap, that is,

$$u_\nu \leq 0,\ \sigma_\nu \leq 0,\ \sigma_\nu u_\nu = 0.$$

Conditions (3.6) represent Coulomb's law of dry friction with adhesion where μ denotes the coefficient of friction. Equation (3.7) represents the ordinary differential equation which describes the evolution of the bonding field and it was already used in [26]. Since $\dot{\beta} \leq 0$ on $\Gamma_3 \times (0, T)$, once debonding occurs bonding cannot be reestablished and, indeed, the adhesive process is irreversible. Also from [18] it must be pointed out clearly that condition (3.7) does not allow for complete debonding in finite time.

We turn now to the variational formulation of Problem P_1. We denote by V the closed subspace of H_1 defined by

$$V = \{v \in H_1 : v = 0 \text{ on } \Gamma_1\},$$

and let the convex subset of admissible displacements given by

$$K = \{v \in V : v_\nu \leq g \ \ a.e. \text{ on } \Gamma_3\}.$$

Since $meas\,(\Gamma_1) > 0$, the following Korn's inequality holds [10],

$$\|\varepsilon (v)\|_Q \geq c_\Omega \|v\|_{H_1} \quad \forall v \in V, \tag{3.9}$$

where $c_\Omega > 0$ is a constant which depends only on Ω and Γ_1. We equip V with the inner product

$$(u, v)_V = \langle \varepsilon (u), \varepsilon (v) \rangle_Q$$

and $\|.\|_V$ is the associated norm. It follows from Korn's inequality (3.9) that the norms $\|.\|_{H_1}$ and $\|.\|_V$ are equivalent on V. Then $(V, \|.\|_V)$ is a real Hilbert space. Moreover by Sobolev's trace theorem, there exists $d_\Omega > 0$ which only depends on the domain Ω, Γ_1 and Γ_3 such that

$$\|v\|_{(L^2(\Gamma_3))^d} \leq d_\Omega \|v\|_V \quad \forall v \in V. \tag{3.10}$$

We suppose that the body forces and surface tractions have the regularity

$$\varphi_1 \in C([0, T]; H), \qquad \varphi_2 \in C\left([0, T]; \left(L^2(\Gamma_2)\right)^d\right). \tag{3.11}$$

We define the function $f : [0, T] \to V$ by

$$(f(t), v)_V = \int_\Omega \varphi_1(t).v dx + \int_{\Gamma_2} \varphi_2(t).v da \ \ \forall v \in V,\ t \in [0, T], \tag{3.12}$$

and we note that (3.11) and (3.12) imply

$$f \in C([0, T]; V).$$

In the study of the mechanical problem P_1 we assume that the elasticity operator F satisfies

$$\left.\begin{array}{l}
(a)\ F : \Omega \times S^d \ \rightarrow S^d; \\[2ex]
(b) \text{ there exists } M > 0 \text{ such that} \\
\quad \left|F\left(x, \varepsilon_1\right) - F\left(x, \varepsilon_2\right)\right| \leq M\left|\varepsilon_1 - \varepsilon_2\right| \quad \forall \varepsilon_1, \varepsilon_2 \in S^d, \\
\quad a.e.\ x \ \in \Omega; \\[2ex]
(c) \text{ there exists } m > 0 \text{ such that} \\
\qquad \left(F\left(x, \varepsilon_1\right) - F\left(x, \varepsilon_2\right)\right).\left(\varepsilon_1 - \varepsilon_2\right) \geq m\left|\varepsilon_1 - \varepsilon_2\right|^2, \\
\qquad \forall \varepsilon_1, \varepsilon_2 \in S^d,\ a.e.\ x \ \in \Omega; \\[2ex]
(d) \text{ the mapping } x \rightarrow F\left(x, \varepsilon\right) \text{ is Lebesgue measurable on } \Omega \\
\quad \text{for any } \varepsilon \in S^d; \\[2ex]
(e) \text{ the mapping } x \rightarrow F\left(x, 0\right) \in Q.
\end{array}\right\} \tag{3.13}$$

We also need to introduce the space of the tensors of fourth order defined by

$$Q_\infty = \left\{\mathcal{E} = \left(\mathcal{E}_{ijkl}\right) : \mathcal{E}_{ijkl} = \mathcal{E}_{jikl} = \mathcal{E}_{klij} \in L^\infty\left(\Omega\right),\ 1 \leq i, j, k, l \leq d\right\},$$

which is the real Banach space with the norm

$$\|\mathcal{E}\|_{Q_\infty} = \max_{1 \leq i,j,k,l \leq d} \left\|\mathcal{E}_{ijkl}\right\|_{L^\infty(\Omega)}.$$

We assume that the tensor of relaxation $\mathcal{F}$ satisfies

$$\mathcal{F} \in C\left(\left[0, T\right]; Q_\infty\right). \tag{3.14}$$

The adhesion coefficients c_ν, c_τ and ε_a satisfy

$$c_\nu, c_\tau \in L^\infty\left(\Gamma_3\right),\ \varepsilon_a \in L^2\left(\Gamma_3\right) \text{ and } c_\nu, c_\tau,\ \varepsilon_a \geq 0 \ a.e. \text{ on } \Gamma_3, \tag{3.15}$$

and we assume that the initial bonding field satisfies

$$\beta_0 \in L^2\left(\Gamma_3\right);\ 0 \leq \beta_0 \leq 1 \ a.e. \text{ on } \Gamma_3. \tag{3.16}$$

Next, we consider the subset W of H_1 defined as

$$W = \left\{v \in H_1 : div\sigma\left(v\right) \in H\right\}$$

and let $j_c : V \times V \rightarrow \mathbb{R}$, $j_f : \left(V \cap W\right) \times V \rightarrow \mathbb{R}$ be the functionals given by

$$j_c\left(u, v\right) = \int_{\Gamma_3} p\left(u_\nu\right) v_\nu da \quad \forall (u, v) \in V \times V,$$

$$j_f\left(u, v\right) = \int_{\Gamma_3} \mu\left|R\sigma_\nu\left(u\right)\right|\left|v_\tau\right| da \quad \forall (u, v) \in \left(V \cap W\right) \times V,$$

where

$$R : H^{-\frac{1}{2}}\left(\Gamma\right) \rightarrow L^2\left(\Gamma_3\right) \text{ is a linear and continuous mapping (see [9]).} \tag{3.17}$$

The coefficient of friction μ is assumed to satisfy

$$\mu \in L^\infty\left(\Gamma_3\right) \text{ and } \mu \geq 0 \ a.e. \text{ on } \Gamma_3. \tag{3.18}$$

Next we let

$$j = j_c + j_f.$$

We also define the functional

$$h : L^2(\Gamma_3) \times V \times V \to \mathbb{R}$$

by

$$h(\beta, u, v) = \int_{\Gamma_3} (-c_\nu \beta^2 R_\nu(u_\nu) v_\nu + c_\tau \beta^2 R_\tau(u_\tau) .v_\tau) da, \ \forall (\beta, u, v) \in L^2(\Gamma_3) \times V \times V,$$

where the normal compliance function p satisfies:

$$\begin{cases} (a) \ \ p : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+; \\ \\ (b) \ \ \text{there exists } L_p > 0 \text{ such that} \\ \qquad |p(x, r_1) - p(x, r_2)| \leq L_p |r_1 - r_2| \\ \qquad \forall r_1,\ r_2 \in \mathbb{R},\ a.e.\ x \in \Gamma_3; \\ \\ (c) \ \ (p(x, r_1) - p(x, r_2))(r_1 - r_2) \geq 0 \\ \qquad \forall r_1,\ r_2 \in \mathbb{R},\ a.e.\ x \in \Gamma_3; \\ \\ (d) \ \ \text{the mapping } x \to p(x, r) \text{ is Lebesgue measurable on } \Gamma_3, \\ \qquad \text{for any } r \in \mathbb{R}; \\ \\ (e) \ \ p(x, r) = 0 \ \forall r \leq 0,\ a.e.\ x \in \Gamma_3. \end{cases} \tag{3.19}$$

Finally, we need to introduce the following set of the bonding field:

$$B = \left\{ \theta : [0, T] \to L^2(\Gamma_3) : 0 \leq \theta(t) \leq 1, \forall t \in [0, T], a.e. \text{ on } \Gamma_3 \right\}.$$

By a standard procedure based on Green's formula we derive the following variational formulation of Problem P_1, in terms of displacement and bonding field.

Problem P_2. Find a displacement field $u \in C([0, T]; V)$ and a bonding field $\beta \in W^{1,\infty}\left(0, T; L^2(\Gamma_3)\right) \cap B$ such that

$$\begin{aligned} & u(t) \in K \cap W,\ \langle F\varepsilon(u(t)), \varepsilon(v) - \varepsilon(u(t)) \rangle_Q \\ & \qquad\qquad + \left\langle \int_0^t \mathcal{F}(t-s) \varepsilon(u(s)) ds, \varepsilon(v) - \varepsilon(u(t)) \right\rangle_Q \\ & + h(\beta(t), u(t), v - u(t)) + j(u(t), v) - j(u(t), u(t)) \\ & \geq (f(t), v - u(t))_V \quad \forall\ v \in K,\ t \in [0, T], \end{aligned} \tag{3.20}$$

$$\dot{\beta}(t) = -\left[\beta(t)\left(c_\nu (R_\nu(u_\nu(t)))^2 + c_\tau |R_\tau(u_\tau(t))|^2\right) - \varepsilon_a\right]_+ \quad a.e.\ t \in (0, T), \tag{3.21}$$

$$\beta(0) = \beta_0. \tag{3.22}$$

4. Existence and uniqueness of solution

Our main result in this section is the following theorem.

Theorem 4.1. *Let* (3.11) , (3.13), (3.14), (3.15), (3.16) , (3.17), (3.18) *and* (3.19) *hold. Then, there exists a constant* $\mu_0 > 0$ *such that Problem* P_2 *has a unique solution if*

$$\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0.$$

The proof of Theorem 4.1 is carried out in several steps. In the first step, we consider the closed subset Z of the space $C\left([0,T];L^2(\Gamma_3)\right)$ defined as

$$Z = \left\{\theta \in C\left([0,T];L^2(\Gamma_3)\right) \cap B : \theta(0) = \beta_0\right\},$$

where the Banach space $C\left([0,T];L^2(\Gamma_3)\right)$ is endowed with the norm

$$\|\beta\|_k = \max_{t\in[0,T]} \left[\exp(-kt)\,\|\beta(t)\|_{L^2(\Gamma_3)}\right],\ k > 0.$$

Next for a given $\beta \in Z$, we consider the following variational problem.

Problem $P_{1\beta}$. Find $u_\beta \in C([0,T];V)$ such that

$$\begin{aligned}
&u_\beta(t) \in K \cap W,\ \langle F\varepsilon(u_\beta(t)), \varepsilon(v) - \varepsilon(u_\beta(t))\rangle_Q \\
&\qquad + \left\langle \int_0^t \mathcal{F}(t-s)\,\varepsilon(u_\beta(s))\,ds, \varepsilon(v) - \varepsilon(u_\beta(t))\right\rangle_Q \\
&+h(\beta(t), u_\beta(t), v - u_\beta(t)) + j(u_\beta(t), v) - j(u_\beta(t), u_\beta(t)) \\
&\geq (f(t), v - u_\beta(t))_V \quad \forall\, v \in K,\ t \in [0,T].
\end{aligned} \tag{4.1}$$

We have the following result.

Theorem 4.2. *There exists a constant* $\mu_1 > 0$ *such that Problem* $P_{1\beta}$ *has a unique solution if*

$$\|\mu\|_{L^\infty(\Gamma_3)} < \mu_1.$$

To prove this theorem, for $\eta \in C([0,T];Q)$ we consider the following intermediate problem.

Problem $P_{1\beta\eta}$. Find $u_{\beta\eta} \in C([0,T];V)$ such that

$$\begin{aligned}
&u_{\beta\eta}(t) \in K \cap W,\ \langle F\varepsilon(u_{\beta\eta}(t)), \varepsilon(v - u_{\beta\eta}(t))\rangle_Q + \langle \eta(t), \varepsilon(v - u_{\beta\eta}(t))\rangle_Q \\
&+h(\beta(t), u_{\beta\eta}(t), v - u_{\beta\eta}(t)) + j(u_{\beta\eta}(t), v) - j(u_{\beta\eta}(t), u_{\beta\eta}(t)) \\
&\geq (f(t), v - u_{\beta\eta}(t))_V \quad \forall v \in K,\ t \in [0,T].
\end{aligned} \tag{4.2}$$

Since Riesz's representation theorem implies that there exists an element $f_\eta \in C([0,T];V)$ such that

$$(f_\eta(t), v)_V = (f(t), v)_V - \langle \eta(t), \varepsilon(v)\rangle_Q,$$

then Problem $P_{1\beta\eta}$ is equivalent to the following problem.

Problem $P_{2\beta\eta}$. Find $u_{\beta\eta} \in C([0,T];V)$ such that

$$u_{\beta\eta}(t) \in K \cap W,\ \langle F\varepsilon(u_{\beta\eta}(t)), \varepsilon(v - u_{\beta\eta}(t))\rangle_Q + h(\beta(t), u_{\beta\eta}(t), v - u_{\beta\eta}(t))$$
$$+j(u_{\beta\eta}(t), v) - j(u_{\beta\eta}(t), u_{\beta\eta}(t)) \geq (f_\eta(t), v - u_{\beta\eta}(t))_V \quad \forall v \in K,\ t \in [0,T]. \tag{4.3}$$

We now show the proposition below.

Proposition 4.3. *There exists a constant* $\mu_1 > 0$ *such that Problem* $P_{2\beta\eta}$ *has a unique solution if*

$$\|\mu\|_{L^\infty(\Gamma_3)} < \mu_1.$$

We shall prove Proposition 4.3 in several steps by using arguments on Banach fixed point theorem. Indeed, let $q \in C_+$ where C_+ is a non-empty closed subset of $L^2(\Gamma_3)$ defined as

$$C_+ = \left\{ s \in L^2(\Gamma_3);\ s \geq 0 \ a.e. \text{ on } \Gamma_3 \right\}$$

and let the functional $j_q : V \to \mathbb{R}$ given by

$$j_q(v) = \int_{\Gamma_3} \mu q |v_\tau| \, da \ \forall v \in V.$$

We consider the following auxiliary problem.

Problem $P_{2\beta\eta q}$. Find $u_{\beta\eta q} \in C([0,T];V)$ such that

$$u_{\beta\eta q}(t) \in K,\ \langle F\varepsilon(u_{\beta\eta q}(t)), \varepsilon(v - u_{\beta\eta q}(t))\rangle_Q + h(\beta(t), u_{\beta\eta q}(t), v - u_{\beta\eta q}(t))$$
$$+j_c(u_{\beta\eta q}(t), v - u_{\beta\eta q}(t)) + j_q(v) - j_q(u_{\beta\eta q}(t)) \geq (f_\eta(t), v - u_{\beta\eta q}(t))_V,$$
$$\forall v \in K,\ t \in [0,T]. \tag{4.4}$$

We have the following lemma.

Lemma 4.4. *Problem* $P_{2\beta\eta q}$ *has a unique solution.*

Proof. Let $t \in [0,T]$ and let $A_{\beta(t)} : V \to V$ be the operator defined by

$$\left(A_{\beta(t)} u, v\right)_V = \langle F\varepsilon(u), \varepsilon(v)\rangle_Q + h(\beta(t), u, v) + j_c(u, v) \quad \forall u, v \in V.$$

As in [28], using (3.13) (b), (3.13) (c), (3.15), (3.19) (b), (3.19) (c) and the properties of R_ν and R_τ, we see that the operator $A_{\beta(t)}$ is Lipschitz continuous and strongly monotone. On the other hand the functional $j_q : V \to \mathbb{R}$ is a continuous seminorm; using standard arguments on elliptic variational inequalties (see [25]), it follows that there exists a unique element $u_{\beta\eta q}(t) \in K$ which satisfies the inequality (4.4).

Now, for each $t \in [0,T]$, we define the map $\Psi_t : C_+ \to C_+$ by

$$\Psi_t(q) = |R\sigma_\nu(u_{\beta\eta q}(t))|.$$

We show the following lemma.

Lemma 4.5. *There exists a constant* $\mu_1 > 0$ *such that the mapping* Ψ_t *has a unique fixed point* q^* *and* $u_{\beta\eta q^*}(t)$ *is a unique solution of the inequality* (4.3) *if*

$$\|\mu\|_{L^\infty(\Gamma_3)} < \mu_1.$$

Proof. Let q_1, $q_2 \in C_+$. Using (3.17), it follows that there exists a constant $c_0 > 0$ such that

$$\left\| \Psi_t\left(q_1\right) - \Psi_t\left(q_2\right) \right\|_{L^2(\Gamma_3)} \leq c_0 \left\| \sigma_\nu\left(u_{\beta\eta q_1}(t)\right) - \sigma_\nu\left(u_{\beta\eta q_2}(t)\right) \right\|_{H^{-\frac{1}{2}}(\Gamma)}. \tag{4.5}$$

Moreover using (3.13) (b) yields

$$\left\| \sigma_\nu\left(u_{\beta\eta q_1}(t)\right) - \sigma_\nu\left(u_{\beta\eta q_2}(t)\right) \right\|_{H^{-\frac{1}{2}}(\Gamma)} \leq M \left\| u_{\beta\eta q_1}(t) - u_{\beta\eta q_2}(t) \right\|_V. \tag{4.6}$$

We also use (3.10), (3.13) (c), (3.19) (c) and the properties of R_ν and R_τ to find after some calculus algebra that

$$\left\| u_{\beta\eta q_1}(t) - u_{\beta\eta q_2}(t) \right\|_V \leq \frac{\|\mu\|_{L^\infty(\Gamma_3)} d_\Omega}{m} \left\| q_1 - q_2 \right\|_{L^2(\Gamma_3)}. \tag{4.7}$$

Hence, taking into account (3.18), we combine (4.5), (4.6) and (4.7) to deduce that

$$\left\| \Psi_t\left(q_1\right) - \Psi_t\left(q_2\right) \right\|_{L^2(\Gamma_3)} \leq \|\mu\|_{L^\infty(\Gamma_3)} \frac{c_0 M d_\Omega}{m} \left\| q_1 - q_2 \right\|_{L^2(\Gamma_3)}.$$

Take $\mu_1 = m / c_0 M d_\Omega$, then this inequality shows that if $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_1$, Ψ is a contraction; thus it has a unique fixed point q^* and $u_{\beta\eta q^*}(t)$ is a unique solution of (4.3).

Denote $u_{\beta\eta q^*} = u_{\beta\eta}$. We now shall see that $u_{\beta\eta} \in C\left([0,T]; V\right)$. Indeed, let t_1, $t_2 \in [0,T]$. Take $v = u_{\beta\eta}(t_2)$ in (4.3) written for $t = t_1$ and then $v = u_{\beta\eta}(t_1)$ in the same inequality written for $t = t_2$. Using (3.13) (c), (3.17), (3.19) (c) and the properties of R_ν and R_τ, and adding the resulting inequalities, it follows that there exists a constant $c_1 > 0$ such that

$$\left\| u_{\beta\eta}(t_2) - u_{\beta\eta}(t_1) \right\|_V \leq$$

$$\frac{c_1}{m - \|\mu\|_{L^\infty(\Gamma_3)} c_0 M d_\Omega} \left(\|\beta(t_2) - \beta(t_1)\|_{L^2(\Gamma_3)} + \|\eta(t_2) - \eta(t_1)\|_Q + \|f(t_2) - f(t_1)\|_V \right).$$

Then, as $\beta \in C\left([0,T]; L^2\left(\Gamma_3\right)\right)$, $\eta \in C\left([0,T]; Q\right)$ and $f \in C\left([0,T]; V\right)$, we immediately conclude. We also have that $u_{\beta\eta}(t) \in W, \forall t \in [0,T]$. Indeed, for each $t \in [0,T]$, denote $\sigma\left(u_{\beta\eta}(t)\right) = F\varepsilon\left(u_{\beta\eta}(t)\right) + \eta(t)$, take $v = u_{\beta\eta}(t) \pm \varphi$ in inequality (4.3) where $\varphi \in \left(C_0^\infty(\Omega)\right)^d$ and use Green's formula with the regularity $\varphi_1(t) \in H$ leads to $div\sigma\left(u_{\beta\eta}(t)\right) \in H$ and then $u_{\beta\eta}(t) \in W$. □

Now we introduce the operator

$$\Lambda_\beta : C\left([0,T]; Q\right) \to C\left([0,T]; Q\right)$$

defined by

$$\Lambda_\beta \eta(t) = \int_0^t \mathcal{F}(t-s)\, \varepsilon\left(u_{\beta\eta}(s)\right) ds \quad \forall \eta \in C\left([0,T]; Q\right), \, t \in [0,T]. \tag{4.8}$$

We have the lemma below.

Lemma 4.6. *The operator Λ_β has a unique fixed point η_β.*

Proof. Let $\eta_1,\ \eta_2\ \in\ C\left([0,T]\,;Q\right)$. Using (4.3), (4.8) and (3.14) we obtain for $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_1$ that

$$\left\|\Lambda_\beta\eta_1\left(t\right)-\Lambda_\beta\eta_2\left(t\right)\right\|_Q \leq c_2\int_0^t \left\|\eta_1\left(s\right)-\eta_2\left(s\right)\right\|_Q ds\ \ \forall t\in[0,T]\,,$$

where $c_2>0$. Reiterating this inequality n times, yields

$$\left\|\Lambda_\beta^n\eta_1-\Lambda_\beta^n\eta_2\right\|_{C([0,T];Q)} \leq \frac{\left(c_2T\right)^n}{n!}\left\|\eta_1-\eta_2\right\|_{C([0,T];Q)}.$$

As $\lim\limits_{n\to+\infty}\dfrac{\left(c_2T\right)^n}{n!}=0$, it follows that for a positive integer n sufficiently large, Λ_β^n is a contraction; then, by using the Banach fixed point theorem, it has a unique fixed point η_β which is also a unique fixed point of Λ_β i.e.,

$$\Lambda_\beta\eta_\beta\left(t\right)=\eta_\beta\left(t\right)\quad\forall t\in[0,T]\,. \tag{4.9}$$

Then by (4.3) and (4.9) we conclude that $u_{\beta\eta_\beta}$ is the unique solution of Problem $P_{1\beta}$. □

Next denote $u_\beta=u_{\beta\eta_\beta}$. In the second step we state the following problem.

Problem P_{ad}. Find $\beta^*:[0,T]\to L^2\left(\Gamma_3\right)$ such that

$$\dot{\beta}^*\left(t\right)=-\left[\beta^*\left(t\right)\left(c_\nu\left(R_\nu\left(u_{\beta^*\nu}\left(t\right)\right)\right)^2+c_\tau\left|R_\tau\left(u_{\beta^*\tau}\left(s\right)\right)\right|^2\right)-\varepsilon_a\right]_+\quad a.e.\ t\in(0,T)\,, \tag{4.10}$$

$$\beta^*\left(0\right)=\beta_0. \tag{4.11}$$

We obtain the following result.

Proposition 4.7. *Problem P_{ad} has a unique solution β^* which satisfies*

$$\beta^*\in W^{1,\infty}\left(0,T;L^2\left(\Gamma_3\right)\right)\cap B.$$

Proof. Let $t\in[0,T]$ and consider the mapping $\Phi:Z\to Z$ defined by

$$\Phi\beta\left(t\right)=\beta_0-\int_0^t\left[\beta\left(s\right)\left(c_\nu(R_\nu\left(u_{\beta\nu}\left(s\right)\right))^2+c_\tau\left|R_\tau\left(u_{\beta\tau}\left(s\right)\right)\right|^2\right)-\varepsilon_a\right]_+ds,$$

where u_β is the solution of Problem $P_{1\beta}$. For $\beta_1,\ \beta_2\in Z$, there exists a constant $c_2>0$ such that

$$\begin{aligned}&\left\|\Phi\beta_1\left(t\right)-\Phi\beta_2\left(t\right)\right\|_{L^2(\Gamma_3)}\\ &\leq c_2\textstyle\int_0^t\left\|\beta_1\left(s\right)\left(R_\nu\left(u_{\beta_1\nu}\left(s\right)\right)\right)^2-\beta_2\left(s\right)\left(R_\nu\left(u_{\beta_2\nu}\left(s\right)\right)\right)^2\right\|_{L^2(\Gamma_3)}ds\\ &+c_2\textstyle\int_0^t\left\|\beta_1\left(s\right)\left|R_\tau\left(u_{\beta_1\tau}\left(s\right)\right)\right|^2-\beta_2\left(s\right)\left|R_\tau\left(u_{\beta_2\tau}\left(s\right)\right)\right|^2\right\|_{L^2(\Gamma_3)}ds.\end{aligned}$$

As in [31] we deduce

$$\begin{aligned}&\left\|\Phi\beta_1\left(t\right)-\Phi\beta_2\left(t\right)\right\|_{L^2(\Gamma_3)}\leq\\ &c_3(\textstyle\int_0^t\left\|\beta_1\left(s\right)-\beta_2\left(s\right)\right\|_{L^2(\Gamma_3)}ds+\int_0^t\left\|u_{\beta_1}\left(s\right)-u_{\beta_2}\left(s\right)\right\|_V ds),\end{aligned} \tag{4.12}$$

for some constant $c_3 > 0$. Now to continue the proof we have needed to prove the following lemma.

Lemma 4.8. *There exists a constant* $\mu_0 > 0$ *such that*

$$\left\| u_{\beta_1}(t) - u_{\beta_2}(t) \right\|_V \leq c \left\| \beta_1(t) - \beta_2(t) \right\|_{L^2(\Gamma_3)} \ \forall t \in [0,T],$$

if

$$\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0.$$

Proof. Let $t \in [0,T]$. Take $u_{\beta_2}(t)$ in (4.1) satisfied by $u_{\beta_1}(t)$, then take $u_{\beta_1}(t)$ in the same inequality satisfied by $u_{\beta_2}(t)$; by adding the resulting inequalities we obtain

$$\begin{aligned}
&\langle F\varepsilon(u_{\beta_1}(t)) - F\varepsilon(u_{\beta_2}(t)), \varepsilon(u_{\beta_1}(t)) - \varepsilon(u_{\beta_2}(t)) \rangle_Q \\
&\leq \left\langle \textstyle\int_0^t \mathcal{F}(t-s)\left(\varepsilon(u_{\beta_1}(s)) - \varepsilon(u_{\beta_2}(s)\right) ds, \varepsilon(u_{\beta_2}(t)) - \varepsilon(u_{\beta_1}(t)) \right\rangle_Q \\
&+h(\beta_1(t), u_{\beta_1}(t), u_{\beta_2}(t) - u_{\beta_1}(t)) + h(\beta_2(t), u_{\beta_2}(t), u_{\beta_1}(t) - u_{\beta_2}(t)) \\
&+j(u_{\beta_1}(t), u_{\beta_2}(t)) - j(u_{\beta_1}(t), u_{\beta_1}(t)) + j(u_{\beta_2}(t), u_{\beta_1}(t)) - j(u_{\beta_2}(t), u_{\beta_2}(t)).
\end{aligned}$$

Using $(3.13)\,(b)$ this inequality implies

$$\begin{aligned}
&m \left\| u_{\beta_1}(t) - u_{\beta_2}(t) \right\|_V^2 \leq \\
&\left\langle \textstyle\int_0^t \mathcal{F}(t-s)\left(\varepsilon(u_{\beta_1}(s)) - \varepsilon(u_{\beta_2}(s)\right) ds, \varepsilon(u_{\beta_2}(t)) - \varepsilon(u_{\beta_1}(t)) \right\rangle_Q \\
&+h(\beta_1(t), u_{\beta_1}(t), u_{\beta_2}(t) - u_{\beta_1}(t)) + h(\beta_2(t), u_{\beta_2}(t), u_{\beta_1}(t) - u_{\beta_2}(t)) \\
&+j(u_{\beta_1}(t), u_{\beta_2}(t)) - j(u_{\beta_1}(t), u_{\beta_1}(t)) + j(u_{\beta_2}(t), u_{\beta_1}(t)) - j(u_{\beta_2}(t), u_{\beta_2}(t)).
\end{aligned} \tag{4.13}$$

We have

$$\begin{aligned}
&\left\langle \textstyle\int_0^t \mathcal{F}(t-s)\left(\varepsilon(u_{\beta_1}(s)) - \varepsilon(u_{\beta_2}(s))\right) ds, \varepsilon(u_{\beta_2}(t) - u_{\beta_1}(t)) \right\rangle_Q \\
&\leq \left(\textstyle\int_0^t \|\mathcal{F}(t-s)\|_{Q_\infty} \left\| u_{\beta_2}(s) - u_{\beta_1}(s) \right\|_V ds \right) \left\| u_{\beta_1}(t) - u_{\beta_2}(t) \right\|_V \\
&\leq c_4 \left(\textstyle\int_0^t \left\| u_{\beta_1}(s) - u_{\beta_2}(s) \right\|_V ds \right) \left\| u_{\beta_1}(t) - u_{\beta_2}(t) \right\|_V,
\end{aligned}$$

for some positive constant c_4. Using Young's inequality, we find

$$\begin{aligned}
&\left\langle \textstyle\int_0^t \mathcal{F}(t-s)\left(\varepsilon(u_{\beta_1}(s)) - \varepsilon(u_{\beta_2}(s))\right) ds, \varepsilon(u_{\beta_2}(t) - u_{\beta_1}(t)) \right\rangle_Q \\
&\leq \frac{c_4^2}{2m} \left(\textstyle\int_0^t \left\| u_{\beta_2}(s) - u_{\beta_1}(s) \right\|_V ds \right)^2 + \frac{m}{2} \left\| u_{\beta_1}(t) - u_{\beta_2}(t) \right\|_V^2.
\end{aligned} \tag{4.14}$$

Using the properties of R_ν and R_τ (see [28]), we have

$$h\left(\beta_1(t), u_{\beta_1}(t), u_{\beta_2}(t)-u_{\beta_1}(t)\right)+h\left(\beta_2(t), u_{\beta_2}(t), u_{\beta_1}(t)-u_{\beta_2}(t)\right)$$
$$\leq c_5\left\|\beta_1(t)-\beta_2(t)\right\|_{L^2(\Gamma_3)}\left\|u_{\beta_1}(t)-u_{\beta_2}(t)\right\|_V,$$

where $c_5>0$. Using also (3.10), (3.17) and (3.19) (c) yields

$$j\left(u_{\beta_1}(t), u_{\beta_2}(t)\right)-j\left(u_{\beta_1}(t), u_{\beta_1}(t)\right)+j\left(u_{\beta_2}(t), u_{\beta_1}(t)\right)-j\left(u_{\beta_2}(t), u_{\beta_2}(t)\right)$$
$$\leq c_0 M d_\Omega\|\mu\|_{L^\infty(\Gamma_3)}\left\|u_{\beta_1}(t)-u_{\beta_2}(t)\right\|_V^2. \tag{4.15}$$

We now combine inequalities (4.13), (4.14) and (4.15) to deduce

$$\begin{aligned} m\left\|u_{\beta_1}(t)-u_{\beta_2}(t)\right\|_V^2 &\leq c_0 M d_\Omega\|\mu\|_{L^\infty(\Gamma_3)}\left\|u_{\beta_1}(t)-u_{\beta_2}(t)\right\|_V^2 \\ &+\frac{c_4^2}{2m}\left(\int_0^t\left\|u_{\beta_1}(s)-u_{\beta_2}(s)\right\|_V ds\right)^2+\frac{m}{2}\left\|u_{\beta_1}(t)-u_{\beta_2}(t)\right\|_V^2 \\ &+c_5\left\|\beta_1(t)-\beta_2(t)\right\|_{L^2(\Gamma_3)}\left\|u_{\beta_1}(t)-u_{\beta_2}(t)\right\|_V. \end{aligned} \tag{4.16}$$

Using Young's inequality we get

$$\begin{aligned} &c_5\left\|\beta_1(t)-\beta_2(t)\right\|_{L^2(\Gamma_3)}\left\|u_{\beta_1}(t)-u_{\beta_2}(t)\right\|_V \\ &\leq c_6\left\|\beta_1(t)-\beta_2(t)\right\|_{L^2(\Gamma_3)}^2+\frac{m}{4}\left\|u_{\beta_1}(t)-u_{\beta_2}(t)\right\|_V^2 \end{aligned} \tag{4.17}$$

for some constant $c_6>0$. Then (4.16) and (4.17) imply that

$$\frac{m}{4}\left\|u_{\beta_1}(t)-u_{\beta_2}(t)\right\|_V^2 \leq$$
$$c_0 M d_\Omega\|\mu\|_{L^\infty(\Gamma_3)}\left\|u_{\beta_1}(t)-u_{\beta_2}(t)\right\|_V^2+\frac{c_4^2}{2m}\left(\int_0^t\left\|u_{\beta_1}(s)-u_{\beta_2}(s)\right\|_V ds\right)^2$$
$$+c_6\left\|\beta_1(t)-\beta_2(t)\right\|_{L^2(\Gamma_3)}^2.$$

Let now

$$\mu_0=\mu_1/4.$$

Then if

$$\|\mu\|_{L^\infty(\Gamma_3)}<\mu_0,$$

we deduce that there exists a constant $c_7>0$ such that

$$\left\|u_{\beta_1}(t)-u_{\beta_2}(t)\right\|_V^2 \leq c_7\left(\int_0^t\left\|u_{\beta_1}(s)-u_{\beta_2}(s)\right\|_V^2 ds+\left\|\beta_1(t)-\beta_2(t)\right\|_{L^2(\Gamma_3)}^2\right).$$

Then using Gronwall's argument, it follows that there exists a constant $c>0$ such that

$$\left\|u_{\beta_1}(t)-u_{\beta_2}(t)\right\|_V \leq c\left\|\beta_1(t)-\beta_2(t)\right\|_{L^2(\Gamma_3)}. \tag{4.18}$$

□

Now to end the proof of Proposition 4.7 we use (4.12) and (4.18) to deduce

$$\|\Phi\beta_1(t)-\Phi\beta_2(t)\|_{L^2(\Gamma_3)} \le c_8 \int_0^t \|\beta_1(s)-\beta_2(s)\|_{L^2(\Gamma_3)}\, ds \quad \forall t\in[0,T],$$

where $c_8>0$, and then we obtain

$$\|\Phi\beta_1-\Phi\beta_2\|_k \le \frac{c_8}{k}\|\beta_1-\beta_2\|_k .$$

This inequality shows that for $k>c_8$, Φ is a contraction. Then it has a unique fixed point β^* which satisfies (4.10) and (4.11). We now have all ingredients to prove Theorem 4.1.

Proof of Theorem 4.1. *Existence.* Let $\beta=\beta^*$ and let u_{β^*} the solution of Problem $P_{1\beta}$. We conclude by (4.1), (4.10) and (4.11) that (u_{β^*},β^*) is a solution to Problem P_2.

Uniqueness. Suppose that (u,β) is a solution of Problem P_2 which satisfies (3.20), (3.21) and (3.22). It follows from (3.20) that u is a solution to Problem $P_{1\beta}$, and from Theorem 4.2 that $u=u_\beta$. Take $u=u_\beta$ in (3.20) and use the initial condition (3.22), we deduce that β is a solution to Problem P_{ad}. Therefore, we obtain from Proposition 4.7 that $\beta=\beta^*$ and then we conclude that (u_{β^*},β^*) is a unique solution to Problem P_2. □

Let now σ^* be the function defined by (3.1) which corresponds to the function u_{β^*}. Then, it results from (3.13) and (3.14) that $\sigma^*\in C([0,T];Q)$. Using also a standard argument, it follows from the inequality (3.20) that

$$div\sigma^*(t)+\varphi_1(t)=0 \text{ in } \Omega, \text{ for all } t\in[0,T].$$

Therefore, using the regularity $\varphi_1\in C([0,T];H)$, we deduce that $div\sigma^*\in C([0,T];H)$ which implies that $\sigma^*\in C([0,T];Q_1)$. The triple $(u_{\beta^*},\sigma^*,\beta^*)$ which satisfies (3.1) and $(3.20)-(3.22)$ is called a weak solution of Problem P_1. We conclude that under the stated assumptions, the problem P_1 has a unique weak solution $(u_{\beta^*},\sigma^*,\beta^*)$. Moreover, the regularity of the weak solution is $u_{\beta^*}\in C([0,T];V)$, $\sigma^*\in C([0,T];Q_1)$, $\beta^*\in W^{1,\infty}\left(0,T;L^2(\Gamma_3)\right)\cap B$.

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