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Tripled fixed point theorems in partially ordered metric spaces

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Abstract. The notion of tripled fixed point is introduced by Berinde and Borcut [1]. In this manuscript, some new tripled fixed point theorems are obtained by using a generalization of the results of Luong and Thuang [11].

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1. Introduction

Existence and uniqueness of a fixed point for contraction type mappings in partially ordered metric spaces were discussed first by Ran and Reurings [15] in 2004. Later, so many results were reported on existence and uniqueness of a fixed point and its applications in partially ordered metric spaces (see e.g. [1]-[18]).

In 1987, Guo and Lakshmikantham [6] introduced the notion of the coupled fixed point. The concept of coupled fixed point reconsidered in partially ordered metric spaces by Bhaskar and Lakshmikantham [5] in 2006. In this remarkable paper, by introducing the notion of a mixed monotone mapping the authors proved some coupled fixed point theorems for mixed monotone mapping and considered the existence and uniqueness of solution for periodic boundary value problem.

The triple $(X, d, \leq)$ is called partially ordered metric spaces if $(X, \leq)$ is a partially ordered set and (X, d) is a metric space. Further, if (X, d) is a complete metric space, then the triple $(X, d, \leq)$ is called partially ordered complete metric spaces. Throughout the manuscript, we assume that $X \neq \emptyset$ and

$$X^k = \underbrace{X \times X \times \cdots X}_{k-\text{many}}.$$

Then the mapping $\rho_k : X^k \times X^k \to [0, \infty)$ such that

$$\rho_k(\mathbf{x}, \mathbf{y}) := d(x_1, y_1) + d(x_2, y_2) + \cdots + d(x_k, y_k),$$

forms a metric on X^k where $\mathbf{x} = (x_1, x_2, ..., x_k), \mathbf{y} = (y_1, y_2, \cdots, y_k) \in X^k$, $k \in \mathbb{N}$.

We state the notions of a mixed monotone mapping and a coupled fixed point as follows.

Definition 1.1. ([5]) *Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$. The mapping F is said to has the mixed monotone property if $F(x, y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is, for any $x, y \in X$,*

$$x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1) \geq F(x, y_2).$$

Definition 1.2. ([5]) *An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if*

$$x = F(x, y) \text{ and } y = F(y, x).$$

In [5] Bhaskar and Lakshmikantham proved the existence of coupled fixed points for an operator $F : X \times X \to X$ having the mixed monotone property on $(X, d, \leq)$ by supposing that there exists a $k \in [0, 1)$ such that

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}\left[d(x, u) + d(y, v)\right], \text{ for all } u \leq x,\ y \leq v. \tag{1.1}$$

under the assumption one of the following condition:

1. Either F is continuous, or
2. (i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$, $\forall n$;
   (ii) if a non-increasing sequence $\{y_n\} \to y$, then $y \leq y_n$, $\forall n$.

Very recently, Borcut and Berinde [1] gave the natural extension of Definition 1.1 and Definition 1.2.

Definition 1.3. *Let $(X, \leq)$ be a partially ordered set and $F : X \times X \times X \to X$. The mapping F is said to has the mixed monotone property if for any $x, y, z \in X$*

$$x_1,\ x_2 \in X, \quad x_1 \leq x_2 \Longrightarrow F(x_1, y, z) \leq F(x_2, y, z),$$

$$y_1,\ y_2 \in X, \quad y_1 \leq y_2 \Longrightarrow F(x, y_1, z) \geq F(x, y_2, z),$$

$$z_1,\ z_2 \in X, \quad z_1 \leq z_2 \Longrightarrow F(x, y, z_1) \leq F(x, y, z_2),$$

Definition 1.4. *Let $F : X^3 \to X$. An element (x, y, z) is called a tripled fixed point of F if*

$$F(x, y, z) = x, \quad F(y, x, y) = y, \quad F(z, y, x) = z.$$

We recall the main theorem of Borcut and Berinde [1] which is inspired by the main theorem in [5].

Theorem 1.5. *Let $(X, \leq, d)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Suppose $F : X \times X \times X \to X$ such that F has the mixed monotone property and*

$$d(F(x,y,z), F(u,v,w)) \leq jd(x,u) + kd(y,v) + ld(z,w), \tag{1.2}$$

for any $x, y, z \in X$ for which $x \leq u$, $v \leq y$ and $z \leq w$. Suppose either F is continuous or X has the following properties:

1. *if a non-decreasing sequence $x_n \to x$, then $x_n \leq x$ for all n,*
2. *if a non-increasing sequence $y_n \to y$, then $y \leq y_n$ for all n,*
3. *if a non-decreasing sequence $z_n \to z$, then $z_n \leq z$ for all n.*

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, z_0)$ and $z_0 \leq F(z_0, y_0, x_0)$, then there exist $x, y, z \in X$ such that

$$F(x,y,z) = x, \quad F(y,x,y) = y, \quad F(z,y,x) = z,$$

that is, F has a tripled fixed point.

In this paper, we prove the existence and uniqueness of a tripled fixed point of $F : X^3 \to X$ satisfying nonlinear contractions in the context of partially ordered metric spaces.

2. Existence of a tripled fixed point

In this section we show the existence of a tripled fixed point. For this purpose, we state the following technical lemma which will be used in the proof of the main theorem efficiently.

Throughout the paper $M = [m_{ij}]$ is a matrix of real numbers and $M^t = [m_{ji}]$ denotes the transpose of M.

Lemma 2.1. *Let* $M = \begin{bmatrix} a & b & c \\ b & a+c & 0 \\ c & b & a \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{12} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$ *with $a + b + c < 1$.*

Then for $M^n = \begin{bmatrix} m_{11}^n & m_{12}^n & m_{13}^n \\ m_{21}^n & m_{22}^n & m_{23}^n \\ m_{31}^n & m_{32}^n & m_{33}^n \end{bmatrix}$ *we have*

$$m_{11}^n + m_{12}^n + m_{13}^n = m_{21}^n + m_{22}^n + m_{23}^n = m_{31}^n + m_{32}^n + m_{33}^n = (a+b+c)^n < 1.$$

Proof. We use mathematical induction. For $n = 1$,

$$M = \begin{bmatrix} a & b & c \\ b & a+c & 0 \\ c & b & a \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{12} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix},$$

then by assumption

$$m_{11} + m_{12} + m_{13} = m_{21} + m_{22} + m_{23} = m_{31} + m_{32} + m_{33} = a + b + c < 1.$$

For $n = 2$,

$$\begin{aligned}
M^2 &= \begin{bmatrix} a & b & c \\ b & a+c & 0 \\ c & b & a \end{bmatrix} \begin{bmatrix} a & b & c \\ b & a+c & 0 \\ c & b & a \end{bmatrix} \\
&= \begin{bmatrix} a^2+b^2+c^2 & b(a+c)+ab+bc & 2ac \\ b(a+c)+ab & (a+c)^2+b^2 & bc \\ b^2+2ac & b(a+c)+ab+bc & a^2+c^2 \end{bmatrix} \\
M^2 &= \begin{bmatrix} m_{11}^2 & m_{12}^2 & m_{13}^2 \\ m_{21}^2 & m_{22}^2 & m_{23}^2 \\ m_{31}^2 & m_{32}^2 & m_{33}^2 \end{bmatrix} = \begin{bmatrix} a^2+b^2+c^2 & b(a+c)+ab+bc & 2ac \\ b(a+c)+ab & (a+c)^2+b^2 & bc \\ b^2+2ac & b(a+c)+ab+bc & a^2+c^2 \end{bmatrix}
\end{aligned}$$

Since $(a+b+c)^2 = a^2+2ab+2ac+b^2+2bc+c^2$ then

$$\begin{aligned}
m_{11}^2+m_{12}^2+m_{13}^2 &= m_{21}^2+m_{22}^2+m_{23}^2 \\
&= m_{31}^2+m_{32}^2+m_{33}^2 \\
&= (a+b+c)^2 < 1.
\end{aligned}$$

Suppose it is true for an arbitrary n, that is, for $M^n = \begin{bmatrix} m_{11}^n & m_{12}^n & m_{13}^n \\ m_{21}^n & m_{22}^n & m_{23}^n \\ m_{31}^n & m_{32}^n & m_{33}^n \end{bmatrix}$ we have

$$\begin{aligned}
m_{11}^n+m_{12}^n+m_{13}^n &= m_{21}^n+m_{22}^n+m_{23}^n \\
&= m_{31}^n+m_{32}^n+m_{33}^n \\
&= (a+b+c)^n < 1.
\end{aligned}$$

Then,

$$\begin{aligned}
M^{n+1} &= M^n M = \begin{bmatrix} m_{11}^n & m_{12}^n & m_{13}^n \\ m_{21}^n & m_{22}^n & m_{23}^n \\ m_{31}^n & m_{32}^n & m_{33}^n \end{bmatrix} \begin{bmatrix} a & b & c \\ b & a+c & 0 \\ c & b & a \end{bmatrix} \\
&= \begin{bmatrix} am_{11}^n+bm_{12}^n+cm_{13}^n & m_{12}^n(a+c)+bm_{11}^n+bm_{13}^n & am_{13}^n+cm_{11}^n \\ am_{21}^n+bm_{22}^n+cm_{23}^n & m_{22}^n(a+c)+bm_{21}^n+bm_{23}^n & am_{23}^n+cm_{21}^n \\ am_{31}^n+bm_{32}^n+cm_{33}^n & m_{32}^n(a+c)+bm_{31}^n+bm_{33}^n & am_{33}^n+cm_{31}^n \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
m_{11}^{n+1}+m_{12}^{n+1}+m_{13}^{n+1} &= am_{11}^n+bm_{12}^n+cm_{13}^n+m_{12}^n(a+c) \\
&\qquad +bm_{11}^n+bm_{13}^n+am_{13}^n+cm_{11}^n \\
&= m_{11}^n(a+b+c)+m_{12}^n(a+b+c)+m_{13}^n(a+b+c) \\
&= (m_{11}^n+m_{12}^n+m_{13}^n)(a+b+c) \\
&= (a+b+c)^n(a+b+c) < 1.
\end{aligned}$$

Analogously we get that

$$\begin{aligned}
m_{21}^{n+1}+m_{22}^{n+1}+m_{23}^{n+1} &= m_{31}^{n+1}+m_{32}^{n+1}+m_{33}^{n+1} \\
&= (a+b+c)^{n+1} < 1
\end{aligned}$$

□

Theorem 2.2. *Let $(X, d, \leq)$ be a partially ordered complete metric space. Let $F : X^3 \to X$ be a mapping having the mixed monotone property on X. Assume that there exist* *non-negative numbers a, b, c and L such that $a+b+c<1$ for which*

$$\begin{aligned} d(F(x,y,z),F(u,v,w)) \quad &\leq ad(x,u)+bd(y,v)+cd(z,w) \\ &+L\min\left\{\begin{array}{c} d(F(x,y,z),x),d(F(x,y,z),y),d(F(x,y,z),z), \\ d(F(x,y,z),u),d(F(x,y,z),v),d(F(x,y,z),w), \\ d(F(u,v,w),x),d(F(u,v,w),y),d(F(u,v,w),z), \\ d(F(u,v,w),u),d(F(u,v,w),v),d(F(u,v,w),w) \end{array}\right\}, \end{aligned} \tag{2.1}$$

for all $x \geq u,\ y \leq v,\ z \geq w$. Assume that X has the following properties:

(a) *F is continuous, or,*

(b) (i) *if non-decreasing sequence $x_n \to x$ (respectively, $z_n \to z$), then $x_n \leq x$ (respectively, $z_n \leq z$), for all n,*

(ii) *if non-increasing sequence $y_n \to y$, then $y_n \geq y$ for all n.*

If there exist $x_0, y_0, z_0 \in X$ such that

$$x_0 \leq F(x_0,y_0,z_0), \quad y_0 \geq F(y_0,x_0,y_0), \quad z_0 \leq F(x_0,y_0,z_0)$$

then there exist $x, y, z \in X$ such that

$$F(x,y,z)=x \text{ and } F(y,x,y)=y \text{ and } F(z,y,x)=z$$

Proof. We construct a sequence $\{(x_n,y_n,z_n)\}$ in the following way: Set

$$x_1=F(x_0,y_0,z_0)\geq x_0, \quad y_1=F(y_0,x_0,y_0)\leq y_0, z_1=F(z_0,y_0,x_0)\geq z_0,$$

and by the mixed monotone property of F, for $n \geq 1$, inductively we get

$$\begin{aligned} x_n &= F(x_{n-1},y_{n-1},z_{n-1}) \geq x_{n-1} \geq \cdots \geq x_0, \\ y_n &= F(y_{n-1},x_{n-1},y_{n-1}) \leq y_{n-1} \leq \cdots \leq y_0, \\ z_n &= F(z_{n-1},y_{n-1},x_{n-1}) \geq z_{n-1} \geq \cdots \geq z_0. \end{aligned} \tag{2.2}$$

Moreover,

$$\begin{aligned} d(x_n,x_{n+1}) \quad &\leq ad(x_{n-1},x_n)+bd(y_{n-1},y_n)+cd(z_{n-1},z_n) \\ &+L\min\left\{\begin{array}{c} d(x_n,x_{n-1}),d(x_n,y_{n-1}),d(x_n,z_{n-1}), \\ d(x_n,x_n),d(x_n,y_n),d(x_n,z_n), \\ d(x_{n+1},x_{n-1}),d(x_{n+1},y_{n-1}),d(x_{n+1},z_{n-1}), \\ d(x_{n+1},x_n),d(x_{n+1},y_n),d(x_{n+1},z_n), \end{array}\right\} \\ &\leq ad(x_{n-1},x_n)+bd(y_{n-1},y_n)+cd(z_{n-1},z_n) \end{aligned} \tag{2.3}$$

$$\begin{aligned} d(y_n,y_{n+1}) \quad &\leq ad(y_{n-1},y_n)+bd(x_{n-1},x_n)+cd(y_{n-1},y_n) \\ &+L\min\left\{\begin{array}{c} d(y_n,y_{n-1}),d(y_n,x_{n-1}),d(y_n,y_{n-1}), \\ d(y_n,y_n),d(y_n,x_n),d(y_n,y_n), \\ d(y_{n+1},y_{n-1}),d(y_{n+1},x_{n-1}),d(y_{n+1},y_{n-1}), \\ d(y_{n+1},y_n),d(y_{n+1},y_n),d(y_{n+1},y_n), \end{array}\right\} \\ &\leq (a+c)d(y_{n-1},y_n)+bd(x_{n-1},x_n) \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} d(z_n, z_{n+1}) &\leq ad(z_{n-1}, z_n) + bd(y_{n-1}, y_n) + cd(x_{n-1}, x_n) \\ &+L \min \left\{ \begin{array}{c} d(z_n, x_{n-1}), d(z_n, y_{n-1}), d(z_n, z_{n-1}), \\ d(z_n, z_n), d(z_n, y_n), d(z_n, x_n), \\ d(z_{n+1}, x_{n-1}), d(z_{n+1}, y_{n-1}), d(z_{n+1}, z_{n-1}), \\ d(z_{n+1}, z_n), d(z_{n+1}, y_n), d(z_{n+1}, x_n), \end{array} \right\} \\ &\leq ad(z_{n-1}, z_n) + bd(y_{n-1}, y_n) + cd(x_{n-1}, x_n) \end{aligned} \tag{2.5}$$

Thus, from by (2.3)-(2.5) and Lemma 2.1, we obtain that

$$D_{n+1} \leq MD_n \leq \cdots \leq M^n D_1 \tag{2.6}$$

where

$$D_{n+1} = \begin{bmatrix} d(x_n, x_{n+1}) \\ d(y_n, y_{n+1}) \\ d(z_n, z_{n+1}) \end{bmatrix} \text{ and } M = \begin{bmatrix} a & b & c \\ b & a+c & 0 \\ c & b & a \end{bmatrix} = [m_{ij}] \text{ as in Lemma 2.1.}$$

We show that $\{x_n\}$,$\{y_n\}$ and $\{z_n\}$ are Cauchy sequences.

Due to Lemma 2.1 and (2.6), we have

$$\begin{aligned} d(x_p, x_q) &\leq d(x_p, x_{p-1}) + d(x_{p-1}, x_{p-2}) + \cdots + d(x_{q+1}, x_p) \\ &\leq \begin{bmatrix} m_{11}^p \\ m_{12}^p \\ m_{13}^p \end{bmatrix}^t D_1 + \begin{bmatrix} m_{11}^{p-1} \\ m_{12}^{p-1} \\ m_{13}^{p-1} \end{bmatrix}^t D_1 + \cdots + \begin{bmatrix} m_{11}^{q+1} \\ m_{12}^{q+1} \\ m_{13}^{q+1} \end{bmatrix}^t D_1 \\ &= \begin{bmatrix} m_{11}^{q+1} + \cdots + m_{11}^{p-1} + m_{11}^p \\ m_{12}^{q+1} + \cdots + m_{12}^{p-1} + m_{12}^p \\ m_{13}^{q+1} + \cdots + m_{13}^{p-1} + m_{13}^p \end{bmatrix}^t D_1 \\ &\leq (m_{11}^{q+1} + \cdots + m_{11}^{p-1} + m_{11}^p) d(x_1, x_0) \\ &\quad +(m_{12}^{q+1} + \cdots + m_{12}^{p-1} + m_{12}^p) d(y_1, y_0) \\ &\quad +(m_{13}^{q+1} + \cdots + m_{13}^{p-1} + m_{13}^p) d(z_1, z_0) \\ &\leq (k^q + k^{q+1} + \cdots + k^{p-1}) d(x_1, x_0) \\ &\quad +(k^q + k^{q+1} + \cdots + k^{p-1}) d(y_1, y_0) \\ &\quad +(k^q + k^{q+1} + \cdots + k^{p-1}) d(z_1, z_0) \\ &= (k^q + k^{q+1} + \cdots + k^{p-1})(d(x_1, x_0) + d(y_1, y_0) + d(z_1, z_0)) \\ &\leq k^q \tfrac{1-k^{p-q}}{1-k} (d(x_1, x_0) + d(y_1, y_0) + d(z_1, z_0)) \end{aligned} \tag{2.7}$$

where $k = a+b+c < 1$. Thus (2.7) yields that $\{x_n\}$ is a Cauchy sequence. Analogously, one can show $\{y_n\}$ and $\{z_n\}$ are Cauchy sequences.

Since X is a complete metric space, there exist $x, y, z \in X$ such that

$$\lim_{n\to\infty} x_n = x, \ \lim_{n\to\infty} y_n = y \text{ and } \lim_{n\to\infty} z_n = z. \tag{2.8}$$

Now, suppose that assumption (a) holds. Taking the limit as $n \to \infty$ in (2.2) and by (2.8), we get

$$\begin{aligned} x &= \lim_{n\to\infty} x_n = \lim_{n\to\infty} F(x_{n-1}, y_{n-1}, z_{n-1}) \\ &= F(\lim_{n\to\infty} x_{n-1}, \lim_{n\to\infty} y_{n-1}, \lim_{n\to\infty} z_{n-1}) \\ &= F(x, y, z) \end{aligned}$$

and

$$\begin{aligned} y &= \lim_{n\to\infty} y_n = lim_{n\to\infty} F(y_{n-1}, x_{n-1}, y_{n-1}) \\ &= F(\lim_{n\to\infty} y_{n-1}, lim_{n\to\infty} x_{n-1}, \lim_{n\to\infty} y_{n-1}) \\ &= F(y, x, y) \end{aligned}$$

$$\begin{aligned} z &= \lim_{n\to\infty} z_n = \lim_{n\to\infty} F(z_{n-1}, y_{n-1}, x_{n-1}) \\ &= F(\lim_{n\to\infty} z_{n-1}, \lim_{n\to\infty} y_{n-1}, \lim_{n\to\infty} x_{n-1}) \\ &= F(z, y, x). \end{aligned}$$

Thus we proved that $x = F(x, y, z), y = F(y, x, y)$ and $z = F(z, y, x)$.

Finally, suppose that (b) holds. Since $\{x_n\}$ (respectively, $\{z_n\}$) is non-decreasing sequence and $x_n \to x$ (respectively, $z_n \to z$) and as $\{y_n\}$ is non-increasing sequence and $y_n \to y$,by assumption (b), we have $x_n \geq x$ (respectively, $z_n \geq z$) and $y_n \leq y$ for all n. We have

$$\begin{aligned} d(F(x_n, y_n, z_n), F(x, y, z)) &\leq ad(x_n, x) + bd(y_n, y) + cd(z_n, z) \\ &+L \min \left\{ \begin{array}{c} d(F(x_n, y_n, z_n), x), d(F(x_n, y_n, z_n), y), \\ d(F(x_n, y_n, z_n), z), d(F(x_n, y_n, z_n), x_n), \\ d(F(x_n, y_n, z_n), y_n), d(F(x_n, y_n, z_n), z_n), \\ d(F(x, y, z), x), d(F(x, y, z), y), \\ d(F(x, y, z), z), d(F(x, y, z), x_n), \\ d(F(x, y, z), y_n), d(F(x, y, z), z_n), \end{array} \right\} \end{aligned} \tag{2.9}$$

Taking $n \to \infty$ in (2.9) we get $d(x, F(x, y, z)) \leq 0$ which implies $F(x, y, z) = x$. Analogously, we can show that $F(y, x, y) = y$ and $F(z, y, x) = z$.
Therefore, we proved that F has a tripled fixed point. □

Corollary 2.3. *(Main Theorem of* [1]*) Let $(X, d, \leq)$ be a partially ordered complete metric space. Suppose $F : X \times X \times X \to X$ such that F has the mixed monotone property and*

$$d(F(x, y, z), F(u, v, w)) \leq jd(x, u) + kd(y, v) + ld(z, w), \tag{2.10}$$

for any $x, y, z \in X$ for which $x \leq u$, $v \leq y$ and $z \leq w$. Suppose either F is continuous or X has the following properties:

1. *if a non-decreasing sequence $x_n \to x$, then $x_n \leq x$ for all n,*
2. *if a non-increasing sequence $y_n \to y$, then $y \leq y_n$ for all n,*
3. *if a non-decreasing sequence $z_n \to z$, then $z_n \leq z$ for all n.*

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, z_0)$ and $z_0 \leq F(z_0, y_0, x_0)$, then there exist $x, y, z \in X$ such that

$$F(x, y, z) = x, \quad F(y, x, y) = y, \quad F(z, y, x) = z,$$

that is, F has a tripled fixed point.

Proof. Taking $L = 0$ in Theorem (2.2), we obtain Corollary 2.3. □

3. Uniqueness of tripled fixed point

In this section we shall prove the uniqueness of tripled fixed point. For a partially ordered set $(X, \leq)$, we endow the product $X \times X \times X$ with the following partial order relation: for all $(x, y, z), (u, v, w) \in (X \times X)$,

$$(x, y, z) \leq (u, v, w) \Leftrightarrow x \leq u, y \geq v \text{ and } z \leq w.$$

We say that (x, y, z) is equal to (u, v, r) if and only if $x = u,\ y = v,\ z = r$.

Theorem 3.1. *In addition to hypotheses of Theorem 2.2, suppose that for all $(x, y, z), (u, v, r) \in X \times X \times X$, there exists $(a, b, c) \in X \times X \times X$ that is comparable to (x, y, z) and (u, v, r), then F has a unique triple fixed point.*

Proof. The set of triple fixed point of F is not empty due to Theorem 2.2. Assume, now, (x, y, z) and (u, v, r) are the triple fixed points of F, that is,

$$\begin{array}{ll} F(x, y, z) = x, & F(u, v, r) = u, \\ F(y, x, y) = y, & F(v, u, v) = v, \\ F(z, y, x) = z, & F(r, v, u) = r, \end{array}$$

We shall show that (x, y, z) and (u, v, r) are equal. By assumption, there exists $(p, q, s) \in X \times X \times X$ that is comparable to (x, y, z) and (u, v, r). Define sequences $\{p_n\}, \{q_n\}$ and $\{s_n\}$ such that

$$p = p_0, \quad q = q_0, \quad s = s_0, \quad \text{and}$$

$$\begin{aligned} p_n &= F(p_{n-1}, q_{n-1}, s_{n-1}), \\ q_n &= F(q_{n-1}, p_{n-1}, q_{n-1}), \\ s_n &= F(s_{n-1}, q_{n-1}, p_{n-1}), \end{aligned} \tag{3.1}$$

for all n. Since (x, y, z) is comparable with (p, q, s), we may assume that $(x, y, z) \geq (p, q, s) = (p_0, q_0, s_0)$. Recursively, we get that

$$(x, y, z) \geq (p_n, q_n, s_n) \quad \text{for all } n. \tag{3.2}$$

By (3.2) and (2.1), we have

$$\begin{aligned} d(x, p_{n+1}) &= d(F(x, y, z), F(p_n, q_n, s_n)) \\ &\leq ad(x, p_n) + bd(y, q_n) + cd(z, s_n) \\ &\quad + L \min \left\{ \begin{array}{c} d(F(x, y, z), x), d(F(x, y, z), y), \\ d(F(x, y, z), z), d(F(x, y, z), p_n), \\ d(F(x, y, z), q_n), d(F(x, y, z), s_n), \\ d(F(p_n, q_n, s_n), x), d(F(p_n, q_n, s_n), y), \\ d(F(p_n, q_n, s_n), z), d(F(p_n, q_n, s_n), p_n), \\ d(F(p_n, q_n, s_n), q_n), d(F(p_n, q_n, s_n), s_n) \end{array} \right\} \\ &\leq ad(x, p_n) + bd(y, q_n) + cd(z, s_n) \\ &\leq k^n [d(x, p_0) + d(y, q_0) + d(z, s_0)] \end{aligned} \tag{3.3}$$

where

$$\begin{aligned}
d(q_{n+1},y) &= d(F(q_n,p_n,q_n),F(y,x,y)) \\
&\le ad(y,q_n)+bd(x,p_n)+cd(y,q_n) \\
&+L\min\left\{\begin{array}{c} d(F(y,x,y),y),d(F(y,x,y),x), \\ d(F(y,x,y),y),d(F(y,x,y),q_n), \\ d(F(y,x,y),p_n),d(F(y,x,y),q_n), \\ d(F(F(q_n,p_n,q_n),y),d(F(q_n,p_n,q_n),x), \\ d(F(q_n,p_n,q_n),y),d(F(q_n,p_n,q_n),q_n), \\ d(F(q_n,p_n,q_n),p_n),d(F(p_n,q_n,s_n),q_n) \end{array}\right\} \\
&\le ad(y,q_n)+bd(x,p_n)+cd(y,q_n) \\
&\le k^n[d(x,p_0)+d(y,q_0)+d(z,s_0)]
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
d(z,s_{n+1}) &= d(F(x,y,z),F(p_n,q_n,s_n)) \\
&\le ad(x,p_n)+bd(y,q_n)+cd(z,s_n) \\
&+L\min\left\{\begin{array}{c} d(F(z,x,y),z),d(F(z,x,y),y), \\ d(F(z,x,y),x),d(F(z,x,y),s_n), \\ d(F(z,x,y),q_n),d(F(z,x,y),p_n), \\ d(F(s_n,q_n,p_n),z),d(F(s_n,q_n,p_n),y), \\ d(F(s_n,q_n,p_n),x),d(F(s_n,q_n,p_n),s_n), \\ d(F(s_n,q_n,p_n),q_n),d(F(s_n,q_n,p_n),p_n) \end{array}\right\} \\
&\le ad(x,p_n)+bd(y,q_n)+cd(z,s_n) \\
&\le k^n[d(x,p_0)+d(y,q_0)+d(z,s_0)]
\end{aligned} \tag{3.5}$$

Letting $n\to\infty$ in (3.3) -(3.5), we get

$$\lim_{n\to\infty} d(x,p_{n+1})=0,\ \lim_{n\to\infty} d(y,q_{n+1})=0,\ \lim_{n\to\infty} d(z,s_{n+1})=0. \tag{3.6}$$

Analogously, we obtain

$$\lim_{n\to\infty} d(u,p_{n+1})=0,\ \lim_{n\to\infty} d(v,q_{n+1})=0,\ \lim_{n\to\infty} d(r,s_{n+1})=0. \tag{3.7}$$

By (3.6) and (3.7) we have $x=u, y=v, z=r$. □

4. Examples

In this sections we give some examples to show that our results are effective.

Example 4.1. Let $X=[0,1]$ with the metric $d(x,y)=|x-y|$, for all $x,y\in X$ and the usual ordering.
Let $F: X^3\to X$ be given by

$$F(x,y,z)=\frac{4x^2-4y^2+8z^2+8}{65},\ \text{for all } x,y,z,w\in X$$

It is easy to check that all the conditions of Corollary 2.3 are satisfied and $(\frac{1}{8},\frac{1}{8},\frac{1}{8})$ is the unique tripled fixed point of F where $a=b=\frac{8}{65}$ and $c=\frac{16}{65}$.

But if we changed the space $X=[0,1]$ with $X=[0,\infty)$, then conditions of Corollary 2.3 are not satisfied anymore. Indeed, $F(8,0,8)=\frac{776}{65}$ and $F(7,0,7)=\frac{596}{65}$. Thus,$d(F(8,0,8),F(7,0,7))=\frac{36}{13}$. On the other hand, $d(8,7)=1$. Thus, there exist no non-negative real number a, b, c with $a + b + c < 1$ and satisfies the conditions of Corollary 2.3. But, for $L = 2$ and $a = b = \frac{8}{65}, c = \frac{16}{65}$, the conditions of Theorem 2.2 are satisfied. Moreover, $(\frac{1}{8}, \frac{1}{8}, \frac{1}{8})$ is the unique tripled fixed point of F.

Example 4.2. Let $X = [0, \infty)$ with the metric $d(x, y) = |x - y|$, for all $x, y \in X$ and the following order relation:

$$x, y \in X, \ \ x \preceq y \ \Leftrightarrow \ x = y = 0 \,\text{or}\ \ (x, y \in (0, \infty) \text{and}\ x \leq y),$$

where $\leq$ be the usual ordering.
Let $F : X^3 \to X$ be given by

$$F(x, y, z) = \begin{cases} 1, & if \quad xyz \neq 0 \\ 0, & if \quad xyz = 0 \end{cases}$$

for all $x, y, z \in X$.
It is easy to check that all the conditions of Corollary 2.3 are satisfied. Applying Corollary 2.3 we conclude that F has a tripled fixed point. In fact, F has two tripled fixed points. They are $(0, 0, 0)$ and $(1, 1, 1)$. Therefore, the conditions of Corollary 2.3 are not sufficient for the uniqueness of a tripled fixed point.

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