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The order of convexity for a new integral operator

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Abstract. In this paper we consider a new integral operator $I(f_1, ..., f_n; g_1, ..., g_n)(z)$ for analytic functions $f_i(z)$, $g_i(z)$ in the open unit disk $\mathcal{U}$. The main object of the present paper is to study the order of convexity for this integral operator.

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1. Introduction and preliminaries

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk

$$\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$$

and satisfy the following usual normalization condition

$$f(0) = f'(0) - 1 = 0.$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which are univalent in $\mathcal{U}$.

Definition 1.1. *A function f belonging to $\mathcal{S}$ is a starlike function by the order α, $0 \leq \alpha < 1$ if f satisfies the inequality*

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \ z \in \mathcal{U}.$$

We denote this class by $\mathcal{S}^*(\alpha)$.

Definition 1.2. *A function f belonging to $\mathcal{S}$ is a convex function by the order $\alpha, 0 \leq \alpha < 1$ if f satisfies the inequality*

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}+1\right) > \alpha,\ \ z \in \mathcal{U}.$$

We denote this class by $\mathcal{K}(\alpha)$.
A function $f \in \mathcal{S}$ is in the class $\mathcal{P}(\alpha)$ if and only if

$$\operatorname{Re}\left(f'(z)\right) > \alpha,\ \ z \in \mathcal{U}.$$

In [1], Frasin and Jahangiri introduced the class $\mathcal{B}(\mu, \alpha)$ defined as follows.

Definition 1.3. *A function $f(z) \in \mathcal{A}$ is said to be a member of the class $\mathcal{B}(\mu, \alpha)$ if and only if*

$$\left|f'(z)\left(\frac{z}{f(z)}\right)^{\mu} - 1\right| < 1 - \alpha, \tag{1.1}$$

$z \in \mathcal{U}; 0 \leq \alpha < 1; \mu \geq 0.$

Note that the condition (1.1) is equivalent to

$$\operatorname{Re}\left(f'(z)\left(\frac{z}{f(z)}\right)^{\mu}\right) > \alpha,$$

for $z \in \mathcal{U}; 0 \leq \alpha < 1; \mu \geq 0.$
Clearly, $\mathcal{B}(1, \alpha) = \mathcal{S}^*(\alpha)$, $\mathcal{B}(0, \alpha) = \mathcal{P}(\alpha)$ and $\mathcal{B}(2, \alpha) = \mathcal{B}(\alpha)$ the class which has been introduced and studied by Frasin and Darus [2] (see also [3]).
Let $\mathcal{S}^*_\beta$ be the subclass of $\mathcal{A}$ consisting of the functions f which satisfy the inequality

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \beta,\ \ 0 < \beta \leq 1; z \in \mathcal{U} \tag{1.2}$$

and let $\mathcal{S}_\beta$ be the subclass of $\mathcal{A}$ consisting of the functions f which satisfy the inequality

$$\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < \beta,\ \ 0 < \beta \leq 1; z \in \mathcal{U}. \tag{1.3}$$

For $f_i(z),\ g_i(z) \in \mathcal{A}$ and $\delta_i, \gamma_i \in \mathbb{C}$, we define the integral operator $I_\beta(f_1, ..., f_n; g_1, ..., g_n)(z)$ given by

$$I_\beta(f_1, ..., f_n; g_1, ..., g_n)(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{\delta_i} \left(e^{g_i(t)}\right)^{\gamma_i} dt. \tag{1.4}$$

In order to prove our main results, we recall the following lemma.

Lemma 1.4. (**General Schwarz Lemma**) (see [4]). *Let the function f be regular in the disk $\mathcal{U}_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$ for fixed M. If f has one zero with multiplicity order bigger than m for $z = 0$, then*

$$|f(z)| \leq \frac{M}{R^m}|z|^m,\ \ z \in \mathcal{U}_R.$$

The equality can hold only if

$$f(z) = e^{i\theta}\frac{M}{R^m}z^m,\ \ \textit{where } \theta \textit{ is constant.}$$

2. The order of convexity for the integral operator $I\left(f_1,...,f_n;g_1,...,g_n\right)$

Theorem 2.1. *Let the functions* $f_i, g_i \in \mathcal{A}$ *and suppose that* $|g_i(z)| \leq M_i,\ M_i \geq 1$ *for all* $i \in \{1,2,...,n\}$. *If* $f_i \in \mathcal{S}^*_{\beta_i},\ 0 < \beta_i \leq 1$ *and* $g_i \in \mathcal{B}\left(\mu_i, \alpha_i\right),\ \mu_i \geq 0, 0 \leq \alpha_i < 1$ *then the integral operator* $I\left(f_1,...,f_n;g_1,...,g_n\right)(z)$ *defined by (1.4) is in* $\mathcal{K}(\lambda)$, *where*

$$\lambda = 1 - \sum_{i=1}^{n}\left[\left|\delta_i\right|\beta_i + \left|\gamma_i\right|\left(2-\alpha_i\right)M_i^{\mu_i}\right]$$

and

$$\sum_{i=1}^{n}\left[\left|\delta_i\right|\beta_i + \left|\gamma_i\right|\left(2-\alpha_i\right)M_i^{\mu_i}\right] < 1,\ \ \delta_i, \gamma_i \in \mathbb{C}$$

for all $i \in \{1,2,...,n\}$.

Proof. From (1.4) we obtain

$$I'\left(f_1,...,f_n;g_1,...,g_n\right)(z) = \prod_{i=1}^{n}\left(\frac{f_i(z)}{z}\right)^{\delta_i}\left(e^{g_i(z)}\right)^{\gamma_i}$$

and

$$I''\left(f_1,...,f_n;g_1,...,g_n\right)(z) =$$

$$\sum_{i=1}^{n}\left[\delta_i\left(\frac{f_i(z)}{z}\right)^{\delta_i-1}\left(\frac{zf_i'(z)-f_i(z)}{z^2}\right)\left(e^{g_i(z)}\right)^{\gamma_i}\right]\prod_{\substack{k=1\\k\neq i}}^{n}\left(\frac{f_k(z)}{z}\right)^{\delta_k}\left(e^{g_k(z)}\right)^{\gamma_k}$$

$$+\sum_{i=1}^{n}\left[\left(\frac{f_i(z)}{z}\right)^{\delta_i}\gamma_i\left(e^{g_i(z)}\right)^{\gamma_i-1}g_i'(z)e^{g_i(z)}\right]\prod_{\substack{k=1\\k\neq i}}^{n}\left(\frac{f_k(z)}{z}\right)^{\delta_k}\left(e^{g_k(z)}\right)^{\gamma_k}.$$

After the calculus we obtain that

$$\frac{zI''\left(f_1,...,f_n;g_1,...,g_n\right)(z)}{I'\left(f_1,...,f_n;g_1,...,g_n\right)(z)} = \sum_{i=1}^{n}\left[\delta_i\left(\frac{zf_i'(z)}{f_i(z)}-1\right)+\gamma_i z g_i'(z)\right]. \tag{2.1}$$

It follows from (2.1) that

$$\left|\frac{zI''\left(f_1,...,f_n;g_1,...,g_n\right)(z)}{I'\left(f_1,...,f_n;g_1,...,g_n\right)(z)}\right| \leq \sum_{i=1}^{n}\left[\left|\delta_i\right|\left|\frac{zf_i'(z)}{f_i(z)}-1\right|+\left|\gamma_i\right|\left|zg_i'(z)\right|\right]$$

$$\leq \sum_{i=1}^{n}\left[\left|\delta_i\right|\left|\frac{zf_i'(z)}{f_i(z)}-1\right|+\left|\gamma_i\right|\left|g_i'(z)\left(\frac{z}{g_i(z)}\right)^{\mu_i}\right|\left|\frac{g_i(z)}{z}\right|^{\mu_i}|z|\right]. \tag{2.2}$$

Since $|g_i(z)| \leq M_i,\ z \in \mathcal{U}$ applying the General Schwarz Lemma for the functions g_i, we have

$$|g_i(z)| \leq M_i\,|z|\,,\ \ z \in \mathcal{U}$$

for all $i \in \{1, 2, ..., n\}$.
Because $f_i \in \mathcal{S}^*_{\beta_i}$, $0 < \beta_i \leq 1$, $i \in \{1, 2, ..., n\}$, we apply in the relation (2.2) the inequalities (1.2) and we obtain

$$\left| \frac{zI''(f_1, ..., f_n; g_1, ..., g_n)(z)}{I'(f_1, ..., f_n; g_1, ..., g_n)(z)} \right| \leq \sum_{i=1}^{n} \left[|\delta_i| \beta_i + |\gamma_i| \left| g_i'(z) \left(\frac{z}{g_i(z)} \right)^{\mu_i} \right| M_i^{\mu_i} \right]. \qquad (2.3)$$

From (2.3) and (1.1), we see that

$$\left| \frac{zI''(f_1, ..., f_n; g_1, ..., g_n)(z)}{I'(f_1, ..., f_n; g_1, ..., g_n)(z)} \right|$$

$$\leq \sum_{i=1}^{n} \left[|\delta_i| \beta_i + |\gamma_i| \left(\left| g_i'(z) \left(\frac{z}{g_i(z)} \right)^{\mu_i} - 1 \right| + 1 \right) M_i^{\mu_i} \right]$$
$$\leq \sum_{i=1}^{n} \left[|\delta_i| \beta_i + |\gamma_i| (2 - \alpha_i) M_i^{\mu_i} \right]$$
$$= 1 - \lambda.$$

So, the integral operator $I(f_1, ..., f_n; g_1, ..., g_n)(z)$ defined by (1.4) is in $\mathcal{K}(\lambda)$. This completes the proof. □

Setting $n = 1$ in Theorem 2.1 we obtain

Corollary 2.2. *Let the functions $f, g \in \mathcal{A}$ and suppose that $|g(z)| \leq M$, $M \geq 1$. If $f \in \mathcal{S}^*_{\beta}$, $0 < \beta \leq 1$ and $g \in \mathcal{B}(\mu, \alpha)$, $\mu \geq 0, 0 \leq \alpha < 1$ then the integral operator*

$$I(f; g)(z) = \int_0^z \left(\frac{f(t)}{t} \right)^{\delta} \left(e^{g(t)} \right)^{\gamma} dt$$

is in $\mathcal{K}(\lambda)$, where

$$\lambda = 1 - [|\delta| \beta + |\gamma| (2 - \alpha) M^{\mu}]$$

and

$$[|\delta| \beta + |\gamma| (2 - \alpha) M^{\mu}] < 1, \; \delta, \gamma \in \mathbb{C}.$$

Theorem 2.3. *Let the functions $f_i, g_i \in \mathcal{A}$ and suppose that $|f_i(z)| \leq M_i$, $|g_i(z)| \leq N_i$, $M_i \geq 1$, $N_i \geq 1$ for all $i \in \{1, 2, ..., n\}$. If $f_i \in \mathcal{S}_{\beta_i}$, $0 < \beta_i \leq 1$ and $g_i \in \mathcal{B}(\mu_i, \alpha_i)$, $\mu_i \geq 0, 0 < \alpha_i < 1$ then the integral operator $I(f_1, ..., f_n; g_1, ..., g_n)(z)$ defined by (1.4) is in $\mathcal{K}(\lambda)$ where*

$$\lambda = 1 - \sum_{i=1}^{n} [|\delta_i| ((\beta_i + 1) M_i + 1) + |\gamma_i| (2 - \alpha_i) N_i^{\mu_i}]$$

and

$$\sum_{i=1}^{n} [|\delta_i| ((\beta_i + 1) M_i + 1) + |\gamma_i| (2 - \alpha_i) N_i^{\mu_i}] < 1, \; \delta_i, \gamma_i \in \mathbb{C}$$

for all $i \in \{1, 2, ..., n\}$.

Proof. If we make the similar operations to the proof of Theorem 2.1, we have

$$\frac{zI''\left(f_1,...,f_n;g_1,...,g_n\right)(z)}{I\left(f_1,...,f_n;g_1,...,g_n\right)(z)}=\sum_{i=1}^{n}\left[\delta_i\left(\frac{zf_i'(z)}{f_i(z)}-1\right)+\gamma_i z g_i'(z)\right]. \tag{2.4}$$

From the relation (2.4), we obtain that

$$\left|\frac{zI''\left(f_1,...,f_n;g_1,...,g_n\right)(z)}{I'\left(f_1,...,f_n;g_1,...,g_n\right)(z)}\right|$$

$$\leq\sum_{i=1}^{n}\left[|\delta_i|\left(\left|\frac{zf_i'(z)}{f_i(z)}\right|+1\right)+|\gamma_i|\left|zg_i'(z)\right|\right]$$

$$\leq\sum_{i=1}^{n}\left[|\delta_i|\left(\left|\frac{z^2f_i'(z)}{f_i^2(z)}\right|\left|\frac{f_i(z)}{z}\right|+1\right)+|\gamma_i|\left|g_i'(z)\left(\frac{z}{g_i(z)}\right)^{\mu_i}\right|\left|\frac{g_i(z)}{z}\right|^{\mu_i}|z|\right]. \tag{2.5}$$

Since $|f_i(z)|\leq M_i$, $|g_i(z)|\leq N_i$, $z\in\mathcal{U}$ applying the General Schwarz Lemma for the functions f_i, g_i, we obtain

$$|f_i(z)|\leq M_i\,|z|\,,\ z\in\mathcal{U}\ \text{and}\ |g_i(z)|\leq N_i\,|z|\,,\ z\in\mathcal{U}$$

for all $i\in\{1,2,...,n\}$.
Because $f_i\in\mathcal{S}_{\beta_i}$, $0<\beta_i\leq 1$ $i\in\{1,2,...,n\}$, we apply in the relation (2.5) the inequality (1.3) and we obtain

$$\left|\frac{zI''\left(f_1,...,f_n;g_1,...,g_n\right)(z)}{I'\left(f_1,...,f_n;g_1,...,g_n\right)(z)}\right|$$

$$\leq\sum_{i=1}^{n}\left[|\delta_i|\left(\left(\left|\frac{z^2f_i'(z)}{f_i^2(z)}-1\right|+1\right)M_i+1\right)+|\gamma_i|\left|g_i'(z)\left(\frac{z}{g_i(z)}\right)^{\mu_i}\right|{N_i}^{\mu_i}\right]$$

$$\leq\sum_{i=1}^{n}\left[|\delta_i|\left(\left(\beta_i+1\right)M_i+1\right)+|\gamma_i|\left|g_i'(z)\left(\frac{z}{g_i(z)}\right)^{\mu_i}\right|{N_i}^{\mu_i}\right] \tag{2.6}$$

From (2.6) and (1.1) we obtain

$$\left|\frac{zI''\left(f_1,...,f_n;g_1,...,g_n\right)(z)}{I'\left(f_1,...,f_n;g_1,...,g_n\right)(z)}\right|$$

$$\leq\sum_{i=1}^{n}\left[|\delta_i|\left(\left(\beta_i+1\right)M_i+1\right)+|\gamma_i|\left(\left|g_i'(z)\left(\frac{z}{g_i(z)}\right)^{\mu_i}-1\right|+1\right){N_i}^{\mu_i}\right]$$

$$\leq\sum_{i=1}^{n}\left[|\delta_i|\left(\left(\beta_i+1\right)M_i+1\right)+|\gamma_i|\left(2-\alpha_i\right){N_i}^{\mu_i}\right]$$

$$=1-\lambda.$$

So, the integral operator $I\left(f_1,...,f_n;g_1,...,g_n\right)(z)$ defined by (1.4) is in $\mathcal{K}\left(\lambda\right)$. This completes the proof. □

Setting $n=1$ in Theorem 2.3 we obtain

Corollary 2.4. *Let the functions* $f, g \in \mathcal{A}$ *and suppose that* $|f(z)| \leq M$, $|g(z)| \leq N$, $M \geq 1$, $N \geq 1$. *If* $f \in \mathcal{S}_\beta$, $0 < \beta \leq 1$ *and* $g \in \mathcal{B}(\mu, \alpha)$, $\mu \geq 0, 0 < \alpha < 1$ *then the integral operator*

$$I(f; g)(z) = \int_0^z \left(\frac{f(t)}{t}\right)^\delta \left(e^{g(t)}\right)^\gamma dt$$

is in $\mathcal{K}(\lambda)$ *where*

$$\lambda = 1 - [|\delta| ((\beta + 1) M + 1) + |\gamma| (2 - \alpha) N^\mu]$$

and

$$[|\delta| ((\beta + 1) M + 1) + |\gamma| (2 - \alpha) N^\mu] < 1, \ \delta, \gamma \in \mathbb{C}.$$

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