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Solving nonlinear oscillators using a modified homotopy analysis method

Mohammad Zurigat, Shaher Momani and Ahmad Alawneh

Abstract. In this paper, a new algorithm called the modified homotopy analysis method (MHAM) is presented to solve a nonlinear oscillators. The proposed scheme is based on the homotopy analysis method (HAM), Laplace transform and Padé approximants. Several illustrative examples are given to demonstrate the effectiveness of the present method. Results obtained using the scheme presented here agree well with those derived from the modified homotopy perturbation method (MHPM). The results reveal that the MHAM is an effective and convenient for solving nonlinear differential equations.

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1. Introduction

The study of nonlinear oscillators is of crucial importance in all areas of physics and engineering, as well as in other disciplines. It is very difficult to solve nonlinear problems and, in general, it is often more difficult to get an analytic approximation than a numerical one to a given nonlinear problem. Several methods have been used to find approximate solutions to strongly nonlinear oscillators. Such methods include variational iteration method [1, 2, 3, 4, 5, 6, 7], Adomian decomposition method [8, 9], differential transform method [10], and harmonic balance based methods [11, 12, 13, 14, 15]. Surveys of the literature with multitudinous references and useful bibliographies have been given in Refs. [16, 17]. Recently, Momani et al [18], proposed a powerful analytic method, namely modified homotopy perturbation method. This method is based on the homotopy perturbation method, the Laplace transformation and Padé approximants. The approximate solution of the MHPM displays the periodic behavior which is characteristic of the oscillatory equations. The homotopy analysis method (HAM) [19] yields rapidly convergent series solutions by using few iterations for both linear and nonlinear differential equations. The HAM was successfully applied to solve many nonlinear problems such as Riccati differential equation of fractional order [20], fractional KdV-Burgers-Kuramoto equation [21], systems of fractional algebraic-differential equations [22], and so on. In this paper, we developed a symbolic algorithm to find the solution of nonlinear oscillators by a modified homotopy analysis method (MHAM). The MHAM is based on the homotopy analysis method (HAM), Laplace transform and Padé approximants. Finally, we make a numerical comparison between our method and the MHPM. The structure of this paper is as follows. In section 2 we describe the homotopy analysis method and briefly discuss Padé approximants. In Section 3 we present three examples to show the efficiency and simplicity of the method.

2. Homotopy analysis method

The HAM has been extended by many authors to solve linear and nonlinear fractional differential equations [19, 20, 21, 22]. In this section the basic ideas of the homotopy analysis method are introduced. To show the basic idea, let us consider the following nonlinear oscillator equation

$$y^{''}(t) + c_1 y(t) + c_2 y^2(t) + c_3 y^3(t) = \epsilon F(t, y(t), y^{'}(t)),\ t \geq 0, \tag{2.1}$$

subject to the initial conditions

$$y(0) = a,\ y^{'}(0) = b, \tag{2.2}$$

where $c_i,\ i = 1, 2, 3$, are positive real numbers and ϵ is a parameter (not necessarily small). We assume that the function $F(t, y(t), y^{'}(t))$ is an arbitrary nonlinear function of its arguments. Now, we can construct the so-called zero-order deformation equations of the equation (2.1) by

$$\begin{aligned} (1-q)L[\phi(t;q) - y_0(t)] &= q\ \hbar[\frac{d^2}{dt^2}\phi(t;q) + c_1\phi(t;q) + c_2\phi^2(t;q) \\ &\quad + c_3\phi^3(t;q) - \epsilon F(t, \phi(t;q), \frac{d}{dt}\phi(t;q))], \end{aligned} \tag{2.3}$$

where $q \in [0, 1]$ is an embedding parameter, L is an auxiliary linear operator satisfy $L(0) = 0$, $y_0(t)$ is an initial guess satisfies the initial condition (2.2), $\hbar \neq 0$ is an auxiliary parameter and $\phi(t; q)$ is an unknown function. Obviously, when $q = 0$ and when $q = 1$, we have $\phi(t; 0) = y_0(t)$ and $\phi(t; 1) = y(t)$.Thus as q increasing from 0 to 1, $\phi(t; q)$ varies from $y_0(t)$ to $y(t)$. Expanding $\phi(t; q)$ in Taylor series with respect to q, one has

$$\phi(t;q) = y_0(t) + \sum_{m=1}^{\infty} y_m(t) q^m, \tag{2.4}$$

where

$$y_m(t) = \frac{1}{m!}\frac{\partial^m \phi(t;q)}{\partial q^m}|_{q=0}. \tag{2.5}$$

If the auxiliary parameter h and the initial guess $y_0(t)$ are so properly chosen, then the series (2.4) converges at $q = 1$, one has

$$y(t) = y_0(t) + \sum_{m=1}^{\infty} y_m(t). \tag{2.6}$$

Define the vector

$$\overrightarrow{y}_m(t) = \{y_0(t), y_1(t), \ldots, y_m(t)\}. \tag{2.7}$$

Differentiating the zero-order deformation equation (2.3) m times with respective to q, then setting $q = 0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equations

$$L[y_m(t) - \chi_m y_{(m-1)}(t)] = \hbar \Re_m(\overrightarrow{y}_{m-1}(t)), \tag{2.8}$$

where

$$\begin{aligned} \Re_m(\overrightarrow{y}_{m-1}(t)) \quad = \quad & y''_{m-1}(t) + c_1 y_{m-1}(t) + c_2 \sum_{i=0}^{m-1} y_i(t) y_{m-i-1}(t) \\ & + c_3 \sum_{i=0}^{m-1} y_{m-i-1}(t) \sum_{j=0}^{i} y_j(t) y_{i-j}(t) \\ & - \frac{\epsilon}{(m-1)!} \frac{\partial^{m-1}}{\partial q^{m-1}} [F(t, \phi(t;q), \frac{d}{dt}\phi(t;q))]|_{q=0}, \end{aligned} \tag{2.9}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1. \end{cases} . \tag{2.10}$$

The mth-order approximation of $y(t)$ is given by $y(t) = \sum_{i=0}^{m} y_i(t)$. This power series can be transformed into Padé series easily. Padé series is defined in the following

$$a_0 + a_1 x + a_2 x^2 + \ldots = \frac{p_0 + p_1 x + \ldots + p_M x^M}{1 + q_1 x + \ldots + q_L x^L}. \tag{2.11}$$

Multiply both sides of (2.11) by the denominator of right-hand side in (2.11). We have

$$\begin{aligned} a_l + \sum_{k=l}^{M} a_{l-k} q_k \quad &= \quad p_l, \ (l = 0, 1, \ldots, M), \\ a_l + \sum_{k=l}^{L} a_{l-k} q_k \quad &= \quad 0, \ (l = M+1, \ldots, M+L). \end{aligned} \tag{2.12}$$

Solving the linear equation in (2.12), we have q_k $(k = 1, \ldots, L)$, and substituting into (2.11), we have p_k $(l = 1, \ldots, L)$ [23]. We use Mathematica to obtain diagonal Padé approximants of various orders, such as [2/2] or [4/4].

3. Numerical results

To demonstrate the effectiveness of the method we consider the following three examples of nonlinear oscillator equation.

3.1. Example 1

Consider the following Helmholtz equation

$$y^{''}(t) + 2y(t) + y^2(t) = 0, \quad t > 0, \tag{3.1}$$

subject to the initial conditions

$$y(0) = 0.1, \; y^{'}(0) = 0. \tag{3.2}$$

We start with initial approximation

$$y_0(t) = 0.1. \tag{3.3}$$

In view of the algorithm presented in the previous section, if we select the auxiliary linear operator as

$$L = \frac{d^2}{dt^2} \tag{3.4}$$

we can construct the homotopy as

$$y_m(t) = \chi_m y_{(m-1)}(t) + \hbar \int_0^t (t-\tau)\Re_m(\overrightarrow{y}_{m-1}(t))d\tau, \tag{3.5}$$

where

$$\Re_m(\overrightarrow{y}_{m-1}(t)) = y^{''}_{m-1}(t) + 2y_{m-1}(t) + \sum_{i=0}^{m-1} y_i(t)y_{m-i-1}(t). \tag{3.6}$$

Using formula (3.5), the fifth-term approximate solution for equation (3.1) is given by

$$\begin{aligned} y(t) \;\; = \;\; & 0.1 + 0.42\hbar t^2 + 0.63\hbar^2 t^2 + 0.42\hbar^3 t^2 + 0.105\hbar^4 t^2 + 0.1155\hbar^2 t^4 \\ & +0.154\hbar^3 t^4 + 0.05775\hbar^4 t^4 + 0.00711667\hbar^3 t^6 + 0.0053375\hbar^4 t^6 \\ & +0.000142083\hbar^4 t^8. \end{aligned} \tag{3.7}$$

Setting $\hbar = -1$ in Eq (3.7), then we have

$$y(t) = 0.1 - 0.105t^2 + 0.0925t^4 - 0.00177917t^6 + 0.000142083t^8. \tag{3.8}$$

In order to improve the accuracy of the homotopy analysis solution of the Helmholtz equation we need to implement the following technique. First applying the Laplace transformation to the previous series solution, then we get

$$\widetilde{y}_m(s) = \frac{0.1}{s} - \frac{0.21}{s^3} + \frac{0.462}{s^5} - \frac{1.281}{s^7} + \frac{5.7288}{s^9}. \tag{3.9}$$

Now, let $s = \frac{1}{t}$ in (3.9), then we have

$$\widetilde{y}_m(t) = 0.1t - 0.21t^3 + 0.462t^5 - 1.281t^7 + 5.7288t^9.$$

The [4/4] Padé approximation gives

$$\left[\frac{4}{4}\right] = \frac{0.1t + 1.27t^3}{1 + 14.8t^2 + 26.46t^4}.$$

Recalling $t = 1/s$, we obtain [4/4] in terms of s

$$\left[\frac{4}{4}\right] = \frac{1.27s + 0.1s^3}{26.46 + 14.8s^2 + s^4}.$$

By using the inverse Laplace transformation to the [4/4] Padé approximation, we obtain the same solution obtained in Momani et al. [18] using the modified homotopy perturbation method

$$y(t) = 0.0998141\cos(1.4423t) + 0.000185858\cos(3.56648t). \quad (3.10)$$

Figure 1 shows the series solution (3.10) exhibit the periodic behavior which is characteristic of the oscillatory Helmholtz equation (3.1) and (3.2).

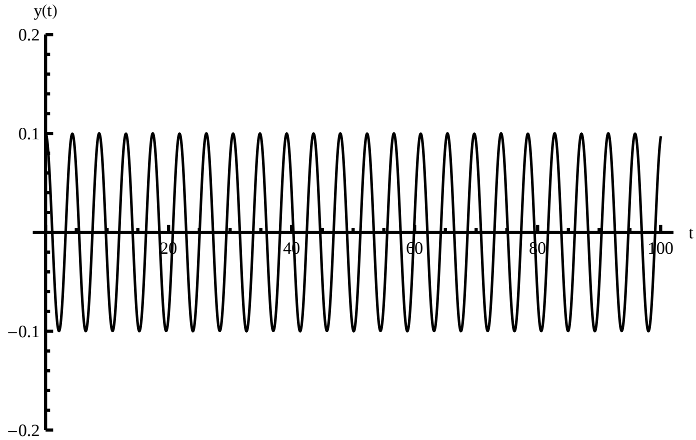

Figure 1. Plots of Eq. (3.10)

3.2. Example 2

Consider the following nonlinear equation

$$y''(t) + y(t) = -0.1y^2(t)y'(t), \qquad t > 0, \quad (3.11)$$

subject to the initial conditions

$$y(0) = 1, \ y'(0) = 0. \quad (3.12)$$

Select the initial guess as

$$y_0(t) = 1, \quad (3.13)$$

and the auxiliary linear operator (3.4), then we have the homotopy (3.5) where

$$\Re_m(\overrightarrow{y}_{m-1}(t)) = y''_{m-1}(t) + y_{m-1}(t) + 0.1\sum_{i=0}^{m-1} y'_{m-i-1}(t)\sum_{j=0}^{i} y_j(t)y_{i-j}(t). \quad (3.14)$$

Using formula (3.5), the fifth-term approximate solution for equation (3.11) is given by

$$\begin{aligned} y(t) &= 1+2\hbar t^2+3\hbar^2t^2+2\hbar^3t^2+0.5\hbar^4t^2+0.1\hbar^2t^3+0.133\hbar^3t^3 \\ &+0.05\hbar^4t^3+0.25\hbar^2t^4+0.34\hbar^3t^4+0.126\hbar^4t^4+0.027\hbar^3t^5 \\ &+0.020008\hbar^4t^5+0.00555556\hbar^3t^6+0.0045694\hbar^4t^6 \\ &+0.001369\hbar^4t^7+0.0000248\hbar^4t^8. \end{aligned} \tag{3.15}$$

Setting $\hbar=-1$ in Eq (3.15), then we have

$$\begin{aligned} y(t) &= 1-0.5t^2+0.01667t^3+0.0413t^4-0.00666t^5 \\ &-0.00098611t^6+0.0013691t^7+0.0000248t^8. \end{aligned} \tag{3.16}$$

Applying the Laplace transformation to the previous series solution, then we get

$$\widetilde{y}_m(s)=\frac{1}{s}-\frac{1}{s^3}+\frac{0.1}{s^4}+\frac{0.99}{s^5}-\frac{0.799}{s^6}-\frac{0.71}{s^7}+\frac{6.9}{s^8}+\frac{1}{s^9}. \tag{3.17}$$

Let $s=\frac{1}{t}$ in (3.17), then we have

$$\widetilde{y}_m(t)=t-t^3+0.1t^4+0.99t^5-0.799t^6-0.71t^7+6.9t^8+t^9.$$

The [4/4] Padé approximation gives

$$\left[\frac{4}{4}\right]=\frac{t+0.3335t^2+9.16122t^3+0.313787t^4}{1+0.3335t+10.1612t^2+0.547287t^3+9.13787t^4}.$$

Recalling $t=1/s$, we obtain [4/4] in terms of s

$$\left[\frac{4}{4}\right]=\frac{0.313787+9.16122s+0.3335s^2+s^3}{9.13787+0.547287s+10.1612s^2+0.3335s^3+s^4}.$$

By using the inverse Laplace transformation to the [4/4] Padé approximation, we obtain the same solution obtained in Momani et al. [18] using the modified homotopy perturbation method

$$\begin{aligned} y(t) &= e^{(-0.013-0.999i)t}((0.5+0.0111i)+(0.5-0.0111i)e^{-1.99it}) \\ &+e^{(-0.15-3.02i)t}((0.0003-0.002i)+(0.0003+0.002i)e^{-6it}). \end{aligned} \tag{3.18}$$

The equation (3.11) called the "unplugged" van der Pol, and all its solutions are expected to oscillate with decreasing to zero. Figure 2 shows the series solution (3.18) of the oscillatory nonlinear equation (3.11) and (3.12).

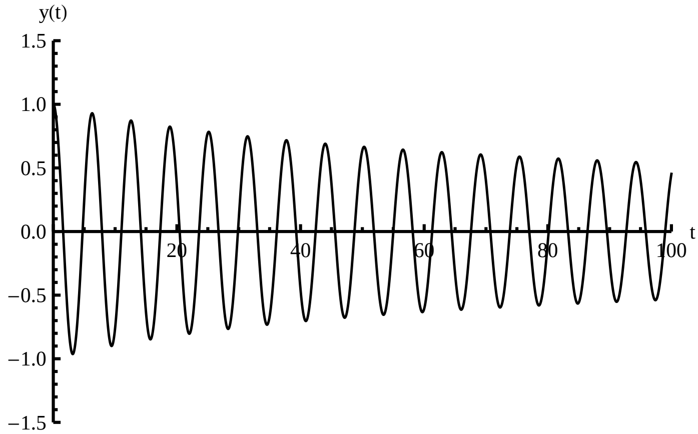

Figure 2. Plots of Eq. (3.18)

3.3. Example 3

Consider the following nonlinear equation

$$y''(t) + y(t) + 0.45y^2(t) = y(t)y'(t), \quad t > 0, \tag{3.19}$$

subject to the initial conditions

$$y(0) = 0.1, \; y'(0) = 0. \tag{3.20}$$

Take

$$y_0(t) = 0.1, \tag{3.21}$$

and the auxiliary linear operator (3.4), then we have the homotopy (3.5) where

$$\begin{aligned}\Re_m(\overrightarrow{y}_{m-1}(t)) \quad &= \quad y''_{m-1}(t) + y_{m-1}(t) + 0.45\sum_{i=0}^{m-1} y_i(t)y_{m-i-1}(t) \\ &\quad -\sum_{i=0}^{m-1} y'_i(t)y_{m-i-1}(t).\end{aligned} \tag{3.22}$$

The fifth-term approximate solution for equation (3.19) is given by

$$\begin{aligned} y(t) \quad = \quad & 0.1 + 0.21\hbar t^2 + 0.31\hbar^2 t^2 + 0.21\hbar^3 t^2 + 0.05\hbar^4 t^2 \\ & -0.01\hbar^2 t^3 - 0.0139\hbar^3 t^3 - 0.005\hbar^4 t^3 + 0.0285\hbar^2 t^4 \\ & +0.038\hbar^3 t^4 + 0.014\hbar^4 t^4 - 0.0019\hbar^3 t^5 - 0.0014\hbar^4 t^5 \\ & +0.0008536\hbar^3 t^6 + 0.000665\hbar^4 t^6 - 0.0000524\hbar^4 t^7 \\ & +8.1389 \times 10^{-6}\hbar^4 t^8. \end{aligned} \tag{3.23}$$

Setting $\hbar = -1$ in Eq (3.23), then we have

$$\begin{aligned} y(t) \quad = \quad & 0.1 - 0.05t^2 - 0.00174t^3 + 0.0047t^4 + 0.00046t^5 \\ & -0.0001889t^6 - 0.0000524t^7 + 8.1 \times 10^{-6}t^8. \end{aligned} \tag{3.24}$$

Applying the Laplace transformation to the previous series solution, then we get

$$\begin{aligned}\widetilde{y}_m(s) &= \frac{0.1}{s} - \frac{0.1045}{s^3} - \frac{0.01045}{s^4} + \frac{0.11286}{s^5} + \frac{0.0554373}{s^6} \\ &\quad - \frac{0.136028}{s^7} - \frac{0.264279}{s^8} + \frac{0.32816}{s^9}. \end{aligned} \tag{3.25}$$

Let $s = \frac{1}{t}$ in (3.25), then we have

$$\begin{aligned}\widetilde{y}_m(t) &= 0.1t - 0.104t^3 - 0.01045t^4 + 0.11286t^5 + 0.0554372t^6 \\ &\quad -0.136028t^7 - 0.264279t^8 + 0.32816t^9.\end{aligned}$$

The [4/4] Padé approximation gives

$$\left[\frac{4}{4}\right] = \frac{0.1t + 0.0643181t^2 + 90.570534t^3 - 0.0226522t^4}{1 + 0.643181t + 6.75034t^2 + 0.550102t^3 + 5.99272t^4}.$$

Recalling $t = 1/s$, we obtain [4/4] in terms of s

$$\left[\frac{4}{4}\right] = \frac{-0.0226522 + 0.570534s + 0.0643181s^2 + 0.1s^3}{5.99272 + 0.550102s + 6.75034s^2 + 0.643181s^3 + s^4}.$$

By using the inverse Laplace transformation to the [4/4] Padé approximation, we obtain

$$\begin{aligned} y(t) &= e^{(-0.3-2.4i)t}((0.0001 + 0.0003i) - (0.0001 - 0.0003i)e^{-4.7it}) \\ &\quad + e^{(-1.02i)t}((0.05 - 0.001i)e^{0.01t} + (0.05 + 0.001i)e^{(0.01+2.1i)t}). \end{aligned} \tag{3.26}$$

The above results are in excellent agreement with the results obtained by Momani et al. [18] using the modified homotopy perturbation method. Figure 3 shows the series solution (3.26) of the oscillatory nonlinear equation (3.19) and (3.20).

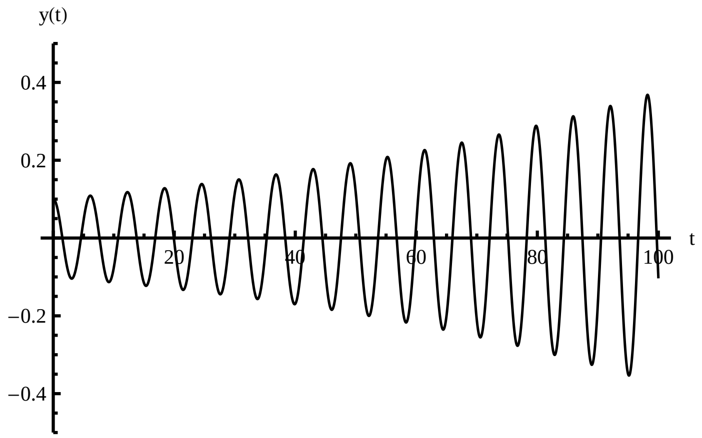

Figure 3. Plots of Eq. (3.26)

4. Conclusions

In this work, we proposed an efficient modification of the HAM which introduces an efficient tool for solving nonlinear oscillatory equations. The modified algorithm has been successfully implemented to find approximate solutions for many problems. The comparison of the result obtained by MHAM with that obtained by MHPM confirms our belief of the efficiency of our techniques. The basic idea described in this paper is expected to be further employed to find periodic solutions to nonlinear fractional oscillatory equations.

References

[1] He, J.H., *Variational iteration method a kind of nonlinear analytical technique: Some examples*, Internat. J. Nonlinear. Mech., **34**(1999), no. 4, 699-708.

[2] He, J.H.., *A review on some new recently developed nonlinear analytical techniques*, Int. J. Nonliear. Sci., **1**(2000), no. 1, 51-70.

[3] He, J.H., *Some asymptotic methods for strongly nonlinear equations*, Int. J. Mod. Phys., **20(10)**(2006), 1141-1199.

[4] He, J.H., Wu, X.H., *Construction of solitary solution and compacton-like solution by variational iteration method*, Chaos. Soliton. Fract., **29**(2006), 108-111.

[5] He, J.H., *Variational iteration method-some recent results and new interpretations*, J. Comput. Appl. Math., **207**(2007), no. 1, 3-17.

[6] Shou, D.H., He, J.H., *Beyond Adomian method: The variational iteration method for solving heat-like and wave-like equations with variable coefficients*, Phys. Lett. A, **372**(2008), no. 3, 233-237.

[7] Bildik, N., Konuralp, A., *The use of variational iteration method, differential transform method and Adomian decomposition method for solving different types of nonlinear partial differential equations*, Int. J. Nonliear. Sci., **7**(2006), no. 1, 65-70.

[8] Ghosh, S., Roy, A., *An adaptation of Adomian decomposition for numeric-analytic integration of strongly nonlinear and chaotic oscillators*, Computer Method in Applied Mechanics and Engineering, **196(4-6)**(2007), 1133-1153.

[9] Momani, S., *Analytical solutions of strongly non-linear oscillators by the decomposition method*, International Journal of Modern Physics C (IJMPC), **15(7)**(2004), 967-979.

[10] Momani, S., Erturk, V.S., *Solutions of non-linear oscillators by the modified differential transform method*, Computers and Mathematics with Applications, **55**(2008), 833-842.

[11] Gottlieb, H.P.W., *Harmonic balance approach to limit cycles for nonlinear jerk equations*, Journal of Sound and Vibration, **297**(2006), 243–250.

[12] Lim, C.W., Wu, B.S., Sun, W.P., *Higher accuracy analytical approximations to the Duffing-harmonic oscillator*, Journal of Sound and Vibration, **296**(2006), 1039–1045.

[13] Beléndez, A., Hernández, A., Márquez, A., Beléndez, T., Neipp, C., *Analytical approximations for the period of a simple pendulum*, Eur. J. Phys., **27**(2006), 539–551.

[14] Itovich, G.R., Moiola, J.L., *On period doubling bifurcations of cycles and the harmonic balance method*, Chaos Solitons Fractals, **27**(2005), 647-665.

[15] Blendez, A., Hernandez, A., Belendez, T., Alvarez, M.L., Gallego, S., Ortuno, M., Neipp, C., *Application of the harmonic balance method to a nonlinear oscillator typified by a mass attached to a stretched wire*, Journal of Sound and Vibration, **302**(2007), 1018-1029.

[16] He, J.H., *Non-perturbation methods for strongly nonlinear problems*, Dissertation, Verlag im Internet GmbH, Berlin, 2006.

[17] He, J.H., *Some asymptotic methods for strongly nonlinear equations*, Internet. J. Modern Phys., **20**(2006), 1141-1199.

[18] Momani, S., Erjaee, G.H., Alnasr, M.H., *The modified homotopy perturbation method for solving strongly nonlinear oscillators*, Computers and Mathematics with Applications, **10**(2009), no. 1, 416-427.

[19] Zurigat, M., Momani, S., Odibat, Z., Alawneh, A., *The homotopy analysis method for handling systems of fractional differential equations*, Applied Mathematical Modelling, **34**(2009), no. 1, 24-35.

[20] Cang, J., Tan, Y., Xu, H., Liao, S., *Series solutions of non-linear Riccati differential equations with fractional order*, Chaos Solitons and Fractals, **40**(2009), no. 1, 1-9.

[21] Song, L., Zhang, H., *Application of homotopy analysis method to fractional KdV-Burgers-Kuramoto equation*, Physics Letters A, **367**(2007), 88-94.

[22] Zurigat, M., Momani, S., Alawneh, A., *Analytical approximate solutions of systems of fractional algebraic-differential equations by homotopy analysis method*, Computer and Mathematics with Applications, **59**(2010), no. 3, 1227-1235.

[23] Celik, E., Bayram, M., *The basic successive substitute approximations method and Padé approximations to solve the elasticity problem of settled of the wronkler ground with variable coefficients*, Appl. Math. Comput., **155**(2004), 495-505.

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