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Rate of convergence for Szász type operators including Sheffer polynomials

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Abstract. In the present paper, we study the rate of convergence of Szász type and Kantorovich-Szász type operators involving Sheffer polynomials with the help of modulus of continuity and examine these type operators including reverse Bessel polynomials which are Sheffer type. Furthermore, we compute error estimation for a function f by operators containing reverse Bessel polynomials.

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1. Introduction

One of the fundamental problems of approximation theory is to approximate function f by functions which have better properties than f. In 1953, Korovkin [4] discovered the most powerful and simplest criterion for positive approximation processes. This theory has widely affected not only classical approximation theory but also such other areas of mathematics as partial differential equations, harmonic analysis, orthogonal polynomials and wavelet analysis.

In 1950, Szász [7] introduced and exhaustively investigated the operator

$$S_n(f;x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right) \tag{1.1}$$

defined on the set of real valued function on $[0,\infty)$.

Jakimovski and Leviatan [3] presented a new type of operators which involves Appell polynomials as follows

$$P_n(f;x) := \frac{e^{-nx}}{A(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right) \ . \tag{1.2}$$

In the above relation p_k are Appell polynomials defined by the generating functions

$$A(u)\,e^{ux}=\sum_{k=0}^{\infty}p_k(x)\,u^k$$

where $A(z)=\sum\limits_{k=0}^{\infty}a_kz^k$ $(a_0\neq 0)$ be an analytic function in the disc $|z|<R$ $(R>1)$ and suppose $A(1)\neq 0$. They obtained important results analogue to Szász [7]. If we take $A(z)=1$, we get Szász operators (1.1) by using above generating functions.

Ismail [2] generalized Jakimovski and Leviatan's work by dealing with the approximation of Szász operators with the help of Sheffer polynomials as follows

$$T_n(f;x):=\frac{e^{-nxH(1)}}{A(1)}\sum_{k=0}^{\infty}p_k(nx)\,f\left(\frac{k}{n}\right) \tag{1.3}$$

whenever the right hand side of (1.3) exists. In the relation (1.3) p_k are Sheffer polynomials given by the generating functions

$$A(u)\,e^{xH(u)}=\sum_{k=0}^{\infty}p_k(x)\,u^k \tag{1.4}$$

where

$$\begin{aligned}A(z)&=\sum_{k=0}^{\infty}a_kz^k\qquad(a_0\neq 0)\\ H(z)&=\sum_{k=1}^{\infty}h_kz^k\qquad(h_1\neq 0)\end{aligned} \tag{1.5}$$

be analytic functions in the disc $|z|<R$ $(R>1)$. Under the restrictions:

$$\begin{array}{ll}\text{(i)} & \text{For } x\in[0,\infty) \text{ and } k\in\mathbb{N}\cup\{0\}\,,p_k(x)\geq 0,\\ \text{(ii)} & A(1)\neq 0 \text{ and } H^{'}(1)=1,\\ \text{(iii)} & (1.4) \text{ relation is valid for } |u|<R \text{ and}\\ & \text{the power series given by (1.5) converges for } |z|<R\,(R>1)\,,\end{array} \tag{1.6}$$

Ismail showed that the same type of results which are obtained by Jakimovski and Leviatan are still valid for the operators including Sheffer polynomials known as more general class of polynomials than Appell polynomials. It is clear that the operators (1.3) contain (1.1) and (1.2). Furthermore, Ismail introduced Kantorovich generalization of the operators (1.3) as

$$T_n^*(f;x):=n\frac{e^{-nxH(1)}}{A(1)}\sum_{k=0}^{\infty}p_k(nx)\int_{k/n}^{(k+1)/n}f(s)\,ds\ . \tag{1.7}$$

Ismail proved that for $f\in C[0,\infty)\,,|f(x)|\leq ce^{Kx}$ $K\in\mathbb{R}$ and $c\in\mathbb{R}^+$, the operators $T_n(f;.)$ converge to the function f in each compact subset of $[0,\infty)$ by using the methods of Szász [7]. And he also investigated that the operators T_n^* converge for the functions $f\in C[0,\infty)\ ,\int\limits_0^t f(s)\,ds=\mathcal{O}\left(e^{Kt}\right),\,(t\to\infty)\,.$

As it is known, there are two main problems in approximation theory. One of them is existence of approximation, the other is rate of convergence.

The purpose of the present paper is to study the rate of approximation of the sequences of operators T_n and T_n^* by means of the modulus of continuity. Moreover, since these operators are of general form, we give an example of these type operators T_n and T_n^* including reverse Bessel polynomials [5] and obtain error estimation for operators T_n including reverse Bessel polynomials with the help of Maple13.

2. Approximation properties of T_n and T_n^* operators

We begin by considering the following definition of the class E as follows:

$$E := \left\{ f : \forall x \in [0, \infty), \ \ |f(x)| \leq ce^{Kx} \quad K \in \mathbb{R} \text{ and } c \in \mathbb{R}^{+} \right\}.$$

In the sequel, we shall need the following auxiliary result.

Lemma 2.1. *For $x \in [0, \infty)$, we have*

$$\begin{aligned} T_n(1; x) &= 1, \\ T_n(\xi; x) &= x + \frac{A'(1)}{nA(1)}, \\ T_n(\xi^2; x) &= x^2 + \left(\frac{2A'(1)}{A(1)} + H''(1) + 1 \right) \frac{x}{n} + \frac{A'(1) + A''(1)}{n^2 A(1)}. \end{aligned}$$

Proof. With the help of generating functions of Sheffer polynomials (1.4), we get the assertion of our lemma. □

Lemma 2.2. *For $x \in [0, \infty)$, the following equality holds:*

$$T_n\left((\xi - x)^2 ; x \right) = \frac{H''(1) + 1}{n} x + \frac{A'(1) + A''(1)}{n^2 A(1)}.$$

Proof. From the linearity property of T_n operators and Lemma 2.1, one can find the above relation. □

Definition 2.3. *The modulus of continuity of a function $f \in \tilde{C}[0, \infty)$ is a function $\omega(f; \delta)$ defined by the relation*

$$\omega(f; \delta) := \sup_{\substack{|x-y| \leq \delta \\ x, y \in [0, \infty)}} |f(x) - f(y)|$$

where $\tilde{C}[0, \infty)$ is uniformly continuous functions space.

Ismail proved that for $f \in C[0, \infty) \cap E$ the operators $T_n(f; .)$ converge to the function f in each compact subset of $[0, \infty)$ by using the methods of Szász [7]. On the other hand, if we consider the Lemma 2.1, we obtain the approximation result through the instrument of universal Korovkin-type theorem whenever functions belong to the convenient set. The following theorem is the quantitative version of the result of Ismail only in a particular case. While the result of Ismail holds for functions $f \in C[0, \infty) \cap E$, Theorem 2.4 is valid for restrictive functions $f \in \tilde{C}[0, \infty) \cap E$. Now we are going to state the degree of convergence of the former operator by using the modulus of continuity.

Theorem 2.4. *If* $f \in \tilde{C}[0,\infty) \cap E$, *then for any* $x \in [0,\infty)$ *we have*

$$|T_n(f;x) - f(x)| \leq \left(1 + \sqrt{\left(H''(1)+1\right)x + \frac{A'(1)+A''(1)}{nA(1)}}\right)\omega\left(f;\frac{1}{\sqrt{n}}\right). \quad (2.1)$$

Proof. By the aid of Lemma 2.1 and property of modulus of continuity, one has the following expression

$$|T_n(f;x) - f(x)| \leq \left\{1 + \frac{1}{\delta}\frac{e^{-nxH(1)}}{A(1)}\sum_{k=0}^{\infty} p_k(nx)\left|\frac{k}{n} - x\right|\right\}\omega(f;\delta). \quad (2.2)$$

On the other hand, making use of the Cauchy-Schwarz inequality we can write

$$\sum_{k=0}^{\infty} p_k(nx)\left|\frac{k}{n} - x\right| \leq A(1)\,e^{nxH(1)}\sqrt{\frac{H''(1)+1}{n}x + \frac{A'(1)+A''(1)}{n^2A(1)}}.$$

Combining the above relation with (2.2), also choosing $\delta = \frac{1}{\sqrt{n}}$, we obtain (2.1). This completes the proof. □

We will need the following lemmas for proving our results about order of convergence for T_n^* operators. Let us consider the class

$$\mathcal{E} := \left\{f : [0,\infty) \to \mathbb{R} \mid \int_0^t f(s)\,ds = \mathcal{O}\left(e^{Kt}\right),\ (t \to \infty)\right\}.$$

Lemma 2.5. *There hold the equalities*

$$\begin{aligned}
T_n^*(1;x) &= 1,\\
T_n^*(\xi;x) &= x + \left(\frac{1}{2} + \frac{A'(1)}{A(1)}\right)\frac{1}{n},\\
T_n^*\left(\xi^2;x\right) &= x^2 + \left(\frac{2A'(1) + \left(H''(1)+2\right)A(1)}{A(1)}\right)\frac{x}{n}\\
&\quad + \frac{3\left(A''(1) + 2A'(1)\right) + A(1)}{3n^2A(1)}.
\end{aligned}$$

Proof. As a consequence of the Lemma 2.1, we immediately get the desired conclusion. □

Therefore by the Lemma 2.5 and by the linearity of T_n^*, we can state the following result.

Lemma 2.6. *For all $x \in [0,\infty)$, we have*

$$T_n^*\left((\xi-x)^2;x\right)=\frac{H''(1)+1}{n}x+\frac{A(1)+3\left(A''(1)+2A'(1)\right)}{3n^2A(1)}.$$

Through the instrument of Lemma 2.5 and universal Korovkin-type theorem, if function f belongs to appropriate set then the operators given by (1.7) convergence uniformly to the function f in each compact subset of $[0,\infty)$. We obtain quantitative version of the theorem of Ismail only in specific case in the following theorem. Hence, we are going to prove the degree of convergence for T_n^* with the help of modulus of continuity.

Theorem 2.7. *Let f be a function of class $\tilde{C}[0,\infty)\cap\mathcal{E}$. Then for any $x\in[0,\infty)$, we have*

$$|T_n^*(f;x)-f(x)|\leq(1+\lambda_n(x))\,\omega\left(f;\frac{1}{\sqrt{n}}\right) \tag{2.3}$$

where

$$\lambda_n(x):=\sqrt{(H''(1)+1)x+\frac{A(1)+3(A''(1)+2A'(1))}{3nA(1)}}.$$

Proof. According to Lemma 2.5 and using the property of modulus of continuity, it follows that

$$|T_n^*(f;x)-f(x)|\leq n\frac{e^{-nxH(1)}}{A(1)}\sum_{k=0}^{\infty}p_k(nx)\left(\frac{1}{n}+\frac{1}{\delta}\int_{k/n}^{(k+1)/n}|s-x|\,ds\right)\omega(f;\delta). \tag{2.4}$$

By a simple application of the Cauchy-Schwarz inequality to the right hand side of (2.4), we obtain

$$|T_n^*(f;x)-f(x)|\leq\left\{1+\frac{1}{\delta}\sqrt{n}\frac{e^{-nxH(1)}}{A(1)}\sum_{k=0}^{\infty}p_k(nx)\sqrt{\int_{k/n}^{(k+1)/n}(s-x)^2\,ds}\right\}\omega(f;\delta).$$

Once again, by the Cauchy-Schwarz inequality and then with the help of Lemma 2.6 we find that

$$|T_n^*(f;x)-f(x)|\leq$$
$$\left(1+\frac{1}{\delta}\frac{1}{\sqrt{n}}\sqrt{(H''(1)+1)x+\frac{A(1)+3(A''(1)+2A'(1))}{3nA(1)}}\right)\omega(f;\delta). \tag{2.5}$$

In the inequality (2.5) with choosing $\delta=\frac{1}{\sqrt{n}}$, we get (2.3). □

Remark 2.8. A general estimate in terms of modulus of continuity was given by Shisha and Mond [6]. In the proof of Theorem 2.4 and Theorem 2.7, we follow the method of proof in this mentioned estimate.

3. Example of these type operators

Example 3.1. The Bessel polynomials [5] which are defined by

$$y_k(x) = \sum_{j=0}^{k} \frac{(k+j)!}{(k-j)!j!}\left(\frac{x}{2}\right)^j$$

are an orthogonal sequence of polynomials. Carlitz [1] subsequently constructed a related set of polynomials known as reverse Bessel polynomials as follows

$$\begin{aligned} \theta_k(x) &= x^k y_{k-1}\left(\frac{1}{x}\right) \\ &= \sum_{j=1}^{k} \frac{(2k-j-1)!}{(j-1)!\,(k-j)!2^{k-j}} x^j \ . \end{aligned} \tag{3.1}$$

The generating function for $\theta_k(x)$ is

$$\exp\left[x\left(1-\sqrt{1-2t}\right)\right] = \sum_{k=0}^{\infty} \frac{\theta_k(x)}{k!} t^k \ . \tag{3.2}$$

If we consider the generating function (3.2), then reverse Bessel polynomials are Sheffer type polynomials. Taking into account of explicit formula (3.1), reverse Bessel polynomials $\theta_k(x)$ are positive for $x \geq 0$. Now, let be

$$p_k(x) = \frac{\theta_k\left(2\sqrt{2}x\right)}{4^k k!}.$$

Then by virtue of (3.2), we easily find $A(t) = 1$ and $H(t) = 2\sqrt{2}\left(1-\sqrt{1-\frac{t}{2}}\right)$. From these facts; $A(1) \neq 0$, $H'(1) = 1$ and $p_k(x) \geq 0$ $(x \geq 0)$ are verified. Therefore, we get operators $\widetilde{T_n}$ and $\widetilde{T_n^*}$ including reverse Bessel polynomials as follows:

$$\begin{aligned} \widetilde{T_n}(f;x) &:= e^{-2\left(\sqrt{2}-1\right)nx} \sum_{k=0}^{\infty} \frac{\theta_k\left(2\sqrt{2}nx\right)}{4^k k!} f\left(\frac{k}{n}\right) \ , \\ \widetilde{T_n^*}(f;x) &:= ne^{-2\left(\sqrt{2}-1\right)nx} \sum_{k=0}^{\infty} \frac{\theta_k\left(2\sqrt{2}nx\right)}{4^k k!} \int_{k/n}^{(k+1)/n} f(s)\,ds \ . \end{aligned}$$

Remark 3.2. Taking into account Theorem 2.4, we obtain quantitative error estimate for the approximation by $\widetilde{T_n}$ positive linear operators as follows

$$\left|\widetilde{T_n}(f;x) - f(x)\right| \leq \left(1+\sqrt{\frac{3}{2}x}\right)\omega\left(f;\frac{1}{\sqrt{n}}\right) \ ; \quad f \in \tilde{C}[0,\infty) \cap E.$$

Remark 3.3. According to Theorem 2.7, quantitative estimate of the rate of convergence is available for $\widetilde{T_n^*}$ positive linear operators as follows

$$\left|\widetilde{T_n^*}(f;x) - f(x)\right| \leq \left(1+\sqrt{\frac{3}{2}x+\frac{1}{3n}}\right)\omega\left(f;\frac{1}{\sqrt{n}}\right) \ ; \quad f \in \tilde{C}[0,\infty) \cap \mathcal{E}.$$

Remark 3.4. Due to reverse Bessel polynomials are not Appell polynomials, Jakimovski and Leviatan's result [3] does not involve the convergence of $\widetilde{T_n}(f;x)$ to $f(x)$ and of $\widetilde{T_n^*}(f;x)$ to $f(x)$.

Example 3.5. Let us take $f(x) = \dfrac{x}{\sqrt{1+x^4}}$. We compute error estimation by using modulus of continuity for operators $\widetilde{T_n}$ which contain reverse Bessel polynomials in the Table 1 with the help of Maple13 and give its algorithm after the Table 1.

n	Error estimate by $\widetilde{T_n}$ operators including $\{\theta_k(x)\}_{k=1}^{\infty}$ sequence
10	0.7000346345
10^2	0.2224633643
10^3	0.0703525749
10^4	0.0222474486
10^5	0.0070352610
10^6	0.0022247448
10^7	0.0007035261
10^8	0.0002224744
10^9	0.0000703526
10^{10}	0.0000222474

Table 1. The error bound of function f by using modulus of continuity.

Algorithm 3.6. The results with the following algorithm are shown in Table 1.

We derive error estimates for the convergence to the function

$$f(x) = \frac{x}{\sqrt{1+x^4}}$$

with $\widetilde{T_n}$ operators including reverse Bessel polynomials.

```
>restart;
>f:=x->sqrt(x^2/(1+x^4));
>n:=1:
>for i from 1 to 10 do
>n:=10*n;
>delta:=evalf(1/sqrt(n));
>omega(f,delta):=evalf(maximize(expand(abs(f(x+h)-f(x))),
x=0..1-delta,h=0..delta)):
>error:=evalf((1+sqrt(3/2))*omega(f,delta));
>end do;
```

Example 3.7. For $n = 10, 20, 50$; the convergence of $\widetilde{T_n}(f;x)$ to function

$$f(x) = 1 + \sin\left(-2x^2\right)$$

is illustrated in Figure 1 and its algorithm is presented after the Figure 1. Because of our machines have not enough speed and power to compute the complicated infinite series, we have to investigate our approximation result for finite sum.

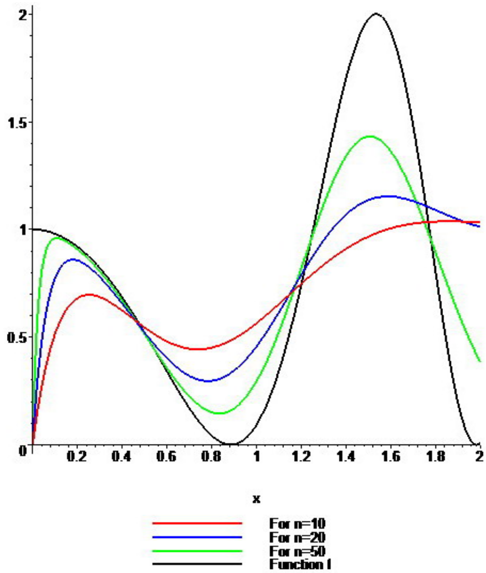

Figure 1. Approximation by $\widetilde{T_n}\,(f;x)$ operator for the function f.

Algorithm 3.8. >restart;

```
>with(plots):
>f:=x->1+sin(-2*x^2);
>Gt:=0:
>m:=100;
>G:=(k,n,x)->sum(f(k/n)*((2*k-j-1)!)/(((j-1)!)*((k-j)!)*((k)!)
*(2^(3*k-5*j/2)))*(n*x)^j,j=1..k);
>for i from 1 to m do
>Gt:=Gt+simplify(G(i,n,x))
>end do:
>B:=unapply(exp(-2*(sqrt(2)-1)*n*x)*Gt,n):
>p1:=plot(evalf(simplify(B(10))),x=0..2,color=red):
>p2:=plot(evalf(simplify(B(20))),x=0..2,color=blue):
>p3:=plot(evalf(simplify(B(50))),x=0..2,color=green):
>p4:=plot(f(x),x=0..2,color=black):
>display(p1,p2,p3,p4);
```

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