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On saturated triples associated to some block algebras of finite groups

Constantin-Cosmin Todea

Abstract. Saturated triples are recently defined in [3]. Block algebras of finite groups give an important example of such saturated triples. In this short article we prove that the principal block and the group algebra viewed as an algebra acted by the group of automorphisms of the finite group provides a new example of saturated triples.

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1. Preliminaries

We follow [4] to recall definitions and basic properties of block algebras of finite groups. Let G be a finite group and let k be an algebraically closed field of characteristic p such that p divides the order of G. A *block algebra* of kG is an indecomposable factor B of kG as an algebra. The block algebras are in bijection with the primitive idempotents of the center $Z(kG)$. We denote by $\mathrm{Bl}(kG)$ the finite set of block algebras (i.e. primitive idempotents in $Z(kG)$). If B is a block algebra we have $B = bkG$, where b is the corresponding primitive idempotent from $Z(kG)$.

kG is a p-permutation G-algebra, where G acts by conjugation and then $\mathrm{Bl}(kG)$ are actually the primitive idempotents of $(kG)^G = Z(kG)$. If G acts on a set X we denote by $Orb_G(X)$ its orbits and if $C \in Orb_G(X)$ we denote by C^+ the sum of all elements in the orbit C. If we use a set of indices I for the orbits we usually mean an arbitrary family of orbits, if not, it means that we consider all the orbits. We follow [2, 2.1] for results and notations regarding permutation algebras. By [2, Lemma 2.2] the set $\{C^+ : C \in Orb_G(G)\}$ is a k-basis of $(kG)^G$, where G acts by conjugation on G. We denote by N the element $\sum_{g\in G} g$.

The augmentation map $\varepsilon : kG \to k$ is the surjective homomorphism of k-algebras defined by $\varepsilon(g) = 1_k$ for any $g \in G$, that is, for an element $\sum_{x\in G} \alpha_x x \in kG$ we have

$$\varepsilon(\sum_{x\in G} \alpha_x x) = \sum_{x\in G} \alpha_x.$$

$ker(\varepsilon)$ is a maximal ideal of kG, the *augmentation ideal* and is generated as a k-vector space by $\{g-1 : g \in G\}$. Let ε^G be the restriction of ε to $(kG)^G$ (as a map $\varepsilon^G : Z(kG) \to k$) and let $\{C_i\}_{i\in I} \subseteq Orb_G(G)$. Then it is easy to see that

$$\varepsilon^G(\sum_{i\in I} \alpha_i C_i^+) = \sum_{i\in I} \mid C_i \mid \alpha_i.$$

Proposition 1.1. *With the above notations we have:*
(1) ε^G *is a surjective homomorphism.*
(2) $ker(\varepsilon^G) \neq 0$ *and* $ker(\varepsilon^G)$ *is a maximal ideal of* $Z(kG)$.

Proof. (1) Obviously ε^G is a k-algebra homomorphism. We only prove that is surjective. Let $\alpha \in k$. Then is easy to see that

$$a = (\alpha + 1_k)1_G + \sum_{C\in Orb_G(G)\setminus\{1_G\}} C^+ \in (kG)^G.$$

We have that

$$\varepsilon^G(a) = (\alpha + 1_k)1_k + \sum_{C\in Orb_G(G)\setminus\{1_G\}} \mid C \mid 1_k = \alpha + 1_k + \mid G \mid 1_k - 1_k = \alpha.$$

(2) Since $\varepsilon^G(N) = \mid G \mid = 0$ then $N \in ker(\varepsilon^G)$. □

2. The action of $\mathrm{Aut}(G)$ on kG

We consider in this section an action of $\mathrm{Aut}(G)$ on kG, which we describe in the following lines. If $f \in \mathrm{Aut}(G)$ then there is $\overline{f} \in \mathrm{Aut}_k(kG)$, where $\mathrm{Aut}_k(kG)$ represents the group of all k-algebra automorphisms of kG. For $\sum_{x\in G} \alpha_x x \in kG$ we have that $\overline{f}$ is defined by

$$\overline{f}(\sum_{x\in G} \alpha_x x) = \sum_{x\in G} \alpha_x f(x).$$

Now kG becomes an $\mathrm{Aut}(G)$-algebra where $f \in \mathrm{Aut}(G)$ acts on $a \in kG$ by ${}^f a = \overline{f}(a)$. By [2] we have that $(kG)^{Aut(G)}$ has as k-basis the set

$$\{C^+ \mid C \in Orb_{\mathrm{Aut}(G)} G\}.$$

As above, let $\varepsilon^{Aut(G)}$ be the restriction of ε to $(kG)^{\mathrm{Aut}(G)}$, that is the map

$$\varepsilon^{\mathrm{Aut}(G)} : (kG)^{\mathrm{Aut}(G)} \to k.$$

Proposition 2.1. *With the above notations we have:*
(1) $\varepsilon^{\mathrm{Aut}(G)}$ *is a surjective homomorphism.*
(2) $ker(\varepsilon^{\mathrm{Aut}(G)}) \neq 0$ *and* $ker(\varepsilon^{\mathrm{Aut}(G)})$ *is a maximal ideal of* $(kG)^{\mathrm{Aut}(G)}$.

Proof. (1) We have the same proof as in Proposition 1.1, using the element

$$a' = (\alpha + 1_k)1_G + \sum_{C\in Orb_{\mathrm{Aut}(G)}(G)\setminus\{1_G\}} C^+ \in (kG)^{\mathrm{Aut}(G)}.$$

(2) Similarly we have that $N \in ker(\varepsilon^{\mathrm{Aut}(G)})$. □

We denote by $b_0 \in \mathrm{Bl}(kG)$ the unique block such that $b_0 N = N$, equivalently b_0 is the unique block such that $\varepsilon(b_0) \neq 0$. We call b_0 the *principal* block of kG, see [4, Section 40].

Proposition 2.2. *Let $f \in \mathrm{Aut}(G)$. If $b \in \mathrm{Bl}(kG)$ then $\overline{f}(b) \in \mathrm{Bl}(kG)$. Moreover for the principal block we have $\overline{f}(b_0) = b_0$.*

Proof. It is easy to verify that $\overline{f}(b)$ is an idempotent. To prove that it is central let $g \in G$. Then

$$\overline{f}(b)g = \overline{f}(b)\overline{f}(f^{-1}(g)) = \overline{f}(bf^{-1}(g)) = \overline{f}(f^{-1}(g)b) = g\overline{f}(b).$$

It is easy to check, by contradiction, that $\overline{f}(b)$ is primitive in $Z(kG)$.

For the second part, since $N \in (kG)^{Aut(G)}$ we have that

$$\overline{f}(b_0)N = \overline{f}(b_0)\overline{f}(N) = \overline{f}(b_0 N) = \overline{f}(N) = N.$$

□

3. Saturated triples

From [2] we know that (A, b, G) is a *saturated triple* if b is a central idempotent, primitive in A^G such that for any (A, b, G)-Brauer pair (Q, e) we have that e is primitive in $A(Q)^{C_G(Q,e)}$, where A is a p-permutation algebra. See [2, IV, Section 2] for more details.

Theorem 3.1. *With the above notation we have that the triple $(kG, \mathrm{Aut}(G), b_0)$ is a saturated triple.*

Proof. We have that $(kG)^G = (kG)^{\mathrm{Inn}(G)}$, where $\mathrm{Inn}(G)$ is the normal subgroup in $\mathrm{Aut}(G)$ of inner automorphisms. Then $(kG)^{\mathrm{Aut}(G)} \subseteq (kG)^G$. By Proposition 2.2 we have that b_0 remains primitive in $(kG)^{\mathrm{Aut}(G)}$.

Let Q be a p-subgroup of $\mathrm{Aut}(G)$ and e a primitive idempotent of $Z(kG(Q))$. Since $\mathrm{Aut}(G)$ acts on G (the action given in Section 2), by [1, 2.5] we have that $kG(Q) \cong kC_G(Q)$, where

$$C_G(Q) = \{g \in G \mid f(g) = g, \forall f \in Q\}.$$

We prove next that e remains primitive in $kC_G(Q)^{C_{\mathrm{Aut}(G)}(Q,e)}$, where

$$C_{\mathrm{Aut}(G)}(Q, e) = \{f \mid f \in \mathrm{Aut}(G), \overline{f}(e) = e,\ f \circ q = q \circ f, \forall q \in Q\}.$$

We consider $\mathrm{Inn}_{C_G(Q)}(G)$ as the following subset of $\mathrm{Aut}(G)$, given by

$$\mathrm{Inn}_{C_G(Q)}(G) = \{c_x \mid x \in C_G(Q),\ c_x : G \to G,\ c_x(g) = xgx^{-1}, \forall g \in G\}.$$

It is easy to check that $\mathrm{Inn}_{C_G(Q)}(G)$ is a subgroup of $\mathrm{Aut}(G)$. If we restrict an element $c_x \in \mathrm{Inn}_{C_G(Q)}(G)$ to $C_G(Q)$ we have that $\mathrm{Im}(c_x \mid_{C_G(Q)}) = C_G(Q)$, then it follows that

$$kC_G(Q)^{C_G(Q)} = kC_G(Q)^{\mathrm{Inn}_{C_G(Q)}(G)} \tag{3.1}$$

Next we prove that $\mathrm{Inn}_{C_G(Q)}(G)$ is a subset of $C_{\mathrm{Aut}(G)}(Q, e)$ (in particular it is a subgroup). Let $c_x \in \mathrm{Inn}_{C_G(Q)}(G), q \in Q, g \in G$. We have that

$$\overline{c_x}(e) = xex^{-1} = e,$$

since $e \in kC_G(Q)^{C_G(Q)}$. The following statements holds

$$(c_x \circ q)(g) = xq(g)x^{-1};$$

$$(q \circ c_x)(g) = q(xgx^{-1}) = q(x)q(g)q(x)^{-1} = xq(g)x^{-1},$$

since $q \in Q$ and $x \in C_G(Q)$.

We now obtain that $\mathrm{Inn}_{C_G(Q)}(G) \leq C_{\mathrm{Aut}(G)}(Q,e)$, hence e remains primitive in

$$kC_G(Q)^{C_{\mathrm{Aut}(G)}(Q,e)} \subseteq kC_G(Q)^{\mathrm{Inn}_{C_G(Q)}(G)} = Z(kC_G(Q)),$$

where the last equality is proved in (3.1). □

Remark 3.2. Theorem 3.1 remains valid if we replace b_0 with any block b such that $b \in (kG)^{\mathrm{Aut}(G)}$, which remains primitive in this smaller algebra $(kG)^{\mathrm{Aut}(G)}$.

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Constantin-Cosmin Todea
Babeş-Bolyai University
Faculty of Mathematics and Computer Science
Str. Mihail Kogălniceanu, 1
400084 Cluj-Napoca,
Romania
e-mail: `Constantin.Todea@math.utcluj.ro`