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On a generalization of Szasz-Durrmeyer operators with some orthogonal polynomials

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Abstract. In this paper, we construct a form of linear positive operators with Brenke type polynomials as a generalization of Szasz-Durrmeyer operators. We obtain convergence properties of our operators with the help of the universal Korovkin-type property and calculate the order of approximation by using classical modulus of continuity. Explicit examples of our operators involving some orthonogal and d-orthogonal polynomials such as the Hermite polynomials $H_k^{(\nu)}(x)$ of variance ν and Gould-Hopper polynomials are given.

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1. Introduction

Several integral modifications of Szasz operators [10] take part in approximation theory. One of them is the Durrmeyer type integral modification i.e. Szasz-Durrmeyer operators discovered by Mazhar and Totik [8]

$$\left(S_n^* f\right)(x) = n \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} \int_0^{\infty} e^{-nt} \frac{(nt)^k}{k!} f(t)\, dt \tag{1.1}$$

where $x \geq 0$ and $f \in C[0,\infty)$. Note that the operators (1.1) are linear positive operators.

On the other hand, Jakimovski and Leviatan [6] gave a generalization for Szasz operators by using Appell polynomials. Later, Ciupa [3] investigated the properties of the following operators as a Durrmeyer type integral modification of the operators given in [6]

$$\left(P_n f\right)(x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) \frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)} \int_0^{\infty} e^{-nt} t^{\lambda+k} f(t)\, dt \tag{1.2}$$

where $\lambda \geq 0$, $g(1) \neq 0$ and $p_k(x)$ are the Appell polynomials. The Appell polynomials are defined with the help of the following generating relation

$$g(u)\, e^{ux} = \sum_{k=0}^{\infty} p_k(x)\, u^k \tag{1.3}$$

where $g(u) = \sum\limits_{k=0}^{\infty} a_k u^k$ $(a_0 \neq 0)$ is an analytic function in the disc $|u| < R$ $(R > 1)$. For ensuring the positivity of the operators (1.2), Ciupa considered the assumptions $\frac{a_k}{g(1)} \geq 0$, $k = 0, 1, \ldots$. Notice that for the special case $\lambda = 0$ and $g(u) = 1$, the operators (1.2) return to the Szasz-Durrmeyer operators given by (1.1).

Recently, Varma et al. [12] constructed linear positive operators including Brenke type polynomials. Brenke type polynomials [2] have generating relation of the form

$$A(t)\, B(xt) = \sum_{k=0}^{\infty} p_k(x)\, t^k \tag{1.4}$$

where A and B are analytic functions

$$A(t) = \sum_{r=0}^{\infty} a_r t^r, \qquad a_0 \neq 0, \tag{1.5}$$

$$B(t) = \sum_{r=0}^{\infty} b_r t^r, \qquad b_r \neq 0 \ (r \geq 0) \tag{1.6}$$

and have the following explicit expression

$$p_k(x) = \sum_{r=0}^{k} a_{k-r} b_r x^r, \qquad k = 0, 1, 2, \ldots \ . \tag{1.7}$$

Using the following restrictions

$$\begin{array}{ll} (i) & A(1) \neq 0, \ \frac{a_{k-r} b_r}{A(1)} \geq 0, \ 0 \leq r \leq k, \ k = 0, 1, 2, \ldots, \\ (ii) & B : [0, \infty) \longrightarrow (0, \infty), \\ (iii) & (1.4) \text{ and the power series } (1.5) \text{ and } (1.6) \text{ converge for} \\ & |t| < R \ (R > 1), \end{array} \tag{1.8}$$

Varma et al. introduced the following linear positive operators involving the Brenke type polynomials

$$L_n(f; x) = \frac{1}{A(1)\, B(nx)} \sum_{k=0}^{\infty} p_k(nx)\, f\left(\frac{k}{n}\right) \tag{1.9}$$

where $x \geq 0$ and $n \in \mathbb{N}$.

In this paper, by using the same restrictions given by (1.8), our aim is to construct the Durrmeyer type integral modification of the operators (1.9) as a generalization of Szasz-Durrmeyer operators (1.1) with

$$L_n^*(f; x) = \frac{1}{A(1)\, B(nx)} \sum_{k=0}^{\infty} p_k(nx) \frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)} \int_0^{\infty} e^{-nt} t^{\lambda+k} f(t)\, dt \tag{1.10}$$

where $x \geq 0$, $\lambda \geq 0$ and $n \in \mathbb{N}$.

Remark 1.1. Let $B(t) = e^t$. The operators (1.10) (resp. (1.4)) return to the operators given by (1.2) (resp. (1.3)).

Remark 1.2. Let $\lambda = 0$, $A(t) = 1$ and $B(t) = e^t$. The operators (1.10) reduce to the well-known Szasz-Durrmeyer operators given by (1.1).

The paper is divided into three sections. In the next section, convergence of the operators (1.10) is investigated with the help of the universal Korovkin-type property [1]. The order of approximation is calculated by means of classical modulus of continuity. In section 3, we design the bridge with the notion of approximation theory and orthogonal polynomials. Namely, we give some illustrations with the help of the Hermite polynomials $H_k^{(\nu)}(x)$ of variance ν and Gould-Hopper polynomials for the operators (1.10).

2. Approximation properties of L_n^* operators

In this section, we state our main theorem with the help of the universal Korovkin-type property [1] and calculate the order of approximation by classical modulus of continuity. First of all, we give some definitions and lemmas used in the sequel.

Definition 2.1. *Let $f \in \tilde{C}[0,\infty)$ and $\delta > 0$. The modulus of continuity $\omega(f;\delta)$ of the function f is defined by*

$$\omega(f;\delta) := \sup_{\substack{x,y \in [0,\infty) \\ |x-y| \leq \delta}} |f(x) - f(y)|$$

where $\tilde{C}[0,\infty)$ is the space of uniformly continuous functions on $[0,\infty)$.

Lemma 2.2. *(Varma et al. [12]) For the operators L_n given by the equality (1.9), it holds*

$$L_n(1;x) = 1 \tag{2.1}$$

$$L_n(t;x) = \frac{B'(nx)}{B(nx)}x + \frac{A'(1)}{nA(1)} \tag{2.2}$$

$$L_n(t^2;x) = \frac{B''(nx)}{B(nx)}x^2 + \frac{\left[A(1) + 2A'(1)\right]B'(nx)}{nA(1)B(nx)}x + \frac{A''(1) + A'(1)}{n^2 A(1)} \tag{2.3}$$

for $x \in [0,\infty)$.

Lemma 2.3. *For the operators L_n^*, we have*

$$L_n^*(1;x) = 1 \tag{2.4}$$

$$L_n^*(t;x) = \frac{B'(nx)}{B(nx)}x + \frac{1}{n}\left(\lambda + 1 + \frac{A'(1)}{A(1)}\right) \tag{2.5}$$

$$\begin{aligned} L_n^*(t^2;x) &= \frac{B''(nx)}{B(nx)}x^2 + \frac{2\left[(\lambda+2)A(1) + A'(1)\right]B'(nx)}{nA(1)B(nx)}x \\ &+ \frac{A''(1) + 2(\lambda+2)A'(1) + (\lambda+1)(\lambda+2)A(1)}{n^2A(1)} \end{aligned} \tag{2.6}$$

for $x \in [0,\infty)$.

Proof. For $f(t) = 1$, by using the definition of gamma function, we get from (1.10)

$$\begin{aligned} L_n^*(1;x) &= \frac{1}{A(1)B(nx)}\sum_{k=0}^{\infty} p_k(nx)\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty} e^{-nt}t^{\lambda+k}dt \\ &= \frac{1}{A(1)B(nx)}\sum_{k=0}^{\infty} p_k(nx) = L_n(1;x) \quad . \end{aligned}$$

In view of the equality (2.1), we easily get the equality (2.4).

For $f(t) = t$, we obtain from (1.10)

$$\begin{aligned} L_n^*(t;x) &= \frac{1}{A(1)B(nx)}\sum_{k=0}^{\infty} p_k(nx)\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty} e^{-nt}t^{\lambda+k+1}dt \\ &= \frac{\lambda+1}{n}\frac{1}{A(1)B(nx)}\sum_{k=0}^{\infty} p_k(nx) + \frac{1}{A(1)B(nx)}\sum_{k=0}^{\infty} p_k(nx)\left(\frac{k}{n}\right) \\ &= L_n(t;x) + \frac{\lambda+1}{n}L_n(1;x). \end{aligned}$$

Taking into account the equalities (2.1)-(2.2), we have the equality (2.5).

For $f(t) = t^2$, by virtue of the equalities $(2.1) - (2.3)$, using similar technique leads us to the equality (2.6). □

Let us define the class of E as follows

$$E := \left\{ f : x \in [0,\infty),\ \frac{f(x)}{1+x^2} \text{ is convergent as } x \to \infty \right\}.$$

Theorem 2.4. *Let $f \in C[0,\infty) \cap E$ and assume that*

$$\lim_{y\to\infty} \frac{B'(y)}{B(y)} = 1 \qquad \textit{and} \qquad \lim_{y\to\infty} \frac{B''(y)}{B(y)} = 1. \tag{2.7}$$

Then,

$$\lim_{n\to\infty} L_n^*(f;x) = f(x)$$

uniformly on each compact subset of $[0,\infty)$.

Proof. According to the Lemma 2.3 and taking into account the assumptions (2.7), we find

$$\lim_{n\to\infty} L_n^*\left(t^i;x\right) = x^i, \qquad i = 0,1,2.$$

Above mentioned convergences are satisfied uniformly in each compact subset of $[0,\infty)$. By applying the universal Korovkin-type property (vi) of Theorem 4.1.4 [1], we get the desired result. □

Theorem 2.5. *Let $f \in \tilde{C}[0,\infty) \cap E$. L_n^* operators satisfy the following inequality*

$$\left|L_n^*(f;x) - f(x)\right| \leq 2\omega\left(f; \sqrt{\gamma_n(x)}\right)$$

where

$$\begin{aligned}\gamma_n(x) &= L_n^*\left((t-x)^2;x\right) = \frac{B''(nx) - 2B'(nx) + B(nx)}{B(nx)}x^2 \\ &+ \frac{2\left[\left[(\lambda+2)A(1) + A'(1)\right]B'(nx) - \left[A'(1) + (\lambda+1)A(1)\right]B(nx)\right]}{nA(1)B(nx)}x \\ &+ \frac{A''(1) + 2(\lambda+2)A'(1) + (\lambda+1)(\lambda+2)A(1)}{n^2A(1)}.\end{aligned}$$

Proof. From (2.4) and the property of modulus of continuity, we deduce

$$\begin{aligned}&\left|L_n^*(f;x) - f(x)\right| \\ \leq\ & \frac{1}{A(1)B(nx)}\sum_{k=0}^{\infty} p_k(nx)\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty} e^{-nt}t^{\lambda+k}\left|f(t) - f(x)\right|dt \\ \leq\ & \left\{1 + \frac{1}{\delta}\frac{1}{A(1)B(nx)}\sum_{k=0}^{\infty} p_k(nx)\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty} e^{-nt}t^{\lambda+k}\left|t-x\right|dt\right\}\omega(f;\delta).\end{aligned}$$

By using the Cauchy-Schwarz inequality for the integral, we have

$$\begin{aligned}&\left|L_n^*(f;x) - f(x)\right| \\ \leq\ & \left\{1 + \frac{1}{\delta}\frac{1}{A(1)B(nx)}\sum_{k=0}^{\infty} p_k(nx)\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\left(\int_0^{\infty} e^{-nt}t^{\lambda+k}dt\right)^{1/2}\right. \\ &\left.\times\left(\int_0^{\infty} e^{-nt}t^{\lambda+k}(t-x)^2dt\right)^{1/2}\right\}\omega(f;\delta). \qquad (2.8)\end{aligned}$$

By applying the Cauchy-Schwarz inequality for the sum, (2.8) leads to

$$\begin{aligned} & \left|L_n^*(f;x)-f(x)\right| \\ \leq & \left\{1+\frac{1}{\delta}\left(\frac{1}{A(1)B(nx)}\sum_{k=0}^{\infty}p_k(nx)\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty}e^{-nt}t^{\lambda+k}dt\right)^{1/2}\right. \\ & \left.\times\left(\frac{1}{A(1)B(nx)}\sum_{k=0}^{\infty}p_k(nx)\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty}e^{-nt}t^{\lambda+k}(t-x)^2dt\right)^{1/2}\right\}\omega(f;\delta) \\ = & \left\{1+\frac{1}{\delta}\left(L_n^*(1;x)\right)^{1/2}\left(L_n^*\left((t-x)^2;x\right)\right)^{1/2}\right\}\omega(f;\delta) \quad . \end{aligned}$$

In view of Lemma 2.3, we get the desired result for $\delta=\delta_n=\sqrt{\gamma_n(x)}$. □

Remark 2.6. Note that in Theorem 2.5, when $n\to\infty$, $\gamma_n(x)$ tends to zero under the assumptions (2.7).

3. Examples

Example 3.1. The Hermite polynomials $H_k^{(\nu)}(x)$ of variance ν [9] have the following generating functions of the form

$$e^{-\frac{\nu t^2}{2}+xt}=\sum_{k=0}^{\infty}\frac{H_k^{(\nu)}(x)}{k!}t^k \tag{3.1}$$

and the explicit representations

$$H_k^{(\nu)}(x)=\sum_{r=0}^{\left[\frac{k}{2}\right]}\left(\frac{-\nu}{2}\right)^r\frac{k!}{r!\,(k-2r)!}x^{k-2r}$$

where, as usual, [.] denotes the integer part. It is obvious that the Hermite polynomials $H_k^{(\nu)}(x)$ of variance ν are Brenke type polynomials for

$$A(t)=e^{-\frac{\nu t^2}{2}} \quad \text{and} \quad B(t)=e^t \quad .$$

Under the assumption $\nu\leq 0$; the restrictions (1.8) and assumptions (2.7) for the operators L_n^* given by (1.10) are satisfied. With the help of generating functions (3.1), we get the explicit form of L_n^* operators involving the Hermite polynomials $H_k^{(\nu)}(x)$ of variance ν by

$$H_n^*(f;x)=e^{-nx+\frac{\nu}{2}}\sum_{k=0}^{\infty}\frac{H_k^{(\nu)}(nx)}{k!}\frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)}\int_0^{\infty}e^{-nt}t^{\lambda+k}f(t)\,dt \tag{3.2}$$

where $x\in[0,\infty)$.

Example 3.2. Gould-Hopper polynomials [5] have the generating functions of the type

$$e^{ht^{d+1}} \exp(xt) = \sum_{k=0}^{\infty} g_k^{d+1}(x,h) \frac{t^k}{k!} \tag{3.3}$$

and the explicit representations

$$g_k^{d+1}(x,h) = \sum_{s=0}^{\left[\frac{k}{d+1}\right]} \frac{k!}{s!\,(k-(d+1)s)!} h^s x^{k-(d+1)s} \quad .$$

Gould-Hopper polynomials $g_k^{d+1}(x,h)$ are d-orthogonal polynomial set of Hermite type [4]. Van Iseghem [11] and Maroni [7] discovered the notion of d-orthogonality. Gould-Hopper polynomials are Brenke type polynomials with

$$A(t) = e^{ht^{d+1}} \quad \text{and} \quad B(t) = e^t \quad .$$

Under the assumption $h \geq 0$; the restrictions (1.8) and assumptions (2.7) for the operators L_n^* given by (1.10) are satisfied. With the help of generating functions (3.3), we obtain the explicit form of L_n^* operators including Gould-Hopper polynomials by

$$G_n^*(f;x) = e^{-nx-h} \sum_{k=0}^{\infty} \frac{g_k^{d+1}(nx,h)}{k!} \frac{n^{\lambda+k+1}}{\Gamma(\lambda+k+1)} \int_0^{\infty} e^{-nt} t^{\lambda+k} f(t)\,dt \tag{3.4}$$

where $x \in [0,\infty)$.

Remark 3.3. It is worthy to note that for $h = 0$ and $\nu = 0$, respectively, we obtain that

$$g_k^{d+1}(nx,0) = (nx)^k \quad \text{and} \quad H_k^{(0)}(nx) = (nx)^k .$$

Substituting $H_k^{(0)}(nx) = (nx)^k$ for $\nu = 0$ in the operators (3.2) and similarly $g_k^{d+1}(nx,0) = (nx)^k$ for $h = 0$ in the operators (3.4), with the special case $\lambda = 0$, we get the well-known Szasz-Durrmeyer operators given by (1.1). By the help of H_n^* and G_n^* operators, we introduce an interesting generalization of Szasz-Durrmeyer operators with the Hermite polynomials $H_k^{(\nu)}(x)$ of variance ν and Gould-Hopper polynomials.

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