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About some links between the Dini-Hadamard-like normal cone and the contingent one

Delia-Maria Nechita

Abstract. The primary goal of this paper is to furnish an alternative description for the contingent normal cone, similar to the one that exists for the Fréchet one, but by using a directional convergence in place of the usual one. In fact, we actually prove that the same description is available not only for the contingent normal cone, but also for the Dini-Hadamard normal cone and the Dini-Hadamard-like one. Furthermore, we show that although in the case of the Dini-Hadamard subdifferential the geometric construction agrees with the analytical one, in the case of the Dini-Hadamard-like one the analytical construction is only greater than the geometrical one.

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1. Introduction

It is well known that nonsmooth functions, sets with nonsmooth boundaries and set-valued mappings appear naturally and frequently in various areas of mathematics and applications, especially in those related to optimization, stability, variational systems and control systems. Actually, the study of the local behavior of nondifferentiable objects is accomplished in the framework of *nonsmooth analysis* whose origin goes back in the early 1960's, when control theorists and nonlinear programmers attempted to deal with necessary optimality conditions for problems with nonsmooth data or with nonsmooth functions (such as the pointwise maximum of several smooth functions) that arise even in many problems with smooth data. Since then, nonsmooth analysis has come to play an important role in functional analysis, optimization, mechanics and plasticity, differential equations (as in the theory of viscosity solutions), control theory etc, becoming an active and fruitful area of mathematics.

One of the most important topics in nonsmooth analysis is the study of different kinds of tangent cones and normal cones to arbitrary sets. It is also worth mentioning here that the most successfully construction used in this framework turns out to be the so-called *contingent cone*, independently introduced in 1930 by Bouligand [5] and by Severi [18] in the context of contingent equations and differential geometry. Later, under the name of *cone of variations admissible by equality constrints*, the same cone was rediscovered and used in optimization theory by Dubovitskii and Milyutin [7, 8] (for more details about related tangential constructions we refer the reader to Aubin and Frankowska [2]). On the other hand, it is interesting to observe that there is a close relationship between the Dini-Hadamard directional derivative and the contingent cone (see [1, 15]). Thus, exploring such well known results but also a variational description available for the Dini-Hadamard subdifferential of calm functions (see [12]), we are able to provide the main result of the paper, a nice characterization of the contingent normal cone (the polar cone to the contingent one) via a sort of directional limes superior. Moreover, we even show that this kind of description holds also true for the Dini-Hadamard-like normal cone and for the Dini-Hadamard one. Finally, we study the relationship between various kinds of geometrical and analytical subdifferential constructions, pointing out the key role of a decoupled construction in characterizing the Dini-Hadamard-like subdifferential.

2. Preliminary notions and results

Consider a Banach space X and its topological dual space X^*. We denote the *open ball* with center $\overline{x} \in X$ and radius $\delta > 0$ in X by $B(\overline{x}, \delta)$, while $\overline{B}_X$ and S_X stand for the *closed unit ball* and the *unit sphere* of X, respectively. Having a set $C \subseteq X$, $\delta_C : X \to \overline{\mathbb{R}} \cup \{+\infty\}$, defined by $\delta_C(x) = 0$ for $x \in C$ and $\delta_C(x) = +\infty$, otherwise, denotes its *indicator function*. Given a function $f : X \to \overline{\mathbb{R}}$ which is finite at $\overline{x}$, we usually denote by $\text{epi}f = \{(x, \alpha) \in X \times \mathbb{R} : \ f(x) \leq \alpha\}$ the *epigraph* of f.

One of the most attractive constructions,

$$d^{DH} f(\overline{x}; h) := \liminf_{\substack{u \to h \\ t \downarrow 0}} \frac{f(\overline{x} + tu) - f(\overline{x})}{t} \tag{2.1}$$

which appeared in the 1970's was called *lower semiderivative* by Penot [15], *contingent derivative/epiderivative* by Aubin [1], *lower Dini* (or *Dini-Hadamard*) *directional derivative* by Ioffe [9, 10] and *subderivative* by Rockafellar and Wets [17].

Since in the case of real functions, the Dini-Hadamard directional derivative goes back to the classical *derivative numbers* by Dini [6], in general it can be described in a geometrical way via the contingent cone as follows

$$d^{DH} f(\overline{x}; h) = \inf\{\alpha \in \mathbb{R} : (h, \alpha) \in T((\overline{x}, f(\overline{x})); \text{epi}f)\}, \tag{2.2}$$

where for a given set $C \subset X$ with $\overline{x} \in C$,

$$T(\overline{x}; C) := \text{Limsup}_{t \downarrow 0} \frac{C - \overline{x}}{t}$$

denotes the *contingent* (or the *Bouligand*) *cone* to C at $\overline{x}$, equivalently described as the collection of those $v \in X$ such that there are sequences $(x_n) \subset C$ and $(\alpha_n) \subset \mathbb{R}_+$

with the property that $(x_n) \to \overline{x}$ and $(\alpha_n(x_n - \overline{x})) \to v$ as $n \to \infty$. Note also here that the contingent cone can be viewed (see [1]) in the following way

$$T(\overline{x}; C) = \bigcap_{\substack{\varepsilon>0 \\ \delta>0}} \bigcup_{t\in(0,\delta)} (t^{-1}(C - \overline{x}) + \varepsilon \overline{B}_X), \tag{2.3}$$

i.e. the set of all vectors v so that one can find sequences $t_n \downarrow 0, u_n \to v$ with the property that $\overline{x} + t_n u_n \in C$ for all $n \in \mathbb{N}$.

Consequently, the Dini-Hadamard subdifferential of f at $\overline{x}$, that is

$$\partial^{DH} f(\overline{x}) := \{x^* \in X^* : \langle x^*, h\rangle \leq d^{DH} f(\overline{x}; h) \text{ for all } h \in X\}, \tag{2.4}$$

can also be expressed by means of the polar cone to the contingent one, i.e.

$$\partial^{DH} f(x) = \{x^* \in X^* : (x^*, -1) \in T^\circ((x, f(x)); \text{epi} f)\}, \tag{2.5}$$

where given a subcone $K \subseteq X$, its polar cone K° is defined by

$$K^\circ := \{x^* \in X^* : \sup_{x\in K} \langle x^*, x\rangle \leq 0\}.$$

It is worth emphasizing here that, accordingly to [1, 15], if f is the indicator function of $C \subseteq X$ and $\overline{x} \in C$, then $d^{DH} f(\overline{x}; h)$ (viewed as a function in the second variable) is the indicator function of the contingent cone $T(\overline{x}; C)$ and hence

$$\partial^{DH} \delta_C(x) = T^\circ(x; C). \tag{2.6}$$

Let us also remark that $d^{DH} f(\overline{x}; \cdot)$ is in general not convex (as it is actually *typical* concave in some particular instances, see Bessis and Clarke [3]), but lower semicontinuous. However, the Dini-Hadamard subdifferential of f at $\overline{x}$ is always a convex set.

Similarly, following the two steps procedure of constructing the Dini-Hadamard subdifferential, but employing a directional convergence in place of the usual one, we can define (see [12]) the *Dini-Hadamard-like subdifferential* of f at $\overline{x}$, i.e. the following set

$$\widetilde{\partial} f(\overline{x}) := \{x^* \in X^* : \langle x^*, h\rangle \leq \widetilde{D}_d f(\overline{x}; h) \ \ \forall h \in X \ \forall d \in X \setminus \{0\}\}, \tag{2.7}$$

where

$$\widetilde{D}_d f(\overline{x}; h) := \sup_{\delta>0} \inf_{\substack{u\in B(h,\delta)\cap(h+[0,\delta]\cdot B(d,\delta)) \\ t\in(0,\delta)}} \frac{f(\overline{x} + tu) - f(\overline{x})}{t}, \tag{2.8}$$

labeled as the *Dini-Hadamard-like directional derivative* of f at $\overline{x}$ in the direction $h \in X$ through $d \in X \setminus \{0\}$, extend somehow the Dini-Hadamard directional derivative, while the essential idea was inspired by the fruitful relationship between sponges and directionally convergent sequences (we refer the reader to [16, Lemma 2.1]). As usual, in case $|f(\overline{x})| = \infty$, we set $\partial^{DH} f(\overline{x}) = \widetilde{\partial} f(\overline{x}) = \emptyset$.

Actually, introduced by Treiman [19], the sponge turns out to be very useful for characterizing the Dini-Hadamard subdifferential. Actually, the idea behind this concept was the fact that a neighborhood is in general not broad enough to characterize this kind of subdifferential constructions.

Definition 2.1. *A set $S \subseteq X$ is said to be a* sponge *around $\overline{x} \in X$ if for all $h \in X \setminus \{0\}$ there exist $\lambda > 0$ and $\delta > 0$ such that $\overline{x} + [0, \lambda] \cdot B(h, \delta) \subseteq S$.*

In fact, as one can easily observe from the definition above, the singular point 0 is ignored. Furthermore, every neighborhood of a point $\overline{x} \in X$ is also a sponge around $\overline{x}$, but the converse is not true (see for instance [4, Example 2.2]). However, in case S is a convex set or X is a finite dimensional space (here one can make use of the fact that the unit sphere is compact), then S is also a neighborhood of $\overline{x}$.

As regards the links between the two subdifferentials above, one can easily observe that the following inclusion

$$\partial^{DH} f(\overline{x}) \subseteq \widetilde{\partial} f(\overline{x}) \tag{2.9}$$

holds always true, but it can be even strict (see the discussion after [12, Theorem 3.1]). However, in case X is finite dimensional or the function f is calm at $\overline{x}$, i.e. there exists $c \geq 0$ and $\delta > 0$ such that $f(x) - f(\overline{x}) \geq -c\|x - \overline{x}\|$ for all $x \in B(\overline{x}, \delta)$, in particular if f is locally Lipschitz at $\overline{x}$, then the Dini-Hadamard subdifferential agrees with the Dini-Hadamard-like one.

Although the Dini-Hadamard-like subdifferential as well as the Dini-hadamard one (if additionally a calmness assumption is fulfilled, too) of a given function $f : X \to \overline{\mathbb{R}}$ at a point $\overline{x}$ with $|f(\overline{x})| < +\infty$ can be described via the following variational description (see [12, Theorem 3.1])

$$\begin{aligned} \widetilde{\partial} f(\overline{x}) := \{x^* \in X^* : \forall \varepsilon > 0 \ \exists S \text{ a sponge around } \overline{x} \text{ such that } \forall x \in S \\ f(x) - f(\overline{x}) \geq \langle x^*, x - \overline{x} \rangle - \varepsilon\|x - \overline{x}\|\}, \end{aligned} \tag{2.10}$$

or, in other words,

$$x^* \in \widetilde{\partial} f(\overline{x}) \Leftrightarrow \begin{array}{c} \forall \varepsilon > 0 \ \forall u \in S_X \ \exists \delta > 0 \text{ such that} \\ \forall s \in (0, \delta) \ \forall v \in B(u, \delta) \text{ for } x := \overline{x} + sv \text{ one has} \\ f(x) - f(\overline{x}) \geq \langle x^*, x - \overline{x} \rangle - \varepsilon\|x - \overline{x}\|. \end{array} \tag{2.11}$$

it seems that the following decoupled construction, introduced in [13], was designed not only to derive exact subdifferential formulae for Dini-Hadamard and Dini-Hadamard-like subgradients (see for instance [14]), as it was especially introduced from the necessity to deal with appropriate derivative-like constructions on product spaces. The reason is that, due to the very special structure of the spongious sets, the cartesian product of two sponges is, in general, not a sponge.

Thus, given a function $f : X \times Y \to \overline{\mathbb{R}}$ defined on a product of two Banach spaces X and Y, the following subdifferential construction

$$\begin{aligned} \widetilde{\partial}_{r} f(\overline{x}, \overline{y}) := \{(x^*, y^*) \in X^* \times Y^* : \forall \varepsilon > 0 \ \exists S_1 \text{ a sponge around } \overline{x}, \\ \exists S_2 \text{ a sponge around } \overline{y} \text{ such that } \forall (x, y) \in S_1 \times S_2 \\ f(x, y) - f(\overline{x}, \overline{y}) \geq \langle (x^*, y^*), (x - \overline{x}, y - \overline{y}) \rangle - \varepsilon\|(x - \overline{x}, y - \overline{y})\|\}, \end{aligned} \tag{2.12}$$

denotes the *decoupled Dini-Hadamard-like (lower) subdifferential* of f at $(\overline{x}, \overline{y})$, where $X \times Y$ is a Banach space with respect to the *sum norm*

$$\|(x, y)\| := \|x\| + \|y\|$$

imposed on $X \times Y$ unless otherwise stated. It is interesting to observe that the last notion is actually quite different than the Dini-Hadamard-like one, since, at first sight, neither $\widetilde{\partial} f(\overline{x}, \overline{y}) \not\subseteq \widetilde{\partial}_{\mathfrak{r}} f(\overline{x}, \overline{y})$ nor the opposite inclusion $\widetilde{\partial}_{\mathfrak{r}} f(\overline{x}, \overline{y}) \not\subseteq \widetilde{\partial} f(\overline{x}, \overline{y})$ is valid.

3. Relationships between subgradients and normal cones

Let us begin our exposure with a few remarks. First of all, following relation (2.6) above, one can easily observe that the Dini-Hadamard normal cone to a set $C \subseteq X$ at $\overline{x} \in C$, naturally introduced via the Dini-Hadamard subdifferential to the indicator function, can also be expressed via the polar cone to the contingent one, in fact the contingent normal cone, i.e.

$$N^{DH}(\overline{x}; C) := \partial^{DH} \delta_C(\overline{x}) = T^\circ(\overline{x}; C) := N(\overline{x}; C). \tag{3.1}$$

On the other hand, relations (2.5) and (3.1) clearly yield

$$\partial^{DH} f(\overline{x}) = \{x^* \in X^* : (x^*, -1) \in N^{DH}((\overline{x}, f(\overline{x})); \text{epi} f))\}. \tag{3.2}$$

In fact, the latter actually says that the analytic Dini-Hadamard subdifferential ∂_a^{DH}, as introduced in (2.4), always agrees with the geometrical one, ∂_g^{DH}, as defined in (3.2). However, this is no longer the case for the Dini-Hadamard-like subdifferential. The reason is that, for this particular construction, one can state a similar result like in (3.2) only by making use of the corresponding decoupled one.

Proposition 3.1. *Let $f : X \to \overline{\mathbb{R}}$ be a given function finite at $\overline{x}$. Then*

$$\widetilde{\partial} f(\overline{x}) = \{x^* \in X^* : (x^*, -1) \in \widetilde{N}_{\mathfrak{r}}((\overline{x}, f(\overline{x})); \textit{epi} f)\}, \tag{3.3}$$

where $\widetilde{N}_{\mathfrak{r}}((\overline{x}, \overline{y}); C) := \widetilde{\partial}_{\mathfrak{r}} \delta((\overline{x}, \overline{y}); C)$ stands for the decoupled Dini-Hadamard-like normal cone *to $C \subset X \times Y$ at $(\overline{x}, \overline{y})$.*

Proof. To justify the inclusion " $\subseteq$ ", we have to show that

$$(x^*, -1) \in \widetilde{\partial}_{\mathfrak{r}} \delta((\overline{x}, f(\overline{x})); \text{epi} f),$$

whenever $x^* \in \widetilde{\partial} f(\overline{x})$. To proceed, pick any $\varepsilon > 0$ and observe that there exists a sponge S_1 around $\overline{x}$ such that for all $x \in S_1$

$$f(x) - f(\overline{x}) \geq \langle x^*, x - \overline{x} \rangle - \varepsilon \|x - \overline{x}\|. \tag{3.4}$$

Further, for any $(x, y) \in (S_1 \times \overline{\mathbb{R}}) \setminus \text{epi} f$ the following estimate

$$\delta_{\text{epi} f}(x, y) \geq \langle (x^*, -1), (x - \overline{x}, y - f(\overline{x})) \rangle - \varepsilon(\|x - \overline{x}\| + \|y - f(\overline{x})\|)$$

holds true. Thus, it remains us to show the latter inequality for an arbitrary $(x, y) \in \text{epi} f$.

Indeed, relation (3.4) above leads to

$$\begin{aligned} \langle x^*, x - \overline{x} \rangle - \varepsilon(\|x - \overline{x}\| + \|y - f(\overline{x})\|) &\leq \langle x^*, x - \overline{x} \rangle - \varepsilon \|x - \overline{x}\| \\ &\leq f(x) - f(\overline{x}) \leq y - f(\overline{x}), \end{aligned}$$

and consequently $(x^*, -1) \in \widetilde{N}_{\mathfrak{r}}((\overline{x}, f(\overline{x})); \text{epi} f)\}$, which ends the proof of the first inclusion.

For the reverse one, take an arbitrary element on the right-hand side of (3.3) and assume on the contrary that $x^* \notin \widetilde{\partial} f(\overline{x})$. Then there exists $\gamma > 0$ such that for any natural number k one can find $x_k \in S_1 \cap B(\overline{x}, \frac{1}{k})$ with the following property

$$f(x_k) - f(\overline{x}) - \langle x^*, x_k - \overline{x}\rangle + \gamma\|x - \overline{x}\| < 0,$$

where (obviously) $x_k \neq \overline{x}$. Hence, taking $a_k := f(\overline{x}) + \langle x^*, x_k - \overline{x}\rangle - \gamma\|x_k - \overline{x}\|$ one clearly gets $(x_k, a_k) \in \operatorname{epi} f$ for all $k \in \mathbb{N}$ and additionally $(a_k)_k \to f(\overline{x})$.

Further

$$\begin{aligned}\frac{\langle x^*, x_k - \overline{x}\rangle - (a_k - f(\overline{x}))}{\|(x_k, a_k) - (\overline{x}, f(\overline{x}))\|} &= \frac{\gamma\|x_k - \overline{x}\|}{\|(x_k - \overline{x}, \langle x^*, x_k - \overline{x}\rangle - \gamma\|x_k - \overline{x}\|)\|}\\ &\geq \frac{\gamma}{1 + \|x^*\| + \gamma} > 0.\end{aligned}$$

On the other hand, taking into account the fact that

$$(x^*, -1) \in \widetilde{N}_r((\overline{x}, f(\overline{x})); \operatorname{epi} f),$$

for a fixed $\gamma' \in (0, \frac{\gamma}{1+\|x^*\|+\gamma})$ there exist two sponges S_1 around $\overline{x}$ and S_2 around $f(\overline{x})$ (which is in fact a neighborhood) such that for any $(x, y) \in S_1 \times S_2$ one has

$$\delta_{\operatorname{epi} f}(x, y) \geq \langle (x^*, -1), (x - \overline{x}, y - f(\overline{x}))\rangle - \gamma'\|(x - \overline{x}, y - f(\overline{x}))\|.$$

Consequently, for all large enough $k \in \mathbb{N}$, $(x_k, a_k) \in S_1 \times S_2 \cap \operatorname{epi} f$, and hence

$$\gamma' \geq \frac{\langle x^*, x_k - \overline{x}\rangle - (a_k - f(\overline{x}))}{\|(x_k, a_k) - (\overline{x}, f(\overline{x}))\|},$$

which is a contradiction. Finally, $x^* \in \widetilde{\partial} f(\overline{x})$ and the proof is complete. □

A valuable characterization of the contingent normal cone, similar to the one that exist for the Fréchet normal cone (see for instance [11, Definition 1.1]), but by replacing the usual convergence with a directional one, will be provided in the sequel. In fact, one can say more.

Theorem 3.2. *Let C be a nonempty subset of X and $\overline{x} \in C$. Then*

$$\begin{aligned} N(\overline{x}; C) &= N^{DH}(\overline{x}; C) = \widetilde{N}(\overline{x}; C)\\ &= \{x^* \in X^* : \inf_{\delta \in (0,1)} \sup_{x \in (\overline{x} + (0,\delta)\cdot B(u,\delta)) \cap C} \frac{\langle x^*, x - \overline{x}\rangle}{\|x - \overline{x}\|} \leq 0 \ \forall u \in S_X\}, \end{aligned} \tag{3.5}$$

where $\widetilde{N}(\overline{x}; C) := \widetilde{\partial}\delta_C(\overline{x})$.

Proof. First, it is sufficient to take into account relation (3.1) above and also to observe that the indicator function $\delta_C(\overline{x})$ is calm at $\overline{x}$, in order to justify the equalities

$$N(\overline{x}; C) = N^{DH}(\overline{x}; C) = \widetilde{N}(\overline{x}; C).$$

Thus, it remains us to show only that

$$\widetilde{N}(\overline{x}; C) = \{x^* \in X^* : \inf_{\delta \in (0,1)} \sup_{x \in (\overline{x} + (0,\delta)\cdot B(u,\delta)) \cap C} \frac{\langle x^*, x - \overline{x}\rangle}{\|x - \overline{x}\|} \leq 0 \ \forall u \in S_X\}.$$

So, let us justify only the inclusion " $\subseteq$ ", since the reverse one can be done similarly, but by reversing the steps ordering. Take $x^* \in \widetilde{N}(\overline{x}; C)$ and consider arbitrary $\varepsilon > 0$

and $u \in S_X$. Then, if we take also into account the variational description (2.11), one gets $\delta \in (0,1)$ such that for all $x \in (\overline{x} + (0,\delta) \cdot B(u,\delta)) \cap C$

$$0 \geq \langle x^*, x - \overline{x} \rangle - \varepsilon \|x - \overline{x}\|.$$

Thus,

$$\sup_{x \in (\overline{x} + (0,\delta) \cdot B(u,\delta)) \cap C} \frac{\langle x^*, x - \overline{x} \rangle}{\|x - \overline{x}\|} \leq \varepsilon$$

and consequently

$$\inf_{\delta \in (0,1)} \sup_{x \in (\overline{x} + (0,\delta) \cdot B(u,\delta)) \cap C} \frac{\langle x^*, x - \overline{x} \rangle}{\|x - \overline{x}\|} \leq \varepsilon.$$

Finally, this gives, by passing to the limit as $\varepsilon \downarrow 0$, that x^* satisfies the inequality in the right-hand side of (3.6) and the proof of the theorem is complete. □

Finally, we illustrate the relationship between the analytic Dini-Hadamard-like subdifferential and the geometrical one, concluding that this kind of construction doesn't follows at all the behavior of the Fréchet subdifferential (see, for instance, the results in [11, Section 1.3]).

Corollary 3.3. *Let $f : X \to \overline{\mathbb{R}}$ be an arbitrary function and $\overline{x} \in X$. Then*

$$\widetilde{\partial}_g f(\overline{x}) \subsetneq \widetilde{\partial}_a f(\overline{x}), \tag{3.6}$$

where $\widetilde{\partial}_g f(\overline{x}) := \{x^* \in X^* : (x^*, -1) \in \widetilde{N}((\overline{x}, f(\overline{x})); epi f)\}$ stands for the geometric Dini-Hadamard-like subdifferential of f at $\overline{x}$, while $\widetilde{\partial}_a f(\overline{x}) := \widetilde{\partial} f(\overline{x})$ denotes the analytical one.

Proof. It is easy to check that

$$\partial^{DH} f(\overline{x}) = \{x^* \in X^* : (x^*, -1) \in \widetilde{N}((\overline{x}, f(\overline{x})); \text{epi} f)\}$$

if we take into account relation (3.2) and Theorem 3.2 above. Hence, in view of Proposition 3.1 the following inclusion

$$\widetilde{N}((\overline{x}, f(\overline{x})); \text{epi} f)\} \subseteq \widetilde{N}_t((\overline{x}, f(\overline{x})); \text{epi} f)\},$$

holds true, but it can be even strict, since $\partial^{DH} f(\overline{x}) \subsetneq \widetilde{\partial} f(\overline{x})$. Consequently, $\widetilde{\partial}_g f(\overline{x}) = \partial^{DH} f(\overline{x}) \subsetneq \widetilde{\partial}_a f(\overline{x})$ and the proof of the corollary is complete. □

Finally, let us illustrate the relationships between various subgradients studied above, which are in fact direct consequences of the discussions made in this subsection.

Corollary 3.4. *Let $f : X \to \overline{\mathbb{R}}$ be an arbitrary function and $\overline{x} \in X$. Then*

$$\partial_a^{DH} f(\overline{x}) = \partial_g^{DH} f(\overline{x}) = \widetilde{\partial}_g f(\overline{x}) \subsetneq \widetilde{\partial}_a f(\overline{x}), \tag{3.7}$$

while the equalities hold true in case f is calm at $\overline{x}$.

References

[1] Aubin, J.-P., *Contingent derivatives of set-valued maps and existence of solutions to nonlinear inclusions and differential inclusions*, in Mathematical Analysis and Applications, edited by L. Nachbin, Academic Press, New York, 1981, 159–229.

[2] Aubin, J.-P., Frankowska, H., *Set-Valued Analysis*, Birkhäuser, Boston, Massachusetts, 1990.

[3] Bessis, D.N., Clarke, F.H., *Partial subdifferentials, derivatives and Rademacher's theorem*, Transactions of the American Mathematical Society, **351(7)**(1999), 2899–2926.

[4] Boţ, R.I., Nechita, D.-M., *On the Dini-Hadamard subdifferential of the difference of two functions*, Journal of Global Optimization, **50(3)**(2011), 485–502.

[5] Bouligand, G., *Sur les surfaces dépourvues de points hyperlimits*, Annales de la Société Polonaise de Mathematique, **9**(1930), 32–41.

[6] Dini, U., *Fondamenti per la Teoria delle Funzioni di Variabili Reali*, Pisa, Italy, 1878.

[7] Dubovitskii, A.Y., Milyutin, A.A., *Extremum problems in the presence of constraints*, Doklady Akademii Nauk SSSR, **149**(1963), 759–762.

[8] Dubovitskii, A.Y., Milyutin, A.A., *Extremum problems in the presence of restrictions*, USSR Computational Mathematics and Mathematical Physics, **5**(1965), 1–80.

[9] Ioffe, A.D., *Calculus of Dini subdifferentials of functions and contingent derivatives of set-valued maps*, Nonlinear Analysis, Theory, Methods and Applications, **8**(1984), 517–539.

[10] Ioffe, A.D., *Codirectional compactness, metric regularity and subdifferential calculus*,in Constructive, Experimental and Nonlinear Analysis, edited by M. Théra, Canadian Mathematical Society Conference Proceedings, **27**(2000), 123–163.

[11] Mordukhovich, B.S., *Variational Analysis and Generalized Differentiation, I. Basic Theory*, Series of Comprehensive Studies in Mathematics, Vol. 330, Springer-Verlag, Berlin Heidelberg, 2006.

[12] Nechita, D.-M., *On an extension of the Dini-Hadamard subdifferential*, Automation Computers Applied Mathematics, **19**(2010), 173–181.

[13] Nechita, D.-M., *A smooth variational description involving Dini-Hadamard-like decoupled subgradients*, submitted.

[14] Nechita, D.-M., *Some results concerning chain rules for Dini-Hadamard constructions*, submitted.

[15] Penot, J.-P., *Calcul sous-différentiel et optimization*, Journal of Fundamental Analysis, **27**(1978), 248–276.

[16] Penot, J.-P., *Favorable classes of mappings and multimappings in nonlinear analysis and optimization*, Journal of Convex Analysis, **3(1)**(1996), 97–116.

[17] Rockafellar, R.T., Wets, R.J-B., *Variational Analysis*, Springer-Verlag, New York, 1997.

[18] Severi, F., *Su alcune questioni di topologia infinitesimale*, Annales de la Société Polonaise des Mathématique, **9**(1930), 97–108.

[19] Treiman, J.S., *Clarke's gradient and epsilon-subgradients in Banach spaces*, Transactions of the American Mathematical Society, **294**(1986), 65–78.

Delia-Maria Nechita
"Babeş-Bolyai" University
Faculty of Mathematics and Computer Sciences
1, Kogălniceanu Street
400084 Cluj-Napoca, Romania
e-mail: delia-nechita@nikolai.ro