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The variation of curves length reported to cone metric

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Abstract. On a Lorentz manifold (M,g) we consider a timelike, parallel and unitary vector field Z. We define the Z-length of a curve and we obtain their first and second variation.

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1. Introduction

In 1988 Dan I. Papuc has started the study of differential manifold endowed with a field of tangent cones. This mathematical structure includes also the Lorentz manifold (M, g) with the cone of future directed, timelike vector fields. The future-directed cone is defined by normalized vector field Z. So, we have in each point $p \in M$ the structure (T_pM, K_p) where $K_p = \{v \in T_pM \mid g(v, v) \leq 0, g(v, Z_p) < 0\}$. This implies a Krein space where the following order relation is defined:

$$v \leq w \text{ if and only if } v - w \in K_p$$

Moreover, this order relation involves the definition of a norm [3], [4] named Z-norm through:

$$|v|_{Z_p} = \inf\{\lambda \geq 0 \mid -\lambda Z_p \leq v \leq \lambda Z_p\}$$

The expression of the Z- norm is by [5]:

$$|v|_{Z_p} = |g(v, Z_p)| + \sqrt{g(v, v) + [g(v, Z_p)]^2}$$

For a smooth curve $\lambda : [a, b] \to M$, we define its Z- length the value:

$$\begin{aligned} L_Z(\lambda) &= \int_a^b |\lambda'(t)|_{Z_p} dt \\ &= \int_a^b \left\{ |g(\lambda'(t), Z_{\lambda(t)})| + \sqrt{g(\lambda'(t), \lambda'(t)) + g^2(\lambda'(t), Z_{\lambda(t)})} \right\} dt \end{aligned}$$

We state that the curve $\lambda : [a,b] \to M$ is Z-global in $x_0 = \lambda(t_0)$ if $\lambda'(t_0)$ and $Z_{\lambda(t)}$ are collinear.
For the calculus of the first variation we need to consider some restrictive hypotheses:
A) The future-directed, normalized, timelike vector field Z is parallel, this meaning

$$\nabla_X Z = 0, \forall X \in \mathcal{X}(M).$$

B) The curve λ, $\lambda : [a,b] \to M$ is not Z-global in any of its points.

For this hypothesis we make the following remarks:

Remark 1.1. The necessary condition for B) hypothesis involves $h(\lambda'(t), \lambda'(t)) \neq 0$ where $h(X,Y) = g(X,Y) + g(X,Z)g(Y,Z)$.

We have $h(X,X) = 0 \Leftrightarrow Gram\{X.Z\} = \begin{vmatrix} g(X,X) & g(X,Z) \\ g(Z,X) & g(Z,Z) \end{vmatrix} = 0 \Leftrightarrow \{X,Z\}$ are collinear.

Remark 1.2. If $\phi : (-\varepsilon, \varepsilon) \times [t_p, t_g] \to M$ is a piecewise smooth variation of a timelike, future directed curve λ which is not Z-global, then it exists $\delta > 0$ with the property that $\phi(u,\ .\ ) : [t_p, t_g] \to M$ is timelike, future directed and not Z-global for every $|u| < \delta$.

Beem [2] (page 253) proved the previous statement for a geodesic segment λ. Without any difficulty, we can give up on the restriction of geodesic segment, considering λ a piecewise smooth timelike future directed curve. It still remains to demonstrate that $\phi(u,\ .\ ) : [t_p, t_g] \to M$ is not Z-global for $|u| < \delta$.

Firstly, the smooth differentiation of ϕ involves the fact that it exists $\varepsilon_1 < \varepsilon$ as $\phi : [-\varepsilon_1, \varepsilon_1] \times [t_p, t_g] \to M$ is differentiable on compact. Consequently, we can extend to an open set which contains $[-\varepsilon_1, \varepsilon_1] \times [t_p, t_g]$. Because λ is timelike and not Z-global, we have $\phi'(u, t_p^+)$, $\phi'(u, t_q^-)$ timelike vectors which are not collinear with $Z_{\phi(u,t_p^+)}$, respectively $Z_{\phi(u,t_q^-)}$ for $\forall\ |u| < \delta_1$. We assume that for any $\delta > 0$, $\phi(u_0,\ .\ ) : [t_p, t_g] \to M$ is not Z-global, for $\forall\ |u_0| < \delta_1$. So it exists a sequence $u_n \to 0$ for which $\phi'(u_n, t_n)$ is collinear with $Z_{\phi(u_n,t_n)}$. Hence $(u_n, t_n) \in [-\varepsilon_1, \varepsilon_1] \times [t_p, t_q]$, which is compact, it results that we have an accumulation point $(0,t)$. So, $\phi'(0,t)$ is collinear with $Z_{\phi(0,t)}$, or $\lambda'(t)$ is collinear with $Z_{\lambda(t)}$, that involves λ being Z-global in $x = \lambda(t)$, affirmation excluded by the hypothesis.

Secondly, we consider the case where $\phi : (-\varepsilon, \varepsilon) \times [t_p, t_g] \to M$ is a piecewise smooth variation of piecewise smooth timelike, future directed and not Z-global curve λ. There is a partition $t_p = t_0 < t_1 < ... < t_k = t_q$ so that $\phi|_{(-\varepsilon,\varepsilon)\times[t_{i-1},t_i]}$ is a smooth timelike not Z-global future directed variation of $\lambda|_{[t_{i-1},t_i]}$, $\forall i = \overline{1,k}$. According to the above results, we have δ_i, $i = \overline{1,k}$ so that $\phi(u,\ ) : [t_{i-1}, t_i] \to M$ is not Z-global and $\phi'(u, t_{i-1}^+)$, $\phi'(u, t_i^-)$ are not collinear with $Z_{\phi(u,t_{i-1}^+)}$, respectively $Z_{\phi(u,t_i^-)}$ for $|u| < \delta_i$. Considering $\delta = \min_{i=\overline{1,k}} \delta_i$, we obtain that $\phi(u,\ ) : [t_p, t_g] \to M$ is not a Z-global curve for $|u| < \delta$.

C) We assume that $\lambda : [t_p, t_q] \to M$ is a timelike future directed curve, h-unitary parametrized, this means $h(\lambda', \lambda') = g(\lambda', \lambda') + g(\lambda', Z)^2 = 1$.

Remark 1.3. Obviously, the h-unitary condition implies that the λ curve is not Z-global as stated in Remark 1.1.

Considering $\Phi : \mathcal{A} \to M$, $\mathcal{A} := (-\varepsilon, \varepsilon) \times [t_p, t_q]$ a proper smooth causal variation of λ, the curves $\phi(u, \ ) : [t_p, t_q] \to M$, $|u| < \varepsilon$ are not Z-global in any of their points. So:

1. $\Phi(0, t) = \lambda(t)$, $\forall t \in [t_p, t_q]$
2. $\Phi(u, t_p) = p$, $\Phi(u, t_q) = q$
3. $\Phi \in \mathcal{C}^3(\mathcal{A})$
4. $g(V, V) < 0$, $g(V, Z) < 0$, $g(V, Z)^2 + g(V, V) \neq 0$ where $V = \phi_*(\frac{\partial}{\partial t})$

We note the variation vector field by $X = \phi_*(\frac{\partial}{\partial u})$ where $\{\frac{\partial}{\partial u}, \frac{\partial}{\partial t}\}$ is a base in $T_{(u,t)}\mathcal{A}$. We note $L_Z(u) = L_Z(\phi(u, \ ))$. Considering (A) and (B) hypotheses to be valid, we have to demonstrate:

Lemma 1.4. *The first variation of Z- length of λ is:*

$$\frac{d}{du} L_Z(0) = -\int_{t_p}^{t_q} \frac{1}{\sqrt{h(\lambda'.\lambda')}} h(\lambda'', P^h_{(\lambda')^\perp} X)|_{(0,t)} dt$$

where $P^h_{(Y)^\perp} X = X - \frac{h(X,Y)}{h(Y,Y)} Y$ is the X projection, respecting the bilinear form h on $Y^\perp$.

If, additionally, we assume that λ is a geodesic segment, and all three A, B, C hypotheses are being satisfied, we prove:

Lemma 1.5. *The second variation of Z- length of λ is:*

$$\frac{d^2}{du^2} L_Z(0) = -\int_{t_p}^{t_q} h(R(X, \lambda')\lambda' + N'', N)|_{(0,t)} dt +$$
$$+\{h(\nabla_X X, \lambda') - g(\nabla_X X, Z) + h(N', N)\}|_{(0,t)}\Big|_{t_p}^{t_q}$$

where $N = X - h(X, V)V$.

2. The first variation

We make the following remark:
$h : TM \times TM \to \mathbb{R}$ is a bilinear metric, positive form, semidefined, degenerated, with its signature $(n-1, 0, 1)$ because:

$$\begin{aligned} X(h(Y_1, Y_2)) &= X[g(Y_1, Y_2) + g(Y_1, Z)g(Y_2, Z)] = \\ &= g(\nabla_X Y_1, Y_2) + g(Y_1, \nabla_X Y_2) + \\ &+ [g(\nabla_X Y_1, Z) + g(Y_1, \nabla_X Z)]g(Y_2, Z) + \\ &+ g(Y_1, Z)[g(\nabla_X Y_2, Z) + g(Y_2, \nabla_X Z)] \\ &= g(\nabla_X Y_1, Y_2) + g(Y_1, \nabla_X Y_2) + g(\nabla_X Y_1, Z)g(Y_2, Z) + \\ &+ g(Y_1, Z)g(\nabla_X Y_2, Z) = h(\nabla_X Y_1, Y_2) + h(Y_1, \nabla_X Y_2) \end{aligned}$$

where we used the $A)$ hypothesis about Z, namely $\nabla_X Z = 0$.

We have for the Z- length of curve $\phi(u,\ ) : [t_p, t_q] \to M$ the expression:

$$L_Z(u) = \int_{t_p}^{t_q} \left\{-g(V,Z) + \sqrt{h(V,V)}\right\} dt$$

and:

$$\frac{d}{du} L_Z(u) = \int_{t_p}^{t_q} \left\{-\frac{\partial}{\partial u}[g(V,Z)] + \frac{\partial}{\partial u}\sqrt{h(V,V)}\right\} dt =$$

$$= \int_{t_p}^{t_q} \left\{-g(\nabla_X V, Z) - g(V, \nabla_X Z) + \frac{1}{\sqrt{h(V,V)}} h(\nabla_X V, V)\right\} dt$$

Since $[X,V] = 0$, then $\nabla_X V = \nabla_V X$ and so:

$$\frac{d}{du} L_Z(u) = \int_{t_p}^{t_q} \left\{-g(\nabla_X V, Z) + \frac{h(\nabla_V X, V)}{\sqrt{h(V,V)}}\right\} dt \tag{2.1}$$

We calculate:

$$\frac{\partial}{\partial t}[g(X,Z)] = g(\nabla_V X, Z) + g(X, \nabla_V Z) = g(\nabla_V X, Z) \tag{2.2}$$

$$\frac{\partial}{\partial t}\left[\frac{h(X,V)}{\sqrt{h(V,V)}}\right] =$$

$$= \frac{[h(\nabla_V X, V) + h(X, \nabla_V V)]\sqrt{h(V,V)} - h(X,V)\frac{h(\nabla_V V,V)}{\sqrt{h(V,V)}}}{h(V,V)}$$

$$= \frac{h(\nabla_V X, V)}{\sqrt{h(V,V)}} + \frac{h(X, \nabla_V V)}{\sqrt{h(V,V)}} - \frac{h(X,V)h(\nabla_V V, V)}{\left[\sqrt{h(V,V)}\right]^3} =$$

$$= \frac{h(\nabla_V X, V)}{\sqrt{h(V,V)}} + \frac{1}{\sqrt{h(V,V)}}\left[h(\nabla_V V, X) - \frac{h(X,V)}{h(V,V)} h(\nabla_V V, V)\right] =$$

$$= \frac{h(\nabla_V X, V)}{\sqrt{h(V,V)}} + \frac{1}{\sqrt{h(V,V)}} h\left(\nabla_V V, X - \frac{h(X,V)}{h(V,V)} V\right) \tag{2.3}$$

Replacing the results from (2.2) and (2.3) in (2.1), we obtain the following:

$$\frac{d}{du} L_Z(0) =$$

$$= \int_{t_p}^{t_q} \left\{ -\frac{\partial}{\partial t}[g(X,Z)] + \frac{\partial}{\partial t}\left[\frac{h(X,V)}{\sqrt{h(V,V)}}\right] - \frac{1}{\sqrt{h(V,V)}} h(\nabla_V V, P^h_{V^\perp} X) \right\}\Bigg|_{(0,t)} dt$$

$$= -\int_{t_p}^{t_q} \left\{ \frac{1}{\sqrt{h(V,V)}} h(\nabla_V V, P^h_{V^\perp} X) \right\}\Bigg|_{(0,t)} dt +$$

$$+ \left\{ \frac{h(X,V)}{\sqrt{h(V,V)}} - g(X,Z) \right\}\Bigg|_{(0,t)}\Bigg|_{t_p}^{t_q}$$

$$= -\int_{t_p}^{t_q} \left\{ \frac{1}{\sqrt{h(V,V)}} h(\nabla_V V, P^h_{V^\perp} X) \right\}\Bigg|_{(0,t)} dt$$

$$= -\int_{t_p}^{t_q} \left\{ \frac{1}{\sqrt{h(\lambda',\lambda')}} h(\lambda'', P^h_{(\lambda')^\perp} X) \right\}\Bigg|_{(0,t)} dt$$

We have used the hypothesis of proper variation, namely: $X(t_p^+) = X(t_q^-) = 0$

Remark 2.1. If λ is a geodesic segment, then $\dfrac{d}{du}L_Z(0) = 0$ because $\nabla_V V|_{(0,t)} = \dfrac{D\lambda'}{\partial t} = 0$. In consequence, the geodesic segments, that are not Z-global, are stationary points for the Z- length functional.

Remark 2.2. Let it be $\lambda : [t_p, t_q] \to M$ a piecewise smooth timelike and future directed curve, $h-$unitary parametrized which is not Z- global. Noting with $L_Z^i(u)$ the Z-length of the uniparametric variation of the curve $\lambda|_{[t_{i-1},t_i]}$, $i = \overline{1,k}$ we have:

$$\frac{dL_Z^i(0)}{du} = -\int_{t_{i-1}}^{t_i} h(\lambda'', P^h_{(\lambda')^\perp} X)\Big|_{(0,t)} dt + \\ + \{h(\lambda', X) - g(X,Z)\}|_{(0,t)}\Big|_{t_{i-1}}^{t_i}$$

therefore

$$\frac{dL_Z}{du}(0) = \sum_{i=1}^{k} \frac{dL_Z^i(0)}{du} = -\int_{t_p}^{t_q} h\left(\lambda'', P^h_{(\lambda')^\perp} X\right)\Big|_{(0,t)} dt - \\ - \sum_{i=1}^{k-1} h\left(X(t_i), \Delta_{t_i}(\lambda')\right)$$

where $\Delta_{t_i}(\lambda') = \lambda'\left(t_i^+\right) - \lambda'\left(t_i^-\right)$, $\forall i = \overline{1, k-1}$ and we have taken into account the fact that the variation is proper, so: $X(\lambda(t_p)) = X(\lambda(t_q)) = 0$.

Remark 2.3. Considering H a spatial hypersurface and assuming that $\lambda : [t_p, t_q] \to M$ is a timelike future directed geodesical segment, not Z-global with $\lambda(t_p) = p \in H$ and

$\lambda(t_p) = q \notin H$, then:

$$0 = \frac{dL_Z(0)}{du} = g(X_p, Z_p) - g(X_p, \lambda'(t_p)) - g(X_p, Z_p)g(\lambda'(t_p), Z_p) =$$
$$= -g(X_p, \lambda'(t_p) + g(\lambda'(t_p), Z_p)Z_p - Z_p)$$

It results that $\lambda'(t_p) + [g(\lambda'(t_p), Z_p) - 1]Z_p$ has to be g orthogonal on H.

If $\lambda : [t_p, t_q] \to M$ is a geodesic segment, an affine parametrization of λ so that $g(\lambda'(t_p), Z_p) = 1$ can be found. In this case, the above condition becomes: $\lambda'(t_p)$ is g orthogonal on H.

3. The second variation

We calculate the second variation of the Z-length of the geodesical curve λ, h-unitary parametrized. Starting with the formula (2.1), we have:

$$\frac{d}{du}[h(\nabla_V X, V)] = h(\nabla_X \nabla_V X, V) + h(\nabla_V X, \nabla_X V)$$
$$\frac{d}{du}[h(V, V)] = 2h(\nabla_X V, V)$$

Then

$$I \stackrel{def}{=} \frac{d}{du}\left\{-g(\nabla_V X, Z) + h(V, V)^{-\frac{1}{2}} h(\nabla_V X, V)\right\} \tag{3.1}$$
$$= -g(\nabla_X \nabla_V X, Z) - h(V, V)^{-\frac{3}{2}} h(\nabla_X V, V) h(\nabla_V X, V) +$$
$$+ h(V, V)^{-\frac{1}{2}} [h(\nabla_X \nabla_V X, V) + h(\nabla_V X, \nabla_X V)]$$
$$= -g(\nabla_X \nabla_V X, Z) - h(V, V)^{-\frac{3}{2}} [h(\nabla_V X, V)]^2 +$$
$$+ h(V, V)^{-\frac{1}{2}} [h(\nabla_X \nabla_V X, V) + h(\nabla_V X, \nabla_X V)]$$

We define

$$N \stackrel{def}{=} X - h(X, V)V \tag{3.2}$$

the h-normal on V vector field.

Next we calculate:

$$\nabla_V [h(X, V)V]|_{(0,t)} = \left\{\frac{d}{dt}[h(X, V)]\right\} V + h(X, V)\nabla_V V|_{(0,t)} \tag{3.3}$$
$$= \left\{\frac{d}{dt}[h(X, V)]\right\} V$$

$$h(\nabla_V N, V)|_{(0,t)} = \frac{d}{dt}\{h(N, V)\} - h(N, \nabla_V V)|_{(0,t)} = 0 \tag{3.4}$$

$$h(\nabla_V N, \nabla_V [h(X, V)V])|_{(0,t)} = h\left(\nabla_V N, \left\{\frac{d}{dt}[h(X, V)]\right\} V\right) \tag{3.5}$$
$$= \frac{d}{dt}[h(X, V)] h(\nabla_V N, V)|_{(0,t)} = 0$$

$$
\begin{aligned}
& h(\nabla_V X, \nabla_X V)|_{(0,t)} = h(\nabla_V X, \nabla_V X)|_{(0,t)} \\
& \overset{(3.2)}{=} h(\nabla_V \{N + h(X,V)V\}, \nabla_V \{N + h(X,V)V\})|_{(0,t)} \\
& \overset{(3.3)}{=} h\left(\nabla_V N + \frac{d}{dt}\{h(X,V)\} V, \nabla_V N + \frac{d}{dt}\{h(X,V)\} V\right)|_{(0,t)} \\
& = h(\nabla_V N, \nabla_V N) + 2\frac{d}{dt}\{h(X,V)\} h(\nabla_V N, V) + \qquad (3.6)\\
& + \left[\frac{d}{dt}\{h(X,V)\}\right]^2 h(V,V)|_{(0,t)} \\
& \overset{(3.4)}{=} h(\nabla_V N, \nabla_V N) + \left[\frac{d}{dt}\{h(X,V)\}\right]^2 |_{(0,t)}
\end{aligned}
$$

$$
\begin{aligned}
& h(\nabla_V N, V)|_{(0,t)} = h(\nabla_V \{N + h(X,V)V\}, V)|_{(0,t)} = h(\nabla_V N, V) + \\
& + h(\nabla_V h(X,V)V, V) \overset{(3.3)}{=} h(\nabla_V N, V) + \frac{d}{dt}\{h(X,V)\}|_{(0,t)} \qquad (3.7)
\end{aligned}
$$

Replacing these results in (3.1) we have:

$$
I|_{(0,t)} = -g(\nabla_X \nabla_V X, Z) + h(\nabla_X \nabla_V X, V) + h(\nabla_V N, \nabla_V N)|_{(0,t)}
$$

We get:

$$
\frac{d^2 L_Z(0)}{du^2} = \int_{t_p}^{t_q} \{-g(\nabla_X \nabla_V X, Z) + h(\nabla_X \nabla_V X, V) + h(\nabla_V N, \nabla_V N)\}|_{(0,t)} dt \quad (3.8)
$$

and for this equality, we calculate:

$$
\nabla_X \nabla_V X = \nabla_V \nabla_X X - R(V,X)X \qquad (3.9)
$$

$$
\begin{aligned}
g(\nabla_V \nabla_X X, Z)|_{(0,t)} &= \frac{d}{dt}\{g(\nabla_X X, Z)\} - g(\nabla_X X, \nabla_V Z)|_{(0,t)} \qquad (3.10)\\
&= \frac{d}{dt}\{g(\nabla_X X, Z)\}|_{(0,t)}
\end{aligned}
$$

$$
\begin{aligned}
-g(\nabla_X \nabla_V X, Z)|_{(0,t)} &= g(R(V,X)X - \nabla_V \nabla_X X, Z)|_{(0,t)} \qquad (3.11)\\
&= g(R(V,X)X, Z) - \frac{d}{dt}\{g(\nabla_X X, Z)\}|_{(0,t)}
\end{aligned}
$$

$$
\begin{aligned}
& h(\nabla_X \nabla_V X, V)|_{(0,t)} = h(\nabla_V \nabla_X X - R(V,X)X, V)|_{(0,t)} \qquad (3.12)\\
& = \frac{d}{dt}\{h(\nabla_X X, V)\} - h(\nabla_X X, \nabla_V V) - h(R(V,X)X, V)|_{(0,t)} \\
& = -h(R(V,X)X, V) + \frac{d}{dt}\{h(\nabla_X X, V)\}|_{(0,t)}
\end{aligned}
$$

$$
h(\nabla_V N, \nabla_V V) = \frac{d}{dt}\{h(N, \nabla_V N)\} - h(N, \nabla_V \nabla_V N) \qquad (3.13)
$$

Replacing the results from (3.11), (3.12) and (3.13), (3.8) becomes:

$$
\frac{d^2 L_Z(0)}{du^2} =
$$

$$= \int_{t_p}^{t_q} \left\{ g(R(V,X)X,Z) - \frac{d}{dt}\{g(\nabla_X X, Z)\} - h(R(V,X)X,V) \right\} |_{(0,t)} dt + \quad (3.14)$$

$$+ \int_{t_p}^{t_q} \left\{ -h(N, \nabla_V \nabla_V N) + \frac{d}{dt}\{h(\nabla_X X, V)\} + \frac{d}{dt}\{h(N, \nabla_V N)\} \right\} |_{(0,t)} dt$$

$$= \int_{t_p}^{t_q} \{g(R(V,X)X,Z) - h(R(V,X)X,V) - h(N, \nabla_V \nabla_V N)\} |_{(0,t)} dt +$$

$$+ \{-g(\nabla_X X, Z) + h(\nabla_X X, V) + h(N, \nabla_V N)\} |_{(0,t)} \Big|_{t_p}^{t_q}$$

For the (3.14) expressions we have:

$$g(R(V,X)X,Z) = -g(R(V,X)Z,X) = \quad (3.15)$$
$$= -g(\nabla_V \nabla_X Z - \nabla_X \nabla_V Z, X) = 0$$

$$h(R(V,X)X,V) = g(R(V,X)X,V) + g(R(V,X)X,Z)g(V,Z) \quad (3.16)$$
$$= g(R(V,X)X,V) = g(R(X,V)V,X) = g(R(X,V)V, N + h(X,V)V)$$
$$= g(R(X,V)V,N) + g(R(X,V)V,V)h(X,V) = g(R(X,V)V,N)$$

$$g(R(X,V)V,Z) = -g(R(X,V)Z,V) = 0 \quad (3.17)$$

With the results from (3.15), (3.16) and (3.17) replaced in (3.14) we obtain:

$$\frac{d^2 L_Z(0)}{du^2} =$$

$$= \int_{t_p}^{t_q} \{-g(R(X,V)V,N) - h(N, \nabla_V \nabla_V N)\} |_{(0,t)} dt +$$
$$+ \left\{ \{h(\nabla_X X, V) + h(N, \nabla_V N) - g(\nabla_X X, Z)\} |_{(0,t)} \right\} \Big|_{t_p}^{t_q}$$

$$= \int_{t_p}^{t_q} \{-g(R(X,V)V,N) - g(R(X,V)V,Z)g(N,Z) - h(N, \nabla_V \nabla_V N)\} |_{(0,t)} dt +$$
$$+ \left\{ \{h(\nabla_X X, V) + h(N, \nabla_V N) - g(\nabla_X X, Z)\} |_{(0,t)} \right\} \Big|_{t_p}^{t_q}$$
$$= \int_{t_p}^{t_q} \{-h(R(X,V)V,N) - h(N, \nabla_V \nabla_V N)\} |_{(0,t)} dt +$$
$$+ \left\{ \{h(\nabla_X X, V) + h(N, \nabla_V N) - g(\nabla_X X, Z)\} |_{(0,t)} \right\} \Big|_{t_p}^{t_q}$$

In conclusion the second variation formula is the following:

$$\frac{d^2 L_Z(0)}{du^2} = -\int_{t_p}^{t_q} h(R(X,\lambda')\lambda' + N'', N)|_{(0,t)} dt +$$
$$+ \{h(\nabla_X X, \lambda') + h(N', N) - g(\nabla_X X, Z)\} |_{(0,t)} \Big|_{t_p}^{t_q}$$

Remark 3.1. In the hypothesis according to which $\phi : (-\varepsilon, \varepsilon)\times[t_p, t_q] \to M$ is a proper variation of λ, we have that: $X(t_p^+) = X(t_q^-) = 0$ and therefore $N(t_p^+) = N(t_q^-) = 0$. This simplifies the second variation formula:

$$\frac{d^2L_Z(0)}{du^2} = -\int_{t_p}^{t_q} h(R(X,\lambda')\lambda' + N'', N)|_{(0,t)}dt+$$
$$+ \{h(\nabla_X X, \lambda') - g(\nabla_X X, Z)\}\,|_{(0,t)}\Big|_{t_p}^{t_q}$$

Remark 3.2. In the hypothesis stating that ϕ is a canonical proper variation of λ, (meaning that $\phi(u,t) = \exp_{\lambda(t)} uY(t)$, with $Y(t)$ a vector field along λ, that respects:

$$Y(t_p) = Y(t_q) = 0, g(Y(t), \lambda'(t)) = 0, \forall t \in [t_p, t_q])$$

we have that the curves: $u \longmapsto \phi(u, t_0)$ are geodesics, namely

$$\nabla_X X = 0 \quad \text{and} \quad X = \phi_*(\frac{\partial}{\partial u})|_{(0,t)} = Y(t).$$

This implies the following expression for the second variation:

$$\frac{d^2L_Z(0)}{du^2} = -\int_{t_p}^{t_q} h(R(Y,\lambda')\lambda' + N'', N)|_{(0,t)}dt.$$

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