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On certain subclasses of meromorphic functions with positive coefficients

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Abstract. In this paper we introduce and study a new subclass of meromorphic univalent functions defined by convolution structure. We investigate various important properties and characteristics properties for this class. Further we obtain partial sums for the same.

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1. Introduction

Let $\mathcal{S}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and univalent in $\mathbb{U} = \{z : z \in \mathbb{C},\ |z| < 1\}$, normalized by $f(0) = f'(0) - 1 = 0$. Denote by $S^*(\gamma)$ and $K(\gamma)$, $(0 \leq \gamma < 1)$ the subclasses of function in $\mathcal{S}$ that are starlike and convex functions of order γ respectively. Analytically, $f \in S^*(\gamma)$ if and only if, f is of the form (1.1) and satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \gamma, \quad z \in \mathbb{U},$$

similarly, $f \in K(\gamma)$, if and only if, f is of the form (1.1) and satisfies

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \gamma, \quad z \in \mathbb{U},$$

Also denote by T the subclass of $\mathcal{S}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \ \ a_n \geq 0, \tag{1.2}$$

introduced and studied by Silverman [18], let $T^*(\gamma) = T \cap \mathcal{S}^*(\gamma)$, $CV(\gamma) = T \cap K^*(\gamma)$. The classes $T^*(\gamma)$ and $K^*(\gamma)$ possess some interesting properties and have been extensively studied by Silverman [18] and others.

In 1991, Goodman [7, 8] introduced an interesting subclass uniformly convex (uniformly starlike) of the class CV of convex functions (ST starlike functions) denoted by UCV (UST). A function $f(z)$ is uniformly convex (uniformly starlike) in $\mathbb{U}$ if $f(z)$ in $CV(ST)$ has the property that for every circular arc γ contained in $\mathbb{U}$, with center ξ also in $\mathbb{U}$, the arc $f(\gamma)$ is a convex arc (starlike arc) with respect to $f(\xi)$. Motivated by Gooodman [7, 8], Rønning [16, 17] introduced and studied the following subclasses of $\mathcal{S}$. A function $f \in \mathcal{S}$ is said to be in the class $S_p(\gamma, k)$ uniformly $k-$starlike functions if it satisfies the condition

$$\mathrm{Re}\ \left(\frac{zf'(z)}{f(z)} - \gamma\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right|,\ (0 \leq \gamma < 1; k \geq 0)\ z \in \mathbb{U} \tag{1.3}$$

and is said to be in the class $UCV(\gamma, k)$, uniformly $k-$convex functions if it satisfies the condition

$$\mathrm{Re}\ \left(1 + \frac{zf''(z)}{f'(z)} - \gamma\right) > k\left|\frac{zf''(z)}{f'(z)}\right|,\ (0 \leq \gamma < 1; k \geq 0)\ z \in \mathbb{U}. \tag{1.4}$$

Indeed it follows from (1.3) and (1.4) that

$$f \in UCV(\gamma, k) \Leftrightarrow zf' \in S_p(\gamma, k). \tag{1.5}$$

Further Ahuja et al.[1], Bharathi et al. [6], Murugusundaramoorthy and Magesh [11] and others have studied and investigated interesting properties for the classes $S_p(\gamma, k)$ and $UCV(\gamma, k)$.

Let Σ denote the class of functions of the form

$$f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n,\ \ a_n \geq 0, \tag{1.6}$$

which are analytic in the punctured open unit disk $\mathbb{U}^* := \{z : z \in \mathbb{C}, 0 < |z| < 1\} =: \mathbb{U} \setminus \{0\}$.

Let $\Sigma_{\mathcal{S}}$, $\Sigma^*(\gamma)$ and $\Sigma_K(\gamma)$, $(0 \leq \gamma < 1)$ denote the subclasses of Σ that are meromorphic univalent, meromorphically starlike functions of order γ and meromophically convex functions of order γ respectively. Analytically, $f \in \Sigma^*(\gamma)$ if and only if, f is of the form (1.6) and satisfies

$$-\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \gamma,\ \ z \in \mathbb{U},$$

similarly, $f \in \Sigma_K(\gamma)$, if and only if, f is of the form (1.6) and satisfies

$$-\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \gamma,\ \ z \in \mathbb{U},$$

and similar other classes of meromorphically univalent functions have been extensively studied by (for example) Altintas et al [2], Aouf [3], Mogra et al. [12], Uralegadi et al. [21, 22, 23] and others(see[10, 13, 14]).

Let $f, g \in \Sigma$, where f is given by (1.6) and g is defined by

$$g(z) = z^{-1} + \sum_{n=1}^{\infty} b_n z^n. \tag{1.7}$$

Then the Hadamard product (or convolution) $f * g$ of the functions f and g is defined by

$$(f * g)(z) := z^{-1} + \sum_{n=1}^{\infty} a_n b_n z^n =: (g * f)(z). \tag{1.8}$$

Motivated by Ravichandaran et al [15] and Atshan et al [5], now, we define a new subclass $\Sigma^*(g, \gamma, k, \lambda)$ of Σ.

Definition 1.1. *For $0 \leq \gamma < 1$, $k \geq 0$ and $0 \leq \lambda < \frac{1}{2}$, we let $\Sigma^*(g, \gamma, k, \lambda)$ be the subclass of $\Sigma_{\mathcal{S}}$ consisting of functions of the form (1.6) and satisfying the analytic criterion*

$$\begin{aligned} -\ Re &\left(\frac{z(f*g)'(z)}{(f*g)(z)} + \lambda \frac{z^2 (f*g)''(z)}{(f*g)(z)} + \gamma \right) \\ &> k \left| \frac{z(f*g)'(z)}{(f*g)(z)} + \lambda \frac{z^2 (f*g)''(z)}{(f*g)(z)} + 1 \right|. \end{aligned} \tag{1.9}$$

Also by suitably choosing $g(z)$ involved in the class, the class $\Sigma^*(g, \gamma, k, \lambda)$ reduces to various new subclasses. These considerations can fruitfully be worked out and we skip the details in this regard.

The main object of this paper is to study some usual properties of the geometric function theory such as the coefficient bounds, extreme points, radii of meromorphic starlikeness and meromorphic convexity for the class $\Sigma^*(g, \gamma, k, \lambda)$. Further, we obtain partial sums for aforementioned class.

2. Coefficients inequalities

In this section we obtain necessary and sufficient condition for a function f to be in the class $\Sigma^*(g, \gamma, k, \lambda)$. In this connection we state the following lemmas without proof.

Lemma 2.1. *If γ is a real number and $w = -(u + iv)$ is a complex number, then $Re\ (w) \geq \gamma \Leftrightarrow |w + (1 - \gamma)| - |w - (1 + \gamma)| \geq 0$.*

Lemma 2.2. *If $w = u + iv$ is a complex number and γ, k are real numbers, then*

$$-\ Re\ (w) \geq k|w + 1| + \gamma \Leftrightarrow\ -\ Re(w(1 + ke^{i\theta}) + ke^{i\theta}) \geq \gamma, \quad -\pi \leq \theta \leq \pi.$$

Theorem 2.3. *Let $f \in \Sigma$ be given by (1.6). Then $f \in \Sigma^*(g, \gamma, k, \lambda)$ if and only if*

$$\sum_{n=1}^{\infty} [n(k+1)(1 + (n-1)\lambda) + (k + \gamma)]\ a_n\ b_n \leq (1 - \gamma) - 2\lambda(k+1). \tag{2.1}$$

Proof. Let $f \in \Sigma^*(g,\gamma,k,\lambda)$. Then by definition and using Lemma 2.2, it is enough to show that

$$-\operatorname{Re}\left\{\left(\frac{z(f*g)'(z)}{(f*g)(z)}+\lambda\frac{z^2(f*g)''(z)}{(f*g)(z)}\right)(1+ke^{i\theta})+ke^{i\theta}\right\}>\gamma,\quad -\pi\leq\theta\leq\pi. \tag{2.2}$$

For convenience, we let

$$\begin{aligned} A(z) &:= -[z(f*g)'(z)+\lambda z^2(f*g)''(z)](1+ke^{i\theta})-ke^{i\theta}(f*g)(z)\\ B(z) &:= (f*g)(z). \end{aligned}$$

That is, the equation (2.2) is equivalent to

$$-\operatorname{Re}\left(\frac{A(z)}{B(z)}\right)\geq\gamma.$$

In view of Lemma 2.1, we only need to prove that

$$|A(z)+(1-\gamma)B(z)|-|A(z)-(1+\gamma)B(z)|\geq 0.$$

Therefore

$$\begin{aligned} |A(z)+(1-\gamma)B(z)| \ &\geq\ (2-\gamma-2\lambda(k+1))\frac{1}{|z|}\\ &\quad -\sum_{n=1}^{\infty}[n(1+(n-1)\lambda)-(k+1)+k+\gamma-1)]b_na_n|z|^n \end{aligned}$$

and

$$|A(z)-(1+\gamma)B(z)|\leq(\gamma+2\lambda(k+1))\frac{1}{|z|}+\sum_{n=1}^{\infty}[n(1+(n-1)\lambda)(k+1)+k+\gamma+1]b_na_n|z|^n$$

It is now easy to show that

$$\begin{aligned} &|A(z)+(1-\gamma)B(z)|-|A(z)-(1+\gamma)B(z)|\\ &\geq(2(1-\gamma)-4\lambda(k+1))\frac{1}{|z|}-2\sum_{n=1}^{\infty}[n(1+(n-1)\lambda)(k+1)+(\gamma+k)]b_na_n|z|^n\\ &\geq 0, \end{aligned}$$

by the given condition (2.1). Conversely, suppose $f \in \Sigma^*(g,\gamma,k,\lambda)$. Then by Lemma 2.2, we have (2.2).

Choosing the values of z on the positive real axis the inequality (2.2) reduces to

$$\operatorname{Re}\left\{\frac{(1-\gamma-2\lambda(ke^{i\theta}+1))\frac{1}{z^2}+\sum\limits_{n=1}^{\infty}[n(1+(n-1)\lambda)(1+ke^{i\theta})+(\gamma+ke^{i\theta})]b_na_nz^{n-1}}{\frac{1}{z^2}+\sum\limits_{n=1}^{\infty}b_na_nz^{n-1}}\right\}\geq 0.$$

Since $\operatorname{Re}(-e^{i\theta})\geq-|e^{i\theta}|=-1$, the above inequality reduces to

$$\operatorname{Re}\left\{\frac{(1-\gamma-2\lambda(k+1))\frac{1}{r^2}+\sum\limits_{n=1}^{\infty}[n(1+k)(1+(n-1)\lambda)+(\gamma+k)b_na_nr^{n-1}}{\frac{1}{r^2}+\sum\limits_{n=1}^{\infty}b_na_nr^{n-1}}\right\}\geq 0.$$

Letting $r \to 1^-$ and by the mean value theorem we have desired inequality (2.1). $\square$

Corollary 2.4. *If $f \in \Sigma^*(g,\gamma,k,\lambda)$ then*

$$a_n \le \frac{(1-\gamma)-2\lambda(k+1)}{[n(k+1)(1+(n-1)\lambda)+(k+\gamma)]\ b_n}. \tag{2.3}$$

By taking $\lambda = 0$, in Theorem 2.3, we get the following corollary.

Corollary 2.5. *Let $f(z) \in \Sigma$ be given by (1.6). Then $f \in \Sigma(\gamma,k)$ if and only if*

$$\sum_{n=1}^{\infty}[n(k+1)+(k+\gamma)]\ b_n\ a_n \le (1-\gamma).$$

Next we obtain the growth theorem for the class $\Sigma^*(g,\gamma,k,\lambda)$.

Theorem 2.6. *If $f \in \Sigma^*(g,\gamma,k,\lambda)$ and $b_n \ge b_1 (n \ge 1)$, then*

$$\frac{1}{r} - \frac{(1-\gamma)-2\lambda(k+1)}{(2k+\gamma+1)\ b_1} r \le |f(z)| \le \frac{1}{r} + \frac{(1-\gamma)-2\lambda(k+1)}{(2k+\gamma+1)\ b_1} r \quad (|z|=r)$$

and

$$\frac{1}{r^2} - \frac{(1-\gamma)-2\lambda(k+1)}{(2k+\gamma+1)\ b_1} \le |f'(z)| \le \frac{1}{r^2} + \frac{(1-\gamma)-2\lambda(k+1)}{(2k+\gamma+1)\ b_1} \quad (|z|=r).$$

The result is sharp for

$$f(z) = \frac{1}{z} + \frac{(1-\gamma)-2\lambda(k+1)}{(2k+\gamma+1)\ b_1} z. \tag{2.4}$$

Proof. Since $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$, we have

$$|f(z)| \le \frac{1}{r} + \sum_{n=1}^{\infty} a_n r^n \le \frac{1}{r} + r\sum_{n=1}^{\infty} a_n. \tag{2.5}$$

Since for $n \ge 1$, $(2k+\gamma+1)\ b_1 \le [n(k+1)+(k+\gamma)]b_n$, using Theorem 2.3, we have

$$\begin{aligned}(2k+\gamma+1)\ b_1 \sum_{n=1}^{\infty} a_n &\le \sum_{n=1}^{\infty}[n(k+1)(1+(n-1)\lambda)+(k+\gamma)]\ a_n\ b_n \\ &\le (1-\gamma)-2\lambda(k+1).\end{aligned}$$

That is,

$$\sum_{n=1}^{\infty} a_n \le \frac{(1-\gamma)-2\lambda(k+1)}{(2k+\gamma+1)\ b_1}.$$

Using the above equation in (2.5), we have

$$|f(z)| \le \frac{1}{r} + \frac{(1-\gamma)-2\lambda(k+1)}{(2k+\gamma+1)\ b_1} r$$

and

$$|f(z)| \ge \frac{1}{r} - \frac{(1-\gamma)-2\lambda(k+1)}{(2k+\gamma+1)\ b_1} r.$$

The result is sharp for $f(z)=\frac{1}{z}+\frac{(1-\gamma)-2\lambda(k+1)}{(2k+\gamma+1)\ b_1}z$. Similarly we have,

$$|f'(z)|\geq \frac{1}{r^2}-\ \frac{(1-\gamma)-2\lambda(k+1)}{(2k+\gamma+1)\ b_1}$$

and

$$|f'(z)|\leq \frac{1}{r^2}+\frac{(1-\gamma)-2\lambda(k+1)}{(2k+\gamma+1)\ b_1}.$$

□

Let the functions $f_j(z)$ $(j=1,2,...,m)$ be given by

$$f_j(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_{n,j}z^n,\quad a_{n,j}\geq 0,\ n\in\mathbb{N}, n\geq 1. \tag{2.6}$$

We state the following closure theorem for the class $\Sigma^*(g,\gamma,k,\lambda)$ without proof.

Theorem 2.7. *Let the function $f_j(z)$ defined by (2.6) be in the class $\Sigma^*(g,\gamma,k,\lambda)$ for every $j=1,2,...,m$. Then the function $f(z)$ defined by*

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n,$$

belongs to the class $\Sigma^*(g,\gamma,k,\lambda)$, where $a_n=\frac{1}{m}\sum_{j=1}^{m}a_{n,j}$, $(n=1,2,..)$.

Theorem 2.8. *(Extreme Points) Let*

$$f_0(z)=\frac{1}{z}\text{ and } f_n(z)=\frac{1}{z}+\frac{(1-\gamma)-2\lambda(k+1)}{[n(k+1)(1+(n-1)\lambda)+(k+\gamma)]\ b_n}z^n,\quad (n\geq 1). \tag{2.7}$$

Then $f\in\Sigma^*(g,\gamma,k,\lambda)$, if and only if it can be represented in the form

$$f(z)=\sum_{n=0}^{\infty}\mu_nf_n(z),\ \ (\mu_n\geq 0,\sum_{n=0}^{\infty}\mu_n=1). \tag{2.8}$$

Proof. Suppose $f(z)$ can be expressed as in (2.8). Then

$$\begin{aligned} f(z)&=\sum_{n=0}^{\infty}\mu_nf_n(z)\\ &=\mu_0f_0(z)+\sum_{n=1}^{\infty}\mu_nf_n(z)\\ &=\frac{1}{z}+\sum_{n=1}^{\infty}\mu_n\frac{(1-\gamma)-2\lambda(k+1)}{[n(1+k)(1+(n-1)\lambda)+(k+\gamma)]\ b_n}z^n. \end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{n=1}^{\infty}\mu_n &\frac{(1-\gamma)-2\lambda(k+1)}{[n(1+k)(1+(n-1)\lambda)+(k+\gamma)]\ b_n}\\ &\times\frac{[n(1+k)(1+(n-1)\lambda)+(k+\gamma)]\ b_n}{(1-\gamma)-2\lambda(k+1)}z^n\\ =\sum_{n=1}^{\infty}\mu_n-1&=1-\mu_0\le 1.\end{aligned}$$

So by Theorem 2.3, $f\in\Sigma^*(g,\gamma,k,\lambda)$.

Conversely, we suppose $f\in\Sigma^*(g,\gamma,k,\lambda)$. Since

$$a_n\le\frac{(1-\gamma)-2\lambda(k+1)}{[n(1+k)(1+(n-1)\lambda)+(\gamma+k)]\ b_n},\quad n\ge 1.$$

We set,

$$\mu_n=\frac{[n(k+1)(1+(n-1)\lambda)+(k+\gamma)]\ b_n}{(1-\gamma)-2\lambda(k+1)}a_n,\quad n\ge 1$$

and $\mu_0=1-\sum\limits_{n=1}^{\infty}\mu_n$. Then we have,

$$\begin{aligned}f(z)&=\sum_{n=0}^{\infty}\mu_n f_n(z)\\ &=\mu_0 f_0(z)+\sum_{n=1}^{\infty}\mu_n f_n(z).\end{aligned}$$

Hence the results follows. □

3. Radii of meromorphically starlikeness and meromorphically convexity

Theorem 3.1. *Let $f\in\Sigma^*(g,\gamma,k,\lambda)$. Then f is meromorphically starlike of order $\delta(0\le\delta<1)$ in the disc $|z|<r_1$, where*

$$r_1=\inf_n\left[\left(\frac{1-\delta}{n+2-\delta}\right)\frac{[n(k+1)(1+(n-1)\lambda)+(k+\gamma)]\ b_n}{(1-\gamma)-2\lambda(k+1)}\right]^{\frac{1}{n+1}}\qquad(n\ge 1),$$

The result is sharp for the extremal function $f(z)$ given by (2.7).

Proof. The function $f\in\Sigma^*(g,\gamma,k,\lambda)$ of the form (1.6) is meromorphically starlike of order δ in the disc $|z|<r_1$, if and only if it satisfies the condition

$$\left|\frac{zf'(z)}{f(z)}+1\right|<1-\delta. \tag{3.1}$$

Since

$$\left|\frac{zf'(z)}{f(z)}+1\right|\le\left|\frac{\sum_{n=1}^{\infty}(n+1)a_nz^{n+1}}{1+\sum_{n=1}^{\infty}a_nz^{n+1}}\right|\le\frac{\sum_{n=1}^{\infty}(n+1)|a_n||z|^{n+1}}{1-\sum_{n=1}^{\infty}|a_n||z|^{n+1}}.$$

The above expression is less than $1-\delta$ if

$$\sum_{n=2}^{\infty} \frac{n+2-\delta}{1-\delta} |a_n|\ |z|^{n-1} < 1.$$

Using the fact, that $f \in \Sigma^*(g, \gamma, k, \lambda)$ if and only if

$$\sum_{n=2}^{\infty} \frac{[n(k+1)(1+(n-1)\lambda)+(k+\gamma)]\ b_n}{(1-\gamma)-2\lambda(k+1)} a_n \ < 1.$$

We say (3.1) is true if

$$\frac{n+2-\delta}{1-\delta} |z|^{n+1} < \frac{[n(k+1)(1+(n-1)\lambda)+(k+\gamma)]\ b_n}{(1-\gamma)-2\lambda(k+1)}.$$

Or, equivalently,

$$|z|^{n+1} < \frac{(1-\delta)}{(n+2-\delta)} \frac{[n(k+1)(1+(n-1)\lambda)+(k+\gamma)]\ b_n}{(1-\gamma)-2\lambda(k+1)}$$

which yields the starlikeness of the family. □

Theorem 3.2. *Let $f \in \Sigma^*(g, \gamma, k, \lambda)$. Then f is meromorphically convex of order δ $(0 \leq \delta < 1)$ in the unit disc $|z| < r_2$, where*

$$r_2 = \inf_n \left[\left(\frac{1-\delta}{n(n+2-\delta)} \right) \frac{[n(k+1)(1+(n-1)\lambda)+(k+\gamma)]\ b_n}{(1-\gamma)-2\lambda(k+1)} \right]^{\frac{1}{n+1}} \qquad (n \geq 1),$$

The result is sharp for the extremal function $f(z)$ given by (2.4).

Proof. The proof is analogous to that of Theorem 3.1, and we omit the details. □

4. Partial sums

Let $f \in \Sigma$ be a function of the form (1.6). Motivated by Silverman [19] and Silvia [20] see also [4], we define the partial sums f_m defined by

$$f_m(z) = \frac{1}{z} + \sum_{n=1}^{m} a_n z^n \quad (m \in \mathbb{N}). \tag{4.1}$$

In this section, we consider partial sums of functions from the class $\Sigma^*(g, \gamma, k, \lambda)$ and obtain sharp lower bounds for the real part of the ratios of f to f_m and f' to f'_m.

Theorem 4.1. *Let $f \in \Sigma^*(g, \gamma, k, \lambda)$ be given by (1.6) and define the partial sums $f_1(z)$ and $f_m(z)$, by*

$$f_1(z) = \frac{1}{z} \ \text{ and } \ f_m(z) = \frac{1}{z} + \sum_{n=1}^{m} |a_n| z^n, \ (m \in \mathbb{N}/\{1\}). \tag{4.2}$$

Suppose also that

$$\sum_{n=1}^{\infty} d_n |a_n| \leq 1,$$

where

$$d_n \geq \begin{cases} 1 & \text{for } n = 1, 2, 3, \ldots, m \\ \frac{[n(1+k)(1+(n-1)\lambda)+(\gamma+k)]\ b_n}{(1-\gamma)-2\lambda(k+1)} & \text{for } n = m+1, m+2, m+3\cdots \end{cases}. \tag{4.3}$$

Then $f \in \Sigma^*(g, \gamma, k, \lambda)$. *Furthermore,*

$$Re\left(\frac{f(z)}{f_m(z)}\right) > 1 - \frac{1}{d_{m+1}} \tag{4.4}$$

and

$$Re\left(\frac{f_m(z)}{f(z)}\right) > \frac{d_{m+1}}{1 + d_{m+1}}. \tag{4.5}$$

Proof. For the coefficients d_n given by (4.3) it is not difficult to verify that

$$d_{n+1} > d_n > 1. \tag{4.6}$$

Therefore we have

$$\sum_{n=1}^{m} |a_n| + d_{m+1} \sum_{n=m+1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} d_n |a_n| \leq 1 \tag{4.7}$$

by using the hypothesis (4.3). By setting

$$\begin{aligned} g_1(z) &= d_{m+1}\left(\frac{f(z)}{f_m(z)} - \left(1 - \frac{1}{d_{m+1}}\right)\right) \\ &= 1 + \frac{d_{m+1} \sum\limits_{n=m+1}^{\infty} a_n z^{n-1}}{1 + \sum\limits_{n=1}^{m} a_n z^{n-1}}, \end{aligned}$$

then it suffices to show that

$$\text{Re}\ (g_1(z)) \geq 0\ \ (z \in \mathbb{U}^*)$$

or,

$$\left|\frac{g_1(z) - 1}{g_1(z) + 1}\right| \leq 1\ \ (z \in \mathbb{U}^*)$$

and applying (4.7), we find that

$$\begin{aligned} \left|\frac{g_1(z) - 1}{g_1(z) + 1}\right| &\leq \frac{d_{m+1} \sum\limits_{n=m+1}^{\infty} |a_n|}{2 - 2\sum\limits_{n=1}^{m} |a_n| - d_{m+1} \sum\limits_{n=m+1}^{\infty} |a_n|} \\ &\leq 1,\ \ z \in \mathbb{U}^*, \end{aligned}$$

which readily yields the assertion (4.4) of Theorem 4.1. In order to see that

$$f(z) = \frac{1}{z} + \frac{z^{m+1}}{d_{m+1}} \tag{4.8}$$

gives sharp result, we observe that for $z = re^{i\pi/m}$ that $\frac{f(z)}{f_m(z)} = 1 - \frac{r^{m+2}}{d_{m+1}} \to 1 - \frac{1}{d_{m+1}}$ as $r \to 1^-$.

Similarly, if we take

$$g_2(z) \quad = \quad (1+d_{m+1})\left(\frac{f_m(z)}{f(z)} - \frac{d_{m+1}}{1+d_{m+1}}\right)$$

and making use of (4.7), we deduce that

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right| \leq \frac{(1+d_{m+1})\sum\limits_{n=m+1}^{\infty}|a_n|}{2-2\sum\limits_{n=1}^{m}|a_n| - (1-d_{m+1})\sum\limits_{n=m+1}^{\infty}|a_n|}$$

which leads us immediately to the assertion (4.5) of Theorem 4.1. The bound in (4.5) is sharp for each $m \in \mathbb{N}$ with the extremal function $f(z)$ given by (4.8). □

Theorem 4.2. *If $f(z)$ of the form (1.6) satisfies the condition (2.1). Then*

$$Re\left(\frac{f'(z)}{f'_m(z)}\right) \geq 1 - \frac{m+1}{d_{m+1}}$$

and

$$Re\left(\frac{f'_m(z)}{f'(z)}\right) \geq \frac{d_{m+1}}{m+1+d_{m+1}},$$

where

$$d_n \geq \begin{cases} n & \text{for } n = 2,3,\ldots,m \\ \frac{n[n(1+k)(1+(n-1)\lambda)+(\gamma+k)]\ b_n}{(1-\gamma)-2\lambda(k+1)} & \text{for } n = m+1, m+2, m+3\cdots \end{cases}.$$

The bounds are sharp, with the extremal function $f(z)$ of the form (2.4).

Proof. The proof is analogous to that of Theorem 4.1, and we omit the details. □

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