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Approximate fixed point theorems for generalized T-contractions in metric spaces

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Abstract. In this paper, we introduce the concept of T-asymptotically regular mapping and establish a lemma for ϵ-fixed points of two commuting mappings in metric spaces. This lemma is used for proving approximate fixed point theorems for various types of contraction mappings in the framework of metric spaces. Our results in this paper extends and improves upon, among others, the corresponding results of Berinde given in [2].

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1. Introduction

Fixed point property of various types of mappings has been used to solve many problems in applied mathematics. In several situations of practical utility, the mapping under consideration may not have an exact fixed point due to some restriction on the space or the map. Besides that, there may arise many situations in real life where the existence of fixed points is not strictly required, but that of 'nearly fixed points' is more than enough. In such cases, we can make use of the concept of ϵ-fixed points (approximate fixed points) which is one type of 'nearly fixed points'. Let us consider the metric space (X, d) and T a self map of this metric space. Suppose that we would like to find an approximate solution of $Tx = x$. If there exists a point $x_0 \in X$ such that $d(Tx_0, x_0) < \epsilon$, where ϵ is a positive number, then x_0 is called an approximate solution of the equation $Tx = x$ or we can say that $x_0 \in X$ is an approximate fixed point (or ϵ-fixed point) of T.

Approximate fixed point property for various types of mappings have been a prominent area of research of many mathematicians for the last few years. In 2006, Berinde [2] proved quantitative and qualitative approximate fixed point theorems for various types of well known contractions on metric spaces. It was proved that even by weakening the conditions by giving up the completeness of the space, the existence of ϵ-fixed points is still guaranteed for operators satisfying Kannan, Chatterjea and Zamfirescu type of conditions on metric spaces.

In 2009, Beiranvand et al. [1], introduced the notions of T-Banach contraction and T-contractive mapping and extended the Banach contraction principle (see [4]) and Edelstein's fixed point theorem [7]. In the same year, Moradi [9] introduced the T-Kannan contractive type mappings, extending in this way the well-known Kannan's fixed point theorem given in [8]. The corresponding versions of T-contractive, T-Kannan mappings and T-Chatterjea contractions on cone metric spaces were studied in [10]. The same authors [12], then studied the existence of fixed points of T-Zamfirescu and T-weak contraction mappings defined on a complete cone metric space. Later, in [11] they studied the existence of fixed points for T-Zamfirescu operators in complete metric spaces and proved a convergence theorem of T-Picard iteration for the class of T-Zamfirescu operators.

Inspired and motivated by the above facts, we prove approximate fixed point theorems for the classes of T-Banach contraction, T-Kannan contraction, T-Chatterjea contraction, T-Zamfirescu operators and T-almost contraction. Here we mention that we consider operators in metric spaces, not in complete metric spaces which is the usual framework for fixed point theorems.

Let (X, d) be a metric space and $T, S : X \to X$ be two commuting mappings. Here we introduce the concept of T-asymptotically regular mapping in a metric space and then establish a lemma regarding approximate fixed points of the commuting mappings in metric spaces. We use this lemma to prove qualitative theorems for various types of contractions on metric spaces.

We need the following definitions to prove our main results:

Definition 1.1. *Let (X, d) be a metric space. Let $f : X \to X, \epsilon > 0$ and $x \in X$. Then x_0 is an* **ϵ-fixed point(approximate fixed point) of** *f if*

$$d(f(x_0), x_0) < \epsilon.$$

Note. The set of all ϵ-fixed points of f, for a given ϵ can be denoted by

$$F_\epsilon(f) = \{x \in X : x \text{ is an } \epsilon - \text{fixed point of } f\}.$$

Definition 1.2. *Let (X, d) be a metric space and $f : X \to X$. Then f has the* **approximate fixed point property (a.f.p.p)** *if for every $\epsilon > 0$,*

$$F_\epsilon(f) \neq \phi.$$

Definition 1.3. *Let (X, d) be a metric space, $f : X \to X$ is said to be asymptotically regular if*

$$d(f^n(x), f^{n+1}(x)) \to 0 \ \text{ as } \ n \to \infty, \quad \text{ for all } \ x \in X.$$

Definition 1.4. *Let (X, d) be a metric space, $T, S : X \to X$ be two functions. S is called T-asymptotically regular if*

$$d(TS^n(x), TS^{n+1}(x)) \to 0 \ \text{ as } \ n \to \infty, \quad \text{ for all } \ x \in X.$$

Definition 1.5. *Let (X, d) be a metric space and $T, S : X \to X$ be two functions. S is said to be T-***Banach contraction** *(TB contraction) if there exists $a \in [0, 1)$ such that*

$$d(TSx, TSy) \leq ad(Tx, Ty), \qquad \text{for all } x, y \in X.$$

If we take $T = I$, the identity map, then we obtain the definition of *Banach's contraction* (see [4]).

Definition 1.6. *Let (X, d) be a metric space and $T, S : X \to X$ be two functions. S is said to be T-***Kannan contraction** *(TK contraction) if there exists $b \in [0, \frac{1}{2})$ such that*

$$d(TSx, TSy) \leq b(d(Tx, TSx) + d(Ty, TSy]), \qquad \text{for all } x, y \in X.$$

Here when $T = I$, the identity map, we get *Kannan operator* [8].

Definition 1.7. *Let (X, d) be a metric space and $T, S : X \to X$ be two functions. S is said to be T-***Chatterjea contraction** *(TC contraction) if there exists $c \in [0, \frac{1}{2})$ such that*

$$d(TSx, TSy) \leq c[d(Tx, TSy) + d(Ty, TSx]), \qquad \text{for all } x, y \in X.$$

When $T = I$, the identity map, in the above definition, it becomes *Chatterjea operator* [6].

Definition 1.8. *Let (X, d) be a metric space and $T, S : X \to X$ be two functions. S is said to be T-***Zamfirescu operator** *(TZ operator) if there are real numbers $0 \leq a < 1, 0 \leq b < \frac{1}{2}, 0 \leq c < \frac{1}{2}$ such that for all $x, y \in X$ at least one of the conditions is true:*
$(TZ_1): \ d(TSx, TSy) \leq ad(Tx, Ty),$
$(TZ_2): \ d(TSx, TSy) \leq b[d(Tx, TSx) + d(Ty, TSy]),$
$(TZ_3): \ d(TSx, TSy) \leq c[d(Tx, TSy) + d(Ty, TSx]).$

When the function T is equated to I, the identity map, we obtain the definition of *Zamfirescu operator* introduced in [13].

Definition 1.9. *Let (X, d) be a metric space and $T, S : X \to X$ be two functions. S is said to be T-***almost contraction** *if there exists $\delta \in [0, 1)$ and $L \geq 0$ such that*

$$d(TSx, TSy) \leq \delta d(Tx, Ty) + Ld(Ty, TSx), \qquad \text{for all } x, y \in X.$$

When $T = I$, the identity map, in the above definition, we obtain the definition of almost contraction, the concept introduced by Berinde (see [3], [5]).

In 2004, this concept was introduced by Berinde as *weak contraction* [3] and later in 2008, it was renamed by himself as *almost contraction* [5].

In order to prove our main results we need the following lemma:

Lemma 1.10. *Let (X, d) be a metric space and $T, S : X \to X$ be two commuting mappings. If S is T-asymptotically regular, then S has approximate fixed point property.*

Proof. Let $x_0 \in X$. Since $S : X \to X$ is T-asymptotically regular, we have

$$d(TS^n(x_0), TS^{n+1}(x_0)) \to 0 \text{ as } n \to \infty, \quad \text{for all } x \in X,$$

which gives that for every $\epsilon > 0$, there exists $n_0(\epsilon) \in N$ such that

$$d(TS^n(x_0), TS^{n+1}(x_0)) < \epsilon, \text{ for all } n \geq n_0(\epsilon)$$

$$i.e., d(TS^n(x_0), TS(S^n(x_0))) < \epsilon, \text{ for all } n \geq n_0(\epsilon).$$

Since T and S are commuting mappings, this implies that for every $\epsilon > 0$, there exists $n_0(\epsilon) \in N$ such that

$$d(TS^n(x_0), ST(S^n(x_0))) < \epsilon, \text{ for all } n \geq n_0(\epsilon).$$

Denote $y_0 = TS^n(x_0)$. Then we get that for every $\epsilon > 0$, there exists $y_0 \in X$ such that

$$d(y_0, Sy_0) < \epsilon, \text{ for all } n \geq n_0(\epsilon).$$

So for each $\epsilon > 0$, there exists an ϵ-fixed point of S in X, namely y_0 which means that S has approximate fixed point property.

2. Main results

Theorem 2.1. *Let (X, d) be a metric space and $T, S : X \to X$ be two commuting mappings where S is a TB-contraction. Then for every $\epsilon > 0$,*

$$F_\epsilon(S) \neq \phi,$$

i.e., S has approximate fixed point property.

Proof. Let $\epsilon > 0, x \in X$. Then

$$\begin{aligned} d(TS^n(x), TS^{n+1}(x)) &= d(TS(S^{n-1}(x)), TS(S^n(x))) \\ &\leq ad(TS^{n-1}(x), TS^n(x)) \\ &\leq a^2 d(TS^{n-2}(x), TS^{n-1}(x)) \\ &\leq \dots\dots\dots\dots \\ &\leq a^n d(Tx, TSx). \end{aligned}$$

Since $a \in [0, 1)$, from the above inequality we get that

$$d(TS^n(x), TS^{n+1}(x)) \to 0 \text{ as } n \to \infty, \quad \text{for all } x \in X,$$

which implies that S is T-asymptotically regular. Now by applying Lemma 1.10 we obtain that for every $\epsilon > 0$,

$$F_\epsilon(S) \neq \phi,$$

which means that S has approximate fixed point property.

Corollary 2.2. [2, *Theorem* 2.1] *Let (X, d) be a metric space and $f : X \to X$ an a-contraction. Then for every $\epsilon > 0$,*

$$F_\epsilon(f) \neq \phi.$$

Theorem 2.3. *Let (X, d) be a metric space and $T, S : X \to X$ be two commuting mappings where S is a TK-contraction. Then for every $\epsilon > 0$,*

$$F_\epsilon(S) \neq \phi,$$

i.e., S has approximate fixed point property.

Proof. Let $\epsilon > 0, x \in X$. Then

$$\begin{aligned} d(TS^n(x), TS^{n+1}(x)) &= d(TS(S^{n-1}(x)), TS(S^n(x))) \\ &\leq b[d(TS^{n-1}(x), TS(S^{n-1}(x))) + d(TS^n(x), TS(S^n(x)))] \\ &= b[d(TS^{n-1}(x), TS^n(x)) + d(TS^n(x), TS^{n+1}(x))]. \end{aligned}$$

Thus we have the inequality

$$(1-b)d(TS^n(x), TS^{n+1}(x)) \leq b[d(TS^{n-1}(x), TS^n(x))],$$

which implies that

$$\begin{aligned} d(TS^n(x), TS^{n+1}(x)) &\leq \frac{b}{1-b} d(TS^{n-1}(x), TS^n(x)) \\ &\leq \left(\frac{b}{1-b}\right)^2 d(TS^{n-2}(x), TS^{n-1}(x)) \\ &\leq \cdots\cdots\cdots\cdots \\ &\leq \left(\frac{b}{1-b}\right)^n d(Tx, TSx). \end{aligned}$$

Since $b \in [0, \frac{1}{2})$, from the above inequality we get that

$$d(TS^n(x), TS^{n+1}(x)) \to 0 \text{ as } n \to \infty, \quad \text{for all } x \in X,$$

which implies that S is T-asymptotically regular. Now by applying Lemma 1.10 we obtain that for every $\epsilon > 0$,

$$F_\epsilon(S) \neq \phi,$$

which means that S has approximate fixed point property.

Corollary 2.4. [2, *Theorem* 2.2] *Let (X, d) be a metric space and $f : X \to X$ a Kannan operator. Then for every $\epsilon > 0$,*

$$F_\epsilon(f) \neq \phi.$$

Theorem 2.5. *Let (X, d) be a metric space and $T, S : X \to X$ be two commuting mappings where S is a TC-contraction. Then for every $\epsilon > 0$,*

$$F_\epsilon(S) \neq \phi,$$

i.e., S has approximate fixed point property.

Proof. Let $\epsilon > 0, x \in X$. Then

$$\begin{aligned} d(TS^n(x), TS^{n+1}(x)) &= d(TS(S^{n-1}(x)), TS(S^n(x))) \\ &\leq c[d(TS^{n-1}(x), TS(S^n(x))) + d(TS^n(x), TS(S^{n-1}(x)))] \\ &= c[d(TS^{n-1}(x), TS^{n+1}(x) + d(TS^n(x), TS^n(x))] \\ &= c[d(TS^{n-1}(x), TS^{n+1}(x))]. \end{aligned}$$

Now we have,

$$d(TS^{n-1}(x), TS^{n+1}(x)) \leq d(TS^{n-1}(x), TS^n(x)) + d(TS^n(x), TS^{n+1}(x)),$$

which implies that

$$d(TS^n(x), TS^{n+1}(x)) \leq c[d(TS^{n-1}(x), TS^n(x)) + d(TS^n(x), TS^{n+1}(x))],$$

which gives

$$(1-c)d(TS^n(x), TS^{n+1}(x)) \leq c[d(TS^{n-1}(x), TS^n(x)].$$

Thus we have the inequality,

$$\begin{aligned} d(TS^n(x), TS^{n+1}(x)) &\leq \frac{c}{1-c} d(TS^{n-1}(x), TS^n(x)) \\ &\leq \left(\frac{c}{1-c}\right)^2 d(TS^{n-2}(x), TS^{n-1}(x)) \\ &\leq \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\leq \left(\frac{c}{1-c}\right)^n d(Tx, TSx). \end{aligned}$$

Since $c \in [0, \frac{1}{2})$, from the above inequality we get that

$$d(TS^n(x), TS^{n+1}(x)) \to 0 \text{ as } n \to \infty, \quad \text{for all } x \in X,$$

which implies that S is T-asymptotically regular. Now by applying Lemma 1.10 we obtain that for every $\epsilon > 0$,

$$F_\epsilon(S) \neq \phi,$$

which means that S has approximate fixed point property.

Corollary 2.6. [2, *Theorem* 2.3] *Let (X, d) be a metric space and $f : X \to X$ a Chatterjea operator. Then for every $\epsilon > 0$,*

$$F_\epsilon(f) \neq \phi.$$

Theorem 2.7. *Let (X, d) be a metric space and $T, S : X \to X$ be two commuting mappings where S is a TZ-operator. Then for every $\epsilon > 0$,*

$$F_\epsilon(S) \neq \phi,$$

i.e., S has approximate fixed point property.

Proof. Let $\epsilon > 0, x \in X$. If (TZ_2) holds, then

$$\begin{aligned} d(TSx, TSy) &\leq b[d(Tx, TSx) + d(Ty, TSy)] \\ &\leq b[d(Tx, TSx)] + b[d(Ty, Tx) + d(Tx, TSx) + d(TSx, TSy)] \\ &= 2b[d(Tx, TSx)] + b[d(Tx, Ty)] + b[d(TSx, TSy)]. \end{aligned}$$

The above inequality gives

$$(1-b)d(TSx, TSy) \leq 2b[d(Tx, TSx)] + b[d(Tx, Ty)],$$

which implies

$$d(TSx, TSy) \leq \frac{2b}{1-b}[d(Tx, TSx)] + \frac{b}{1-b}[d(Tx, Ty)]. \tag{2.1}$$

If (TZ_3) holds, then

$$\begin{aligned} d(TSx,TSy) &\leq c[d(Tx,TSy)+d(Ty,TSx)] \\ &\leq c[d(Tx,TSx)+d(TSx,TSy)+d(Ty,Tx)+d(Tx,TSx)] \\ &= c[d(TSx,TSy)]+2c[d(Tx,TSx)]+c[d(Tx,Ty)]. \end{aligned}$$

The above inequality gives

$$(1-c)d(TSx,TSy) \leq 2c[d(Tx,TSx)]+c[d(Tx,Ty)],$$

which implies

$$d(TSx,TSy) \leq \frac{2c}{1-c}[d(Tx,TSx)]+\frac{c}{1-c}[d(Tx,Ty)]. \tag{2.2}$$

Denote

$$\delta = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}.$$

Then we have $0 \leq \delta < 1$ and in view of (TZ_1), (2.1) and (2.2), it results that the inequality

$$d(TSx,TSy) \leq 2\delta[d(Tx,TSx)]+\delta[d(Tx,Ty)] \tag{2.3}$$

holds for all $x,y \in X$.
Using (2.3), we get

$$\begin{aligned} d(TS^n(x),TS^{n+1}(x)) &= d(TS(S^{n-1}(x)),TS(S^n(x))) \\ &\leq 2\delta[d(TS^{n-1}(x),TS(S^{n-1}(x))]+\delta[d(TS^{n-1}(x),TS^n(x))] \\ &= 3\delta[d(TS^{n-1}(x),TS^n(x))]. \end{aligned}$$

Thus we obtain that

$$\begin{aligned} d(TS^n(x),TS^{n+1}(x)) &\leq 3\delta[d(TS^{n-1}(x),TS^n(x))] \\ &\leq (3\delta)^2\,[d(TS^{n-2}(x),TS^{n-1}(x))] \\ &\leq \ldots\ldots\ldots\ldots\ldots\ldots \\ &\leq (3\delta)^n\,[d(Tx,TSx)]. \end{aligned}$$

Since $\delta \in [0,1)$, the above inequality gives

$$d(TS^n(x),TS^{n+1}(x)) \to 0 \text{ as } n \to \infty, \quad \text{for all } x \in X,$$

which implies that S is T-asymptotically regular. Now by applying Lemma 1.10 we obtain that for every $\epsilon > 0$,

$$F_\epsilon(S) \neq \phi,$$

which means that S has approximate fixed point property.

Corollary 2.8. [2,*Theorem* 2.4] *Let* (X,d) *be a metric space and* $f : X \to X$ *a Zamfirescu operator. Then for every* $\epsilon > 0$,

$$F_\epsilon(f) \neq \phi.$$

Theorem 2.9. *Let (X, d) be a metric space and $T, S : X \to X$ be two commuting mappings where S is a T-almost contraction. Then for every $\epsilon > 0$,*

$$F_\epsilon(S) \neq \phi,$$

i.e., S has approximate fixed point property.

Proof. Let $\epsilon > 0, x \in X$. Then

$$\begin{aligned} d(TS^n(x), TS^{n+1}(x)) &= d(TS(S^{n-1}(x)), TS(S^n(x))) \\ &\leq \delta d(TS^{n-1}(x), TS^n(x)) + Ld(TS^n(x), TS^n(x)) \\ &= \delta d(TS^{n-1}(x), TS^n(x)) \\ &\leq \delta^2 d(TS^{n-2}(x), TS^{n-1}(x)) \\ &\leq \ldots\ldots\ldots\ldots\ldots\ldots \\ &\leq \delta^n d(Tx, TSx). \end{aligned}$$

Since $\delta \in [0, 1)$, from the above inequality we get that

$$d(TS^n(x), TS^{n+1}(x)) \to 0 \text{ as } n \to \infty, \quad \text{for all } x \in X,$$

which implies that S is T-asymptotically regular. Now by applying Lemma 1.10 we obtain that for every $\epsilon > 0$,

$$F_\epsilon(S) \neq \phi,$$

which means that S has approximate fixed point property.

Corollary 2.10. [2, *Theorem* 2.5] *Let (X, d) be a metric space and $f : X \to X$ an almost contraction. Then for every $\epsilon > 0$,*

$$F_\epsilon(f) \neq \phi.$$

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