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Subordination results for a class of Bazilević functions with respect to symmetric points

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Abstract. In this paper, using the principle of subordination we introduce the class of Bazilević functions with respect to k-symmetric points. Several subordination results are obtained for this classes of functions involving a certain family of linear operators.

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1. Introduction, definitions and preliminaries

Let $\mathcal{H}$ be the class of functions analytic in the open unit disc $\mathcal{U} = \{z :\mid z \mid< 1\}$. Let $\mathcal{H}(a, n)$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ \ldots .$$

Let

$$\mathcal{A}_n = \{f \in \mathcal{H},\ f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \ldots\}$$

and let $\mathcal{A} = \mathcal{A}_1$.

Let $\mathcal{S}$ denote the class of functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$. Also let $\mathcal{P}$ to denote the class of functions of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \quad (z \in \mathcal{U}),$$

which satisfy the condition $Re(p(z)) > 0$.

We denote by $\mathcal{S}^*$, $\mathcal{C}$, $\mathcal{K}$ and $\mathcal{C}^*$ the familiar subclasses of $\mathcal{A}$ consisting of functions which are respectively starlike, convex, close-to-convex and quasi-convex in $\mathcal{U}$. One of our favorite reference of the field is [4] which covers most of the topics in a lucid and economical style.

Let the functions $f(z)$ and $g(z)$ be members of $\mathcal{A}$. we say that the function g is subordinate to f (or f is superordinate to g), written $g \prec f$, if there exists a function

w analytic in $\mathcal{U}$, with $w(0) = 0$ and $|w(z)| < 1$ and such that $g(z) = f(w(z))$. In particular, if g is univalent, then $f \prec g$ if $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$. Using the concept of subordination of analytic functions, Ma and Minda[6] introduced the class $\mathcal{S}^*(\phi)$ of functions in $\mathcal{A}$ satisfying $\frac{zf'(z)}{f(z)} \prec \phi$ where $\phi \in \mathcal{P}$ with $\phi'(0) > 0$ maps $\mathcal{U}$ onto a region starlike with respect to 1 and symmetric with respect to real axis.

For a fixed non zero positive integer k and $f_k(z)$ defined by the following equality

$$f_k(z) = \frac{1}{k}\sum_{\nu=0}^{k-1} \varepsilon_k^{-\nu} f(\varepsilon_k^{\nu} z) \qquad \left(\varepsilon_k = \exp\left(\frac{2\pi i}{k}\right)\right), \tag{1.1}$$

a function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{S}_s^{(k)}(\phi)$ if and only if it satisfies the condition

$$\frac{zf'(z)}{f_k(z)} \prec \phi(z) \quad (z \in \mathcal{U}), \tag{1.2}$$

where $\phi \in \mathcal{P}$, the class of functions with positive real part.

Similarly, a function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{C}_s^{(k)}(\phi)$ if and only if it satisfies the condition

$$\frac{(zf'(z))'}{f_k'(z)} \prec \phi(z) \quad (z \in \mathcal{U}), \tag{1.3}$$

where $\phi \in \mathcal{P}$, $k \geq 1$ is a fixed positive integer and $f_k(z)$ is defined by equality (1.1). The classes $\mathcal{S}_s^{(k)}(\phi)$ and $\mathcal{C}_s^{(k)}(\phi)$ were introduced and studied by Wang et. al. [11]. Motivated by the class of univalent Bazilević functions, we introduce the following: For $0 \leq \gamma < \infty$, a function $f(z) \in \mathcal{A}$ is said to be in $\mathcal{B}_k(\gamma;\, \phi)$ if and only if it satisfies the condition

$$\frac{zf'(z)}{[f_k(z)]^{1-\gamma}\,[g_k(z)]^{\gamma}} \prec \phi(z), \quad (z \in \mathcal{U};\, g \in \mathcal{S}_s^{(k)}(\phi)) \tag{1.4}$$

where $\phi \in \mathcal{P}$ and $g_k(z) \neq 0$ for all $z \in \mathcal{U}$ is defined as in (1.1).

For complex parameters $\alpha_1, \ldots, \alpha_q$ and $\beta_1, \ldots, \beta_s$ ($\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-;\, \mathbb{Z}_0^- = 0, -1, -2, \ldots;\, j = 1, \ldots, s$), we define the generalized hypergeometric function ${}_qF_s(\alpha_1, \ldots, \alpha_q;\, \beta_1, \ldots, \beta_s; z)$ by

$${}_qF_s(\alpha_1, \alpha_2, \ldots, \alpha_q;\, \beta_1, \beta_2, \ldots, \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ \ldots\ (\alpha_q)_n}{(\beta_1)_n \ \ldots\ (\beta_s)_n} \frac{z^n}{n!}$$

$$(q \leq s+1;\, q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\};\, z \in \mathcal{U}),$$

where $\mathbb{N}$ denotes the set of positive integers and $(x)_k$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = \begin{cases} 1 & if\, k = 0 \\ x(x+1)(x+2)\ \ldots\ (x+k-1) & if\, k \in \mathbb{N} = \{1, 2, , \ldots\}. \end{cases}$$

Corresponding to a function $\mathcal{G}_{q,\,s}(\alpha_1,\, \beta_1;\, z)$ defined by

$$\mathcal{G}_{q,\,s}(\alpha_1,\, \beta_1;\, z) := z\, {}_qF_s(\alpha_1, \alpha_2, \ldots, \alpha_q;\, \beta_1, \beta_2, \ldots, \beta_s; z), \tag{1.5}$$

Selvaraj and Karthikeyan in [9] recently introduced the following operator $D_{\lambda,\,s}^{m,\,q}(\alpha_1,\,\beta_1)f\,:\,\mathcal{A}\longrightarrow\mathcal{A}$ by

$$D_{\lambda,\,s}^{0,\,q}(\alpha_1,\,\beta_1)f(z)=f(z)*\mathcal{G}_{q,\,s}(\alpha_1,\,\beta_1;\,z)$$

$$D_{\lambda,\,s}^{1,\,q}(\alpha_1,\,\beta_1)f(z)=(1-\lambda)(f(z)*\mathcal{G}_{q,\,s}(\alpha_1,\,\beta_1;\,z))+\lambda\,z(f(z)*\mathcal{G}_{q,\,s}(\alpha_1,\,\beta_1;\,z))' \quad (1.6)$$

$$D_{\lambda,\,s}^{m,\,q}(\alpha_1,\,\beta_1)f(z)=D_{\lambda,\,s}^{1,\,q}(D_{\lambda,\,s}^{m-1,\,q}(\alpha_1,\,\beta_1)f(z)) \quad (1.7)$$

If f of the form $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$, then from (1.6) and (1.7) we may easily deduce that

$$D_{\lambda,\,s}^{m,\,q}(\alpha_1,\,\beta_1)f(z)=z+\sum_{n=2}^{\infty}\left[1+(n-1)\lambda\right]^m\frac{(\alpha_1)_{n-1}\,\dots\,(\alpha_q)_{n-1}}{(\beta_1)_{n-1}\,\dots\,(\beta_s)_{n-1}}\,\frac{a_nz^n}{(n-1)!} \quad (1.8)$$

where $m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ and $\lambda\geq 0$. We remark that, for choice of the parameter $m=0$, the operator $D_{\lambda,\,s}^{m,\,q}(\alpha_1,\,\beta_1)f(z)$ reduces to the well-known Dziok- Srivastava operator [1] and for $q=2$, $s=1$; $\alpha_1=\beta_1$, $\alpha_2=1$ and $\lambda=1$, we get the operator introduced by G. Ş. Sălăgean [8]. Also many (well known and new) integral and differential operators can be obtained by specializing the parameters.

Throughout this paper we assume that

$$m,\,q,\,s\in N_0,\quad \varepsilon_k=\exp\left(\frac{2\pi i}{k}\right)$$

and

$$f_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)=\frac{1}{k}\sum_{\nu=0}^{k-1}\varepsilon_k^{-\nu}D_{\lambda,\,s}^{m,\,q}(\alpha_1,\,\beta_1)f(\varepsilon_k^{\nu}z). \quad (1.9)$$

Clearly, for $k=1$, we have

$$f_{1,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)=D_{\lambda,\,s}^{m,\,q}(\alpha_1,\,\beta_1)f(z).$$

Lemma 1.1. [3]*Let h be convex in $\mathcal{U}$, with $h(0)=a$, $\delta\neq 0$ and $Re\,\delta\geq 0$. If $p\in\mathcal{H}(a,\,n)$ and*

$$p(z)+\frac{zp'(z)}{\delta}\prec h(z),$$

then

$$p(z)\prec q(z)\prec h(z),$$

where

$$q(z)=\frac{\delta}{n\,z^{\delta/n}}\int_0^z h(t)\,t^{(\delta/n)-1}dt.$$

The function q is convex and is the best $(a,\,n)$-dominant.

Lemma 1.2. [7]*Let h be starlike in $\mathcal{U}$, with $h(0)=0$. If $p\in\mathcal{H}(a,\,n)$ satisfies*

$$zp'(z)\prec h(z),$$

then

$$p(z)\prec q(z)=a+n^{-1}\int_0^z h(t)\,t^{-1}\,dt.$$

The function q is convex and is the best $(a,\,n)$-dominant.

Remark 1.3. The Lemma 1.1 for the case of $n=1$ was earlier given by Suffridge [10].

2. Main results

We begin with the following

Theorem 2.1. *Let* $f,\, g \in \mathcal{A}$ *with* $f(z),\ f^{'}(z),\ f_k(z) \neq 0$ *and* $g_k(z) \neq 0$ *for all* $z \in \mathcal{U} \setminus \{0\}$. *Also let* h *be convex in* $\mathcal{U}$ *with* $h(0) = 1$ *and* $Re\, h(z) > 0$. *Further suppose that* $g \in \mathcal{S}_s^{(k)}(\phi)$ *and*

$$\left(\frac{z(D_{\lambda,\, s}^{m,\, q}(\alpha_1,\, \beta_1)f(z))^{'}}{[f_{k,\,\lambda}^{q,\, s}(\alpha_1,\, \beta_1;\, m;\, z)]^{1-\gamma}\,[g_{k,\,\lambda}^{q,\, s}(\alpha_1,\, \beta_1;\, m;\, z)]^{\gamma}}\right)^2 \left[3 + 2\left\{\frac{z(D_{\lambda,\, s}^{m,\, q}(\alpha_1,\, \beta_1)f(z))^{''}}{(D_{\lambda,\, s}^{m,\, q}(\alpha_1,\, \beta_1)f(z))^{'}} - (1-\gamma)\frac{z\left(f_{k,\,\lambda}^{q,\, s}(\alpha_1,\, \beta_1;\, m;\, z)\right)^{'}}{f_{k,\,\lambda}^{q,\, s}(\alpha_1,\, \beta_1;\, m;\, z)} - \gamma\frac{z\left(g_{k,\,\lambda}^{q,\, s}(\alpha_1,\, \beta_1;\, m;\, z)\right)^{'}}{g_{k,\,\lambda}^{q,\, s}(\alpha_1,\, \beta_1;\, m;\, z)}\right\}\right] \prec h(z). \tag{2.1}$$

Then

$$\frac{z(D_{\lambda,\, s}^{m,\, q}(\alpha_1,\, \beta_1)f(z))^{'}}{[f_{k,\,\lambda}^{q,\, s}(\alpha_1,\, \beta_1;\, m;\, z)]^{1-\gamma}\,[g_{k,\,\lambda}^{q,\, s}(\alpha_1,\, \beta_1;\, m;\, z)]^{\gamma}} \prec \phi(z) = \sqrt{Q(z)} \tag{2.2}$$

where

$$Q(z) = \frac{1}{z}\int_0^z h(t)\, dt$$

and ϕ *is convex and is the best dominant.*

Proof. Let

$$p(z) = \frac{z(D_{\lambda,\, s}^{m,\, q}(\alpha_1,\, \beta_1)f(z))^{'}}{[f_{k,\,\lambda}^{q,\, s}(\alpha_1,\, \beta_1;\, m;\, z)]^{1-\gamma}\,[g_{k,\,\lambda}^{q,\, s}(\alpha_1,\, \beta_1;\, m;\, z)]^{\gamma}} \quad (z \in \mathcal{U};\ \gamma \geq 0),$$

then $p(z) \in \mathcal{H}(1,\, 1)$ with $p(z) \neq 0$.

Since h is convex, it can be easily seen that Q is convex and univalent in $\mathcal{U}$. If we make the change of the variables $P(z) = p^2(z)$, then $P(z) \in \mathcal{H}(1,\, 1)$ with $P(z) \neq 0$ in $\mathcal{U}$.

By a straight forward computation, we have

$$\frac{zP^{'}(z)}{P(z)} = 2\left[1 + \frac{z(D_{\lambda,\, s}^{m,\, q}(\alpha_1,\, \beta_1)f(z))^{''}}{(D_{\lambda,\, s}^{m,\, q}(\alpha_1,\, \beta_1)f(z))^{'}} - (1-\gamma)\frac{z\left(f_{k,\,\lambda}^{q,\, s}(\alpha_1,\, \beta_1;\, m;\, z)\right)^{'}}{f_{k,\,\lambda}^{q,\, s}(\alpha_1,\, \beta_1;\, m;\, z)} - \gamma\frac{z\left(g_{k,\,\lambda}^{q,\, s}(\alpha_1,\, \beta_1;\, m;\, z)\right)^{'}}{g_{k,\,\lambda}^{q,\, s}(\alpha_1,\, \beta_1;\, m;\, z)}\right].$$

Thus by (2.1), we have

$$P(z) + zP^{'}(z) \prec h(z) \quad (z \in \mathcal{U}). \tag{2.3}$$

Now by Lemma 1.1, we deduce that

$$P(z) \prec Q(z) \prec h(z).$$

Since $Re\, h(z) > 0$ and $Q(z) \prec h(z)$ we also have $Re\, Q(z) > 0$. Hence the univalence of Q implies the univalence of $\sqrt{Q(z)}$, $p^2(z) \prec Q(z)$ implies that $p(z) \prec \sqrt{Q(z)}$ and the proof is complete. $\square$

Corollary 2.2. *Let $f, g \in \mathcal{A}$ with $f^{'}(z)$, $f_k(z)$ and $g_k(z) \neq 0$ for all $z \in \mathcal{U} \setminus \{0\}$. If $g \in \mathcal{S}_s^{(k)}$ and $Re\,[\Omega(z)] > \eta \quad (0 \leq \eta < 1)$, where*

$$\Omega(z) = \left(\frac{z(D_{\lambda,\, s}^{m,\, q}(\alpha_1,\, \beta_1) f(z))^{'}}{[f_{k,\, \lambda}^{q,\, s}(\alpha_1,\, \beta_1;\, m;\, z)]^{1-\gamma}\, [g_{k,\, \lambda}^{q,\, s}(\alpha_1,\, \beta_1;\, m;\, z)]^{\gamma}} \right)^2$$

$$\left[3 + 2 \left\{ \frac{z(D_{\lambda,\, s}^{m,\, q}(\alpha_1,\, \beta_1) f(z))^{''}}{(D_{\lambda,\, s}^{m,\, q}(\alpha_1,\, \beta_1) f(z))^{'}} - (1-\gamma) \frac{z \left(f_{k,\, \lambda}^{q,\, s}(\alpha_1,\, \beta_1;\, m;\, z) \right)^{'}}{f_{k,\, \lambda}^{q,\, s}(\alpha_1,\, \beta_1;\, m;\, z)} \right.\right.$$
$$\left.\left. -\gamma \frac{z \left(g_{k,\, \lambda}^{q,\, s}(\alpha_1,\, \beta_1;\, m;\, z) \right)^{'}}{g_{k,\, \lambda}^{q,\, s}(\alpha_1,\, \beta_1;\, m;\, z)} \right\} \right],$$

then

$$Re \left[\frac{z(D_{\lambda,\, s}^{m,\, q}(\alpha_1,\, \beta_1) f(z))^{'}}{[f_{k,\, \lambda}^{q,\, s}(\alpha_1,\, \beta_1;\, m;\, z)]^{1-\gamma}\, [g_{k,\, \lambda}^{q,\, s}(\alpha_1,\, \beta_1;\, m;\, z)]^{\gamma}} \right] > \lambda(\eta),$$

where $\lambda(\eta) = [2(1-\eta) \cdot \log 2 + (2\eta - 1)]^{\frac{1}{2}}$. This result is sharp.

Proof. If we let $h(z) = \dfrac{1 + (2\eta - 1)z}{1+z}$ $0 \leq \eta < 1$ in Theorem 2.1.
It follows that $Q(z)$ is convex and $Re\, Q(z) > 0$. Therefore

$$\min_{|z| \leq 1} Re \sqrt{Q(z)} = \sqrt{Q(1)} = [2(1-\eta) \cdot \log 2 + (2\eta - 1)]^{\frac{1}{2}}.$$

Hence the proof of the Corollary.

If we let $m = \gamma = 0$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1$ and $\alpha_2 = 1$ in the Corollary 2.2, then we have the following

Corollary 2.3. *Let $f \in \mathcal{A}$ with $f^{'}(z)$ and $f_k(z) \neq 0$ for all $z \in \mathcal{U} \setminus \{0\}$. If*

$$Re \left\{ \left(\frac{z f^{'}(z)}{f_k(z)} \right)^2 \left[3 + \frac{2\, z f^{''}(z)}{f^{'}(z)} - \frac{2\, z f_k^{'}(z)}{f_k(z)} \right] \right\} > \eta,$$

then

$$Re \frac{z f^{'}(z)}{f_k(z)} > \lambda(\eta),$$

where $\lambda(\eta) = [2(1-\eta) \cdot \log 2 + (2\eta - 1)]^{\frac{1}{2}}$. This result is sharp.

If we let $\gamma = 1$, $m = 0$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1$ and $\alpha_2 = 1$ in the Corollary 2.2, then we have the following

Corollary 2.4. *Let $f,\, g \in \mathcal{A}$ with $f'(z)$ and $g_k(z) \neq 0$ for all $z \in \mathcal{U} \setminus \{0\}$. If $g \in \mathcal{S}_s^{(k)}$ and*

$$Re \left\{ \left(\frac{zf'(z)}{g_k(z)} \right)^2 \left[3 + \frac{2\,zf''(z)}{f'(z)} - \frac{2\,zg_k'(z)}{g_k(z)} \right] \right\} > \eta,$$

then

$$Re\, \frac{zf'(z)}{g_k(z)} > \lambda(\eta),$$

where $\lambda(\eta) = [2(1-\eta)\cdot \log 2 + (2\eta - 1)]^{\frac{1}{2}}$. This result is sharp.

Remark 2.5. If we let $k = 1$ in Corollary 2.4 and in Corollary 2.3, then we have the condition for usual starlikeness and close-to-convex respectively.

Theorem 2.6. *Let $f,\, g \in \mathcal{A}$ with $f(z)$, $f'(z)$ and $g_k(z) \neq 0$ for all $z \in \mathcal{U} \setminus \{0\}$. Further suppose h is starlike with $h(0) = 0$ in the unit disk $\mathcal{U}$, $g \in \mathcal{S}_s^{(k)}(\phi)$ and*

$$1 + \frac{z(D_{\lambda,\,s}^{m,\,q}(\alpha_1,\,\beta_1)f(z))''}{(D_{\lambda,\,s}^{m,\,q}(\alpha_1,\,\beta_1)f(z))'} - (1-\gamma)\frac{z\left(f_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)\right)'}{f_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)} - \gamma\frac{z\left(g_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)\right)'}{g_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)} \prec h(z) \quad (z \in \mathcal{U};\, \gamma \geq 0). \tag{2.4}$$

Then

$$\frac{z(D_{\lambda,\,s}^{m,\,q}(\alpha_1,\,\beta_1)f(z))'}{[f_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)]^{1-\gamma}\,[g_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)]^{\gamma}} \prec \phi(z) = \exp\left(\int_0^z \frac{h(t)}{t}\,dt\right) \tag{2.5}$$

where ϕ is convex and is the best dominant.

Proof. Let

$$\Psi(z) = 1 + \frac{z(D_{\lambda,\,s}^{m,\,q}(\alpha_1,\,\beta_1)f(z))''}{(D_{\lambda,\,s}^{m,\,q}(\alpha_1,\,\beta_1)f(z))'} - (1-\gamma)\frac{z\left(f_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)\right)'}{f_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)} - \gamma\frac{z\left(g_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)\right)'}{g_{k,\,\lambda}^{q,\,s}(\alpha_1,\,\beta_1;\,m;\,z)}. \tag{2.6}$$

Since $f,\, g \in \mathcal{A}$ with $f'(z)$, $f_k(z)$ and $g_k(z) \neq 0$ for all $z \in \mathcal{U} \setminus \{0\}$, therefore

$$\Psi(z) = z + b_1 z + b_2 z^2 + \ldots .$$

Obviously Ψ is analytic in $\mathcal{U}$. Thus we have

$$\Psi(z) = h(z) \quad (z \in \mathcal{U}).$$

Now by Lemma, we deduce that

$$\int_0^z \frac{\Psi(t)}{t}\,dt \prec \int_0^z \frac{h(t)}{t}\,dt. \tag{2.7}$$

Hence using

$$\frac{\Psi(z)}{z} = \frac{d}{dz}\left[\log\left\{\frac{z(D^{m,\,q}_{\lambda,\,s}(\alpha_1,\,\beta_1)f(z))'}{[f^{q,\,s}_{k,\,\lambda}(\alpha_1,\,\beta_1;\,m;\,z)]^{1-\gamma}\,[g^{q,\,s}_{k,\,\lambda}(\alpha_1,\,\beta_1;\,m;\,z)]^{\gamma}}\right\}\right]$$

in (2.7), we arrive at the desired result. □

If we let $m = 0$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1$ and $\alpha_2 = 1$ in the Theorem 2.6, then we have the following

Corollary 2.7. *Let $f,\, g \in \mathcal{A}$ with $f(z)$, $f'(z)$ and $g_k(z) \neq 0$ for all $z \in \mathcal{U}\setminus\{0\}$. Further suppose h is starlike with $h(0) = 0$ in the unit disk $\mathcal{U}$ and*

$$1 + \frac{zf''(z)}{f'(z)} - (1-\gamma)\frac{zf_k'(z)}{f_k(z)} - \gamma\frac{zg_k'(z)}{g_k(z)} \prec h(z) \quad (z \in \mathcal{U};\, \gamma \geq 0).$$

Then

$$\frac{zf'(z)}{[f_k(z)]^{1-\gamma}\,[g_k(z)]^{\gamma}} \prec \phi(z) = \exp\left(\int_0^z \frac{h(t)}{t}\,dt\right)$$

where ϕ is convex and is the best dominant.

For $k = 1$ in the Corollary 2.7, we get result obtained by Goyal and Goswami in [2].

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