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On a subalgebra of $L^1_w(G)$

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Abstract. Let G be a locally compact abelian group with Haar measure. We define the spaces $B_{1,w}(p,q) = L^1_w(G) \cap (L^p, \ell^q)(G)$ and discuss some properties of these spaces. We show that $B_{1,w}(p,q)$ is an $S_w(G)$ space. Furthermore we investigate compact embeddings and the multipliers of $B_{1,w}(p,q)$.

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1. Introduction

Let G be a locally compact abelian group with Haar measure μ. An amalgam space $(L^p, \ell^q)(G)$ $(1 \leq p, q \leq \infty)$ is a Banach space of measurable (equivalence classes of) functions on G which belong locally to L^p and globally to ℓ^q. Several authors have introduced special cases of amalgams. Among others N. Wiener [28], [29], P. Szeptycki [25], T. S. Liu, A. Van Rooij and J. K. Wang [19], H. E. Krogstad [17] and H. G. Feichtinger [8]. For a historical background of amalgams see [11]. The first systematic study of amalgams on the real line was undertaken by F. Holland [16]. In 1979 J. Stewart [24] extended the definition of Holland to locally compact abelian groups using the Structure Theorem for locally compact groups.

For $1 \leq p < \infty$, the spaces $B^p(G) = L^1(G) \cap L^p(G)$ is a Banach algebra with respect to the norm $\|.\|_{B^p(G)}$ defined by $\|f\|_{B^p(G)} = \|f\|_1 + \|f\|_p$ and usual convolution product. The Banach algebras $B^p(G)$ have been studied by C. R. Warner [27], L. Y. H. Yap [30], and others. L. Y. H. Yap [31] extended some of the results on $B^p(G)$ to the Segal algebras

$$B(p,q)(G) = L^1(G) \cap L(p,q)(G),$$

where $L(p,q)(G)$ is Lorentz spaces. The purpose of this paper is to discuss some properties of the spaces $B_{1,w}(p,q) = L^1_w(G) \cap (L^p, \ell^q)(G)$. Also we investigate the spaces of all multipliers from $L^1_w(G)$ into $B_{1,w}(p,q)$ and $(B_{1,w}(p,q))^*$ over $L^1_w(G)$.

2. Preliminaries

The translation operator T_y is given by $T_y f(x) = f(x-y)$ for $x \in G$. $(B, \|.\|_B)$ is called (strongly) translation invariant if one has $T_y f \in B$ $\left(\text{and } \|T_y f\|_B = \|f\|_B \right)$ for all $f \in B$ and $y \in G$. A space $(B, \|.\|_B)$ is called strongly character invariant if one has $M_t f(x) = \langle x, t \rangle f(x) \in B$ and $\|M_t f\|_B = \|f\|_B$ for all $f \in B$, $x \in G$ and $t \in \widehat{G}$, where $\widehat{G}$ is the dual group of G. A Banach function space (shortly BF-space) on G is a Banach space $(B, \|.\|_B)$ of measurable functions which is continuously embedded into $L^1_{loc}(G)$, i.e. for any compact subset $K \subset G$ there exists some constant $C_K > 0$ such that $\|f\chi_K\|_1 \le C_K \|f\|_B$ for all $f \in B$. A BF-space is called solid if $g \in B$, $f \in L^1_{loc}(G)$ and $|f(x)| \le |g(x)|$ locally almost every where (shortly l.a.e) implies $f \in B$ and $\|f\|_B \le \|g\|_B$. It is easy to see that $(B, \|.\|_B)$ is solid iff it is a $L^\infty-$module. $C_c(G)$ will denote the linear space of continuous functions on G, which have compact support.

Definition 2.1. A strictly positive, continous function w satisfying $w(x) \ge 1$ and $w(x+y) \le w(x)w(y)$ for all $x, y \in G$ will be called a weight function. Let $1 \le p < \infty$. Then the weighted Lebesgue space $L^p_w(G) = \{f : fw \in L^p(G)\}$ is a Banach space with norm $\|f\|_{p,w} = \|fw\|_p$ and its dual space $L^{p'}_{w^{-1}}(G)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover, if $1 < p < \infty$, then $L^p_w(G)$ is a reflexive Banach space. Particularly, for $p = 1$, $L^1_w(G)$ is a Banach algebra under convolution, called a Beurling algebra. It is obvious that $\|.\|_1 \le \|.\|_{1,w}$ and $L^1_w(G) \subset L^1(G)$. We say that $w_1 \prec w_2$ if and only if there exists a $C > 0$ such that $w_1(x) \le C w_2(x)$ for all $x \in G$. Two weight functions are called equivalent and written $w_1 \approx w_2$, if $w_1 \prec w_2$ and $w_2 \prec w_1$. It is known that $L^p_{w_2}(G) \subset L^p_{w_1}(G)$ iff $w_1 \prec w_2$. A weight function w is said to satisfy the Beurling-Domar (shortly BD) condition, if

$$\sum_{n \ge 1} n^{-2} \log w(nx) < \infty$$

for all $x \in G$ [6].

Definition 2.2. Let V and W be two Banach modules over a Banach algebra A. Then a multiplier from V into W is a bounded linear operator T from V into W, which commutes with module multiplication, i.e. $T(av) = aT(v)$ for $a \in A$ and $v \in V$. We denote by $Hom_A(V, W)$ the space of all multipliers from V into W. Also we write $Hom_A(V, V) = Hom_A(V)$. It is known that

$$Hom_A(V, W^*) \cong (V \otimes_A W)^*,$$

where W^* is dual of W and $V \otimes_A W$ is the $A-$module tensor product of V and W [Corollary 2.13, 21].

We will denote by $M(G)$ the space of bounded regular Borel measures on G. We let

$$M(w) = \left\{ \mu \in M(G) : \int_G w d|\mu| < \infty \right\}.$$

It is known that the space of multipliers from $L^1_w(G)$ to from $L^1_w(G)$ is homeomorphic to $M(w)$ [12].

A kind of generalization of Segal algebra was defined in [3], as follows:

Definition 2.3. Let $S_w(G) = S_w$ be a subalgebra of $L^1_w(G)$ satisfying the following conditions:

S1) S_w is dense in $L^1_w(G)$.

S2) S_w is a Banach algebra under some norm $\|.\|_{S_w}$ and invariant under translations.

S3) $\|T_a f\|_{S_w} \leq w(a)\|f\|_{S_w}$ for all $a \in G$ and for each $f \in S_w$.

S4) If $f \in S_w$, then for every $\varepsilon > 0$ there exists a neighborhood U of the identity element of G such that $\|T_y f - f\|_{S_w} < \varepsilon$ for all $y \in U$.

S5) $\|f\|_{1,w} \leq \|f\|_{S_w}$ for all $f \in S_w$.

Definition 2.4. We denote by $L^p_{loc}(G)$ $(1 \leq p \leq \infty)$ the space of (equivalence classes of) functions on G such that f restricted to any compact subset E of G belongs to $L^p(G)$. Let $1 \leq p, q \leq \infty$. The amalgam of L^p and ℓ^q on the real line is the normed space

$$(L^p, \ell^q) = \left\{ f \in L^p_{loc}(\mathbb{R}) : \|f\|_{pq} < \infty \right\},$$

where

$$\|f\|_{pq} = \left[\sum_{n=-\infty}^{\infty} \left[\int_n^{n+1} |f(x)|^p \, dx \right]^{q/p} \right]^{1/q}. \tag{2.1}$$

We make the appropriate changes for p, q infinite. The norm $\|.\|_{pq}$ makes (L^p, ℓ^q) into a Banach space [16].

The following definition of $(L^p, \ell^q)(G)$ is due to J. Stewart [24]. By the Structure Theorem [Theorem 24.30, 15], $G = \mathbb{R}^a \times G_1$, where a is a nonnegative integer and G_1 is a locally compact abelian group which contains an open compact subgroup H. Let $I = [0,1)^a \times H$ and $J = \mathbb{Z}^a \times T$, where T is a transversal of H in G_1, i.e. $G_1 = \bigcup_{t \in T} (t + H)$ is a coset decomposition of G_1. For $\alpha \in J$ we define $I_\alpha = \alpha + I$, and therefore G is equal to the disjoint union of relatively compact sets I_α. We normalize μ so that $\mu(I) = \mu(I_\alpha) = 1$ for all α. Let $1 \leq p, q \leq \infty$. The amalgam space $(L^p, \ell^q)(G) = (L^p, \ell^q)$ is a Banach space

$$\left\{ f \in L^p_{loc}(G) : \|f\|_{pq} < \infty \right\},$$

where

$$\begin{aligned} \|f\|_{pq} &= \left[\sum_{\alpha \in J} \|f\|^q_{L^p(I_\alpha)} \right]^{1/q} \quad \text{if } 1 \leq p, q < \infty, \qquad (2.2) \\ \|f\|_{\infty q} &= \left[\sum_{\alpha \in J} \sup_{x \in I_\alpha} |f(x)|^q \right]^{1/q} \quad \text{if } p = \infty,\ 1 \leq q < \infty, \\ \|f\|_{p\infty} &= \sup_{\alpha \in J} \|f\|_{L^p(I_\alpha)} \quad \text{if } 1 \leq p < \infty,\ q = \infty. \end{aligned}$$

If $G = \mathbb{R}$, then we have $J = \mathbb{Z}$, $I_\alpha = [\alpha, \alpha + 1)$ and (2.2) becomes (2.1).

The amalgam spaces (L^p, ℓ^q) satisfy the following relations and inequalities [24]:

$$(L^p, \ell^{q_1}) \subset (L^p, \ell^{q_2}) \quad q_1 \leq q_2 \tag{2.3}$$

$$\left(L^{p_1},\ell^q\right)\subset\left(L^{p_2},\ell^q\right)\quad p_1\geq p_2 \tag{2.4}$$

$$\left(L^p,\ell^p\right)=L^p \tag{2.5}$$

$$\left(L^p,\ell^q\right)\subset L^p\cap L^q,\quad p\geq q \tag{2.6}$$

$$L^p\cup L^q\subset\left(L^p,\ell^q\right),\quad p\leq q \tag{2.7}$$

$$\|f\|_{pq_2}\leq\|f\|_{pq_1},\quad q_1\leq q_2 \tag{2.8}$$

$$\|f\|_{p_2q}\leq\|f\|_{p_1q},\quad p_1\geq p_2. \tag{2.9}$$

Note that $C_c(G)$ is included in all amalgam spaces. If $1\leq p,q<\infty$, then the dual space of (L^p,ℓ^q) is isometrically isomorphic to $\left(L^{p'},\ell^{q'}\right)$, where $1/p+1/p'=1/q+1/q'=1$.

Definition 2.5. Let A be a Banach algebra. A Banach space B is said to be a Banach $A-$module if there exists a bilinear operation $\cdot:A\times B\to B$ such that

(*i*) $(f\cdot g)\cdot h=f\cdot(g\cdot h)$ for all $f,g\in A$, $h\in B$.

(*ii*) For some constant $C\geq1$, $\|f\cdot h\|_B\leq C\|f\|_A\|h\|_B$ for all $f\in A$, $h\in B$ [7].

Theorem 2.6. If p,q,r,s are exponents such that $1/p+1/r-1=1/m\leq1$ and $1/q+1/s-1=1/n\leq1$, then

$$\left(L^p,\ell^q\right)*\left(L^r,\ell^s\right)\subset\left(L^m,\ell^n\right).$$

Moreover, if $f\in(L^p,\ell^q)$ and $g\in(L^r,\ell^s)$, then

$$\begin{aligned}\|f*g\|_{mn}&\leq 2^a\|f\|_{pq}\|g\|_{rs}\ \text{ if } m\neq1\\ \|f*g\|_{1n}&\leq 2^{2a}\|f\|_{1q}\|g\|_{1s}\end{aligned} \tag{2.10}$$

$([1],[2],[23])$.

Theorem 2.7. Let $1\leq p,q\leq\infty$. If for each $a\in G$ and $f\in(L^p,\ell^q)$, then

$$\|T_af\|_{pq}\leq2^a\|f\|_{pq},$$

i.e. the amalgam space (L^p,ℓ^q) is translation invariant $([23])$.

Theorem 2.8. Let $1\leq p,q<\infty$. Then the mapping $y\to T_y$ is continuous from G into (L^p,ℓ^q) $([23])$.

Now we use the fact that (L^p,ℓ^q) has an equivalent translation-invariant norm $\|.\|_{pq}^{\sharp}$. The following theorem was first introduced in [1].

Theorem 2.9. A function f belongs to (L^p,ℓ^q), $1\leq p,q\leq\infty$, iff the function $f^\sharp$ on G defined by

$$f^\sharp(x)=\|f\|_{L^p(x+E)}$$

belongs to $L^q(G)$. If $\|f\|_{pq}^\sharp=\left\|f^\sharp\right\|_q$, then

$$2^{-a}\|f\|_{pq}\leq\|f\|_{pq}^\sharp\leq2^a\|f\|_{pq},$$

where E is open precompact neighborhood of 0 and

$$\|f\|_{pq}^\sharp=\left[\int_G\|f\|_{L^p(x+E)}^q\,dx\right]^{1/q}$$

$([1],[23],[11])$.

Definition 2.10. A net $\{e_\alpha\}$ in a commutative, normed algebra A is an approximate identity, abbreviated a.i., if for all $a \in A$, $\lim_\alpha e_\alpha a = a$ in A.

Proposition 2.11. Let $1 \leq p, q < \infty$. If $\{e_\alpha\}$ is an a.i. in $L^1(G)$, then $\{e_\alpha\}$ is also an a.i. in (L^p, ℓ^q), i.e.

$$\lim_\alpha \|e_\alpha * f - f\|_{pq} = 0$$

for all $f \in (L^p, \ell^q)$ ([23]).

The proof the following Lemma is easy.

Lemma 2.12. Let $1 \leq p, q < \infty$. Let $\{f_n\}$ be a sequence in (L^p, ℓ^q) and $\|f_n - f\|_{pq} \to 0$, where $f \in (L^p, \ell^q)$. Then $\{f_n\}$ has a subsequence which converges pointwise almost everywhere to f.

3. The space $B_{1,w}(p, q)$

Let $1 \leq p, q < \infty$. We define the vector space $B_{1,w}(p, q) = L_w^1(G) \cap (L^p, \ell^q)(G)$ and equip this space with the sum norm

$$\|f\|_{pq}^{1,w} = \|f\|_{1,w} + \|f\|_{pq}$$

where $f \in B_{1,w}(p, q)$. In this section we will discuss some properties of this space.

Theorem 3.1. The space $\left(B_{1,w}(p, q), \|.\|_{pq}^{1,w}\right)$ is a Banach algebra with respect to convolution.

Proof. Let $\{f_n\}$ be a Cauchy sequence in $B_{1,w}(p, q)$. Clearly $\{f_n\}$ is a Cauchy sequence in $L_w^1(G)$ and (L^p, ℓ^q). Since $L_w^1(G)$ and (L^p, ℓ^q) are Banach spaces, then there exist $f \in L_w^1(G)$ and $g \in (L^p, \ell^q)$ such that $\|f_n - f\|_{1,w} \to 0$, $\|f_n - g\|_{pq} \to 0$. Hence there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ which convergence pointwise to f almost everywhere. Also we obtain $\|f_{n_k} - g\|_{pq} \to 0$ and there exists a subsequence $\left\{f_{n_{k_l}}\right\}$ of $\{f_{n_k}\}$ which convergence pointwise to g almost everywhere by Lemma 2.12. Therefore $f = g$ almost everywhere, $\|f_n - f\|_{pq}^{1,w} \to 0$ and $f \in B_{1,w}(p, q)$. That means $B_{1,w}(p, q)$ is a Banach space.

Let $f, g \in B_{1,w}(p, q)$ be given. Since $L_w^1(G)$ is a Banach algebra under convolution, then $f * g \in L_w^1(G)$ and

$$\|f * g\|_{1,w} \leq \|f\|_{1,w} \|g\|_{1,w}. \tag{3.1}$$

Since the amalgam space (L^p, ℓ^q) is a Banach $L^1(G)-$module by [23], then we write

$$\|f * g\|_{pq} \leq C \|f\|_1 \|g\|_{pq}, \tag{3.2}$$

where $C \geq 1$. By using (3.1), (3.2) and the definition of $\|.\|_{pq}^{1,w}$ we have

$$\begin{aligned}
\|f * g\|_{pq}^{1,w} &= \|f * g\|_{1,w} + \|f * g\|_{pq} \\
&\leq \|f\|_{1,w} \|g\|_{1,w} + C \|f\|_1 \|g\|_{pq} \\
&= C \|f\|_{1,w} \left(\|g\|_{1,w} + \|g\|_{pq}\right) \\
&\leq C \|f\|_{pq}^{1,w} \|g\|_{pq}^{1,w}.
\end{aligned}$$

□

Proposition 3.2. The space $\left(B_{1,w}(p,q), \|.\|_{pq}^{1,w}\right)$ is a solid BF-space on G.

Proof. Let $K \subset G$ be given a compact subset and $f \in B_{1,w}(p,q)$. Then we have

$$\int_K |f(x)|\, dx \le \|f\|_1 \le \|f\|_{pq}^{1,w}.$$

Let $f \in B_{1,w}(p,q)$ and $g \in L^\infty(G)$. Since $L_w^1(G)$ and (L^p, ℓ^q) are solid BF-space [9], then

$$\begin{aligned}\|fg\|_{pq}^{1,w} &= \|fg\|_{1,w} + \|fg\|_{pq} \\ &\le \|f\|_{1,w}\|g\|_\infty + \|f\|_{pq}\|g\|_\infty = \|f\|_{pq}^{1,w}\|g\|_\infty.\end{aligned}$$

This completes the proof. □

Proposition 3.3. (**i**) The space $B_{1,w}(p,q)$ is translation invariant and for every $f \in B_{1,w}(p,q)$ the inequality $\|T_a f\|_{pq}^{1,w} \le w(a)\|f\|_{pq}^{1,w}$ holds.

(**ii**) The mapping $y \to T_y f$ is continuous from G into $B_{1,w}(p,q)$ for every $f \in B_{1,w}(p,q)$.

Proof. (**i**) Let $f \in B_{1,w}(p,q)$. Then it is easy to show that $T_a f \in L_w^1(G)$ and $\|T_a f\|_{1,w} \le w(a)\|f\|_{1,w}$ for all $a \in G$. By Theorem 2.9, we write

$$(T_y f)^\sharp(x) = \|T_y f\|_{L^p(x+E)} = \|f\|_{L^p(x+y+E)} = f^\sharp(x+y) = T_{-y} f^\sharp(x).$$

This implies that

$$\|T_y f\|_{pq}^\sharp = \left\|(T_y f)^\sharp\right\|_q = \left\|T_{-y} f^\sharp\right\|_q = \left\|f^\sharp\right\|_q = \|f\|_{pq}^\sharp.$$

Hence we have

$$\|T_a f\|_{pq}^{1,w} \le w(a)\|f\|_{pq}^{1,w} + \|f\|_{pq}^\sharp \le w(a)\|f\|_{pq}^{1,w}.$$

(**ii**) Let $f \in B_{1,w}(p,q)$. Then $f \in L_w^1(G)$ and $f \in (L^p, \ell^q)$. It is well known that the translation operator is continuous from G into $L_w^1(G)$ ([10], [20]). Thus for any $\varepsilon > 0$, there exists a neighbourhood U_1 of unit element of G such that

$$\|T_y f - f\|_{1,w} < \frac{\varepsilon}{2} \tag{3.3}$$

for all $y \in U_1$. Also by using Theorem 2.8, there exists a neighbourhood U_2 of unit element of G such that

$$\|T_y f - f\|_{pq} < \frac{\varepsilon}{2} \tag{3.4}$$

for all $y \in U_2$. Let $U = U_1 \cap U_2$. By using (3.3) and (3.4), then we obtain

$$\begin{aligned}\|T_y f - f\|_{pq}^{1,w} &= \|T_y f - f\|_{1,w} + \|T_y f - f\|_{pq} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

for all $y \in U$. This completes the proof. □

Theorem 3.4. The space $B_{1,w}(p,q)$ is a S_w algebra.

Proof. We have already proved the some conditions in Theorem 3.1 and Proposition 3.3 for S_w algebra. We now prove that $B_{1,w}(p,q)$ is dense in $L^1_w(G)$. Since $C_c(G) \subset B_{1,w}(p,q)$ and $C_c(G)$ is dense in $L^1_w(G)$, then $B_{1,w}(p,q)$ is dense in $L^1_w(G)$. □

Proposition 3.5. The space $\left(B_{1,w}(p,q), \|.\|^{1,w}_{pq}\right)$ is strongly character invariant and the map $t \to M_t f$ is continuous from $\widehat{G}$ into $B_{1,w}(p,q)$ for all $f \in B_{1,w}(p,q)$.

Proof. The spaces $L^1_w(G)$ and (L^p, ℓ^q) are strongly character invariant and the map $t \to M_t f$ is continuous from $\widehat{G}$ into this spaces ([10], [22]). Hence the proof is completed. □

Proposition 3.6. $B_{1,w}(p,q)$ is a essential Banach $L^1_w(G)-$module.

Proof. Let $f \in B_{1,w}(p,q)$ and $g \in L^1_w(G)$. Since (L^p, ℓ^q) is an essential Banach $L^1(G)-$module, then we have

$$\begin{aligned} \|f*g\|^{1,w}_{pq} &= \|f*g\|_{1,w} + \|f*g\|_{pq} \\ &\leq \|f\|_{1,w}\|g\|_{1,w} + \|f\|_{pq}\|g\|_1 \\ &= \|f\|^{1,w}_{pq}\|g\|_{1,w}. \end{aligned}$$

Also, by using Proposition 2.11, then $\|e_\alpha * f - f\|^{1,w}_{pq} \to 0$. Hence $L^1_w(G)*B_{1,w}(p,q) = B_{1,w}(p,q)$ by Module Factorization Theorem [26]. This completes the proof. □

Consider the mapping Φ from $B_{1,w}(p,q)$ into $L^1_w(G)\times(L^p,\ell^q)$ defined by $\Phi(f) = (f,f)$. This is a linear isometry of $B_{1,w}(p,q)$ into $L^1_w(G)\times(L^p,\ell^q)$ with the norm

$$|||(f,f)||| = \|f\|_{1,w} + \|f\|_{pq}, \ (f \in B_{1,w}(p,q)).$$

Hence it is easy to see that $B_{1,w}(p,q)$ is a closed subspace of the Banach space $L^1_w(G)\times(L^p,\ell^q)$. Let

$$H = \{(f,f) : f \in B_{1,w}(p,q)\}$$

and

$$K = \left\{ \begin{array}{c} (\varphi,\psi) : (\varphi,\psi) \in L^\infty_{w^{-1}}(G) \times \left(L^{p'}, \ell^{q'}\right), \\ \int\limits_G f(x)\varphi(x)dx + \int\limits_G f(y)\psi(y)dy = 0, \text{ for all } (f,f) \in H \end{array} \right\},$$

where $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$.

The following Proposition is easily proved by Duality Theorem 1.7 in [18].

Proposition 3.7. The dual space $(B_{1,w}(p,q))^*$ of $B_{1,w}(p,q)$ is isomorphic to

$$L^\infty_{w^{-1}}(G) \times \left(L^{p'}, \ell^{q'}\right)/K.$$

Proposition 3.8. If p,q,r,s are exponents such that $1/p + 1/r - 1 = 1/m \leq 1$ and $1/q + 1/s - 1 = 1/n \leq 1$, then

$$B_{1,w}(p,q) * B_{1,w}(r,s) \subset B_{1,w}(m,n).$$

Moreover, if $f \in B_{1,w}(p,q)$ and $g \in B_{1,w}(r,s)$, then there exists a $C \geq 1$ such that

$$\|f*g\|^{1,w}_{mn} \leq C\|f\|^{1,w}_{pq}\|g\|^{1,w}_{rs}.$$

Proof. Let $f \in B_{1,w}(p,q)$ and $g \in B_{1,w}(r,s)$. By Theorem 2.6 we have

$$\begin{aligned}\|f * g\|_{mn}^{1,w} &= \|f * g\|_{1,w} + \|f * g\|_{mn} \\ &\leq \|f\|_{1,w}\|g\|_{1,w} + C\|f\|_{pq}\|g\|_{rs} \\ &\leq C\|f\|_{1,w}\|g\|_{rs}^{1,w} + C\|f\|_{pq}\|g\|_{rs}^{1,w} \\ &= C\|f\|_{pq}^{1,w}\|g\|_{rs}^{1,w}.\end{aligned}$$

Hence $B_{p,q}^{1}(G) * B_{r,s}^{1}(G) \subset B_{m,n}^{1}(G)$. □

4. Inclusions of the spaces $B_{1,w}(p,q)$

Proposition 4.1. (**i**) If $q_1 \leq q_2$ and $w_2 \prec w_1$, then $B_{1,w_1}(p,q_1) \subset B_{1,w_2}(p,q_2)$.
(**ii**) If $p_1 \geq p_2$ and $w_2 \prec w_1$, then $B_{1,w_1}(p_1,q) \subset B_{1,w_2}(p_2,q)$.

Proof. By using (2.8) and (2.9), then the proof is completed. □

Lemma 4.2. For any $f \in B_{1,w}(p,q)$ and $z \in G$ there exist constants $C_1(f), C_2(f) > 0$ such that

$$C_1(f)w(z) \leq \|T_z f\|_{pq}^{1,w} \leq C_2(f)w(z).$$

Proof. Let $f \in B_{1,w}(p,q)$. Then by Lemma 2.2 in [10], there exists a constant $C_1(f) > 0$ such that

$$C_1(f)w(z) \leq \|T_z f\|_{1,w}. \tag{4.1}$$

By using (4.1), we have

$$C_1(f)w(z) \leq \|T_z f\|_{1,w} + \|T_z f\|_{pq} = \|T_z f\|_{pq}^{1,w} \leq w(z)\|f\|_{pq}^{1,w}. \tag{4.2}$$

If we combine (4.1) and (4.2), we obtain the inequality

$$C_1(f)w(z) \leq \|T_z f\|_{pq}^{1,w} \leq C_2(f)w(z),$$

with $C_2(f) = \|f\|_{pq}^{1,w}$. □

The following lemma is easily proved by using the closed graph theorem.

Lemma 4.3. Let w_1 and w_2 be two weights. Then $B_{1,w_1}(p,q) \subset B_{1,w_2}(p,q)$ if and only if there exists a constant $C > 0$ such that $\|f\|_{pq}^{1,w_2} \leq C\|f\|_{pq}^{1,w_1}$ for all $f \in B_{1,w_1}(p,q)$.

Proposition 4.4. Let w_1 and w_2 be two weights. Then $B_{1,w_1}(p,q) \subset B_{1,w_2}(p,q)$ if and only if $w_2 \prec w_1$.

Proof. The sufficiency of condition is obvious. Suppose that $B_{1,w_1}(p,q) \subset B_{1,w_2}(p,q)$. By Lemma 4.2, there exist C_1, C_2, C_3 and $C_4 > 0$ such that

$$C_1 w_1(z) \leq \|T_z f\|_{pq}^{1,w_1} \leq C_2 w_1(z) \tag{4.3}$$

and

$$C_3 w_2(z) \leq \|T_z f\|_{pq}^{1,w_2} \leq C_4 w_2(z) \tag{4.4}$$

for $z \in G$. Since $T_z f \in B_{1,w_1}(p,q)$ for all $f \in B_{1,w_1}(p,q)$, then there exists a constant $C > 0$ such that

$$\|T_z f\|_{pq}^{1,w_2} \leq C\|T_z f\|_{pq}^{1,w_1} \tag{4.5}$$

by Lemma 4.3. If one using (4.3), (4.4) and (4.5),we obtain

$$C_3 w_2(z) \leq \|T_z f\|_{pq}^{1,w_2} \leq C \|T_z f\|_{pq}^{1,w_1} \leq C C_2 w_1(z).$$

That means $w_2 \prec w_1$. □

Corollary 4.5. Let w_1 and w_2 be two weights. Then $B_{1,w_1}(p,q) = B_{1,w_2}(p,q)$ if and only if $w_1 \approx w_2$.

Now by using the techniques in [14], we investigate compact embeddings of the spaces $B_{1,w}(p,q)$. Also we will take $G = \mathbb{R}^d$ with Lebesgue measure dx for compact embedding.

Lemma 4.6. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $B_{1,w}(p,q)$. If $\{f_n\}$ converges to zero in $B_{1,w}(p,q)$, then $\{f_n\}$ converges to zero in the vague topology (which means that

$$\int_{\mathbb{R}^d} f_n(x)k(x)dx \to 0$$

for $n \to \infty$ for all $k \in C_c(\mathbb{R}^d)$, see [4]).

Proof. Let $k \in C_c(\mathbb{R}^d)$. We write

$$\left| \int_{\mathbb{R}^d} f_n(x)k(x)dx \right| \leq \|k\|_\infty \|f_n\|_1 \leq \|k\|_\infty \|f_n\|_{pq}^{1,w}. \tag{4.6}$$

Hence by (4.6) the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges to zero in vague topology. □

Theorem 4.7. Let w, ν be two weights on $\mathbb{R}^d$. If $\nu \prec w$ and $\frac{\nu(x)}{w(x)}$ doesn't tend to zero in $\mathbb{R}^d$ as $x \to \infty$, then the embedding of the space $B_{1,w}(p,q)$ into $L^1_\nu(\mathbb{R}^d)$ is never compact.

Proof. Firstly we assume that $w(x) \to \infty$ as $x \to \infty$. Since $\nu \prec w$, there exists $C_1 > 0$ such that $\nu(x) \leq C_1 w(x)$. This implies $B_{1,w}(p,q) \subset L^1_\nu(\mathbb{R}^d)$. Let $(t_n)_{n\in\mathbb{N}}$ be a sequence with $t_n \to \infty$ in $\mathbb{R}^d$. Also since $\frac{\nu(x)}{w(x)}$ doesn't tend to zero as $x \to \infty$ then there exists $\delta > 0$ such that $\frac{\nu(x)}{w(x)} \geq \delta > 0$ for $x \to \infty$. For the proof the embedding of the space $B_{1,w}(p,q)$ into $L^1_\nu(\mathbb{R}^d)$ is never compact, take any fixed $f \in B_{1,w}(p,q)$ and define a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$, where $f_n = w(t_n)^{-1} T_{t_n} f$. This sequence is bounded in $B_{1,w}(p,q)$. Indeed we write

$$\|f_n\|_{pq}^{1,w} = \left\|w(t_n)^{-1} T_{t_n} f\right\|_{pq}^{1,w} = w(t_n)^{-1} \|T_{t_n} f\|_{pq}^{1,w}. \tag{4.7}$$

By Lemma 4.2, we know $\|T_y f\|_{pq}^{1,w} \approx w(y)$. Hence there exists $M > 0$ such that $\|T_y f\|_{pq}^{1,w} \leq M w(y)$. By using (4.7), we write

$$\|f_n\|_{pq}^{1,w} = w(t_n)^{-1} \|T_{t_n} f\|_{pq}^{1,w} \leq M w(t_n)^{-1} w(t_n) = M.$$

Now we will prove that there wouldn't exists norm convergence of subsequence of $\{f_n\}_{n\in\mathbb{N}}$ in $L^1_\nu(\mathbb{R}^d)$. The sequence obtained above certainly converges to zero in the vague topology. Indeed for all $k \in C_c(\mathbb{R}^d)$ we write

$$\left|\int\limits_{\mathbb{R}^d} f_n(x)k(x)dx\right| \leq \frac{1}{w(t_n)}\int\limits_{\mathbb{R}^d} |T_{t_n}f(x)|\,|k(x)|\,dx = \frac{1}{w(t_n)}\|k\|_\infty \|T_{t_n}f\|_1 = \frac{1}{w(t_n)}\|k\|_\infty \|f\|_1 . \tag{4.8}$$

Since right hand side of (4.8) tends zero for $n \to \infty$, then we have

$$\int\limits_{\mathbb{R}^d} f_n(x)k(x)dx \to 0.$$

Finally by Lemma 4.6, the only possible limit of $\{f_n\}_{n\in\mathbb{N}}$ in $L^1_\nu(\mathbb{R}^d)$ is zero. It is known by Lemma 2.2 in [10] that $\|T_y f\|_{1,\nu} \approx \nu(y)$. Hence there exists $C_2 > 0$ and $C_3 > 0$ such that

$$C_2\nu(y) \leq \|T_y f\|_{1,\nu} \leq C_3\nu(y). \tag{4.9}$$

From (4.9) and the equality

$$\|f_n\|_{1,\nu} = \left\|w(t_n)^{-1}T_{t_n}f\right\|_{1,\nu} = w(t_n)^{-1}\|T_{t_n}f\|_{1,\nu}$$

we obtain

$$\|f_n\|_{1,\nu} = w(t_n)^{-1}\|T_{t_n}f\|_{1,\nu} \geq C_2 w(t_n)^{-1}\nu(t_n). \tag{4.10}$$

Since $\frac{\nu(t_n)}{w(t_n)} \geq \delta > 0$ for all t_n, by using (4.10) we write

$$\|f_n\|_{1,\nu} \geq C_2 w(t_n)^{-1}\nu(t_n) \geq C_2\delta.$$

It means that there would not be possible to find norm convergent subsequence of $\{f_n\}_{n\in\mathbb{N}}$ in $L^1_\nu(\mathbb{R}^d)$.

Now we assume that w is a constant or bounded weight function. Since $\nu \prec w$, then $\frac{\nu(x)}{w(x)}$ is also constant or bounded and doesn't tend to zero as $x \to \infty$. We take a function $f \in B_{1,w}(p,q)$ with compactly support and define the sequence $\{f_n\}_{n\in\mathbb{N}}$ as in (4.7). Thus $\{f_n\}_{n\in\mathbb{N}} \subset B_{1,w}(p,q)$. It is easy to show that $\{f_n\}_{n\in\mathbb{N}}$ is bounded in $B_{1,w}(p,q)$ and converges to zero in the vague topology. Then there would not possible to find norm convergent subsequence of $\{f_n\}_{n\in\mathbb{N}}$ in $L^1_\nu(\mathbb{R}^d)$. This completes the proof. □

Proposition 4.8. Let w_1, w_2 be Beurling weight functions on $\mathbb{R}^d$. If $w_2 \prec w_1$ and $\frac{w_2(x)}{w_1(x)}$ doesn't tend to zero in $\mathbb{R}^d$ then the embedding $i : B_{1,w_1}(p,q) \hookrightarrow B_{1,w_2}(p,q)$ is never compact.

Proof. The proof can be obtained by means of Proposition 4.4, Proposition 4.3 and Theorem 4.7. □

5. Multipliers of $B_{1,w}(p,q)$

Now we discuss multipliers of the spaces $B_{1,w}(p,q)$. We define the space

$$M_{B_{1,w}(p,q)} = \{\mu \in M(w) : \|\mu\|_M \leq C(\mu)\}$$

where

$$\|\mu\|_M = \sup\left\{ \frac{\|\mu * f\|^{1,w}_{pq}}{\|f\|_{1,w}} : f \in L^1_w(G),\ f \neq 0,\ \widehat{f} \in C_c(\widehat{G}) \right\}.$$

By the Proposition 2.1 in [13], we have $M_{B_{1,w}(p,q)} \neq \{0\}$.

Proposition 5.1. If w satisfies (BD), then for a linear operator $T : L^1_w(G) \to B_{1,w}(p,q)$ the following are equivalent:

(**i**) $T \in Hom_{L^1_w(G)}\left(L^1_w(G), B_{1,w}(p,q)\right)$.

(**ii**) There exists a unique $\mu \in M_{B_{1,w}(p,q)}$ such that $Tf = \mu * f$ for every $f \in L^1_w(G)$. Moreover the correspondence between T and μ defines an isomorphism between $Hom_{L^1_w(G)}\left(L^1_w(G), B_{1,w}(p,q)\right)$ and $M_{B_{1,w}(p,q)}$.

Proof. It is known that $B_{1,w}(p,q)$ is a S_w space by Theorem 3.4. Thus, the proof is completed by Proposition 2.4 in [13]. □

Theorem 5.2. If w satisfies (BD) and $T \in Hom_{L^1_w(G)}(B_{1,w}(p,q))$, then there exists a unique pseudo measure $\sigma \in \left(A(\widehat{G})\right)^*$ (see [20]), such that $Tf = \sigma * f$ for all $f \in B_{1,w}(p,q)$.

Proof. It is known that $B_{1,w}(p,q)$ is a S_w space by Theorem 3.4 and an essential Banach module over $L^1_w(G)$ by Proposition 3.6. Thus, the proof is completed by Theorem 5 in [5]. □

Proposition 5.3. The multiplier space $Hom_{L^1_w(G)}\left(L^1_w(G), (B_{1,w}(p,q))^*\right)$ is isomorphic to $L^\infty_{w^{-1}}(G) \times \left(L^{p'}, \ell^{q'}\right)/K$.

Proof. By Proposition 3.6, we write $L^1_w(G) * B_{1,w}(p,q) = B_{1,w}(p,q)$. Hence by Corollary 2.13 in [21] and Proposition 3.7, we have

$$\begin{aligned} Hom_{L^1_w(G)}\left(L^1_w(G), (B_{1,w}(p,q))^*\right) &= \left(L^1_w(G) * B_{1,w}(p,q)\right)^* = (B_{1,w}(p,q))^* \\ &= L^\infty_{w^{-1}}(G) \times \left(L^{p'}, \ell^{q'}\right)/K. \end{aligned}$$ □

References

[1] Bertrandis, J.P., Darty, C., Dupuis, C., *Unions et intersections d'espaces L^p invariantes par translation ou convolution*, Ann. Inst. Fourier Grenoble, **28**(1978), no. 2, 53-84.

[2] Busby, R.C., Smith, H.A., *Product-convolution operators and mixed-norm spaces*, Trans. Amer. Math. Soc., **263**(1981), no. 2, 309-341.

[3] Cigler, J., *Normed ideals in $L^1(G)$*, Indag Math., **31**(1969), 272-282.

[4] Dieudonné, J., *Treatise on analysis*, Vol 2, Academic Press, New York - San Francisco - London, 1976.

[5] Doğan, M., Gürkanli, A.T., *Multipliers of the spaces $S_w(G)$*, Math. Balcanica, New Series 15, **3-4**(2001), 199-212.

[6] Domar, Y., *Harmonic analysis based on certain commutative Banach algebras*, Acta Math., **96**(1956), 1-66.

[7] Doran, R.S., Wichmann, J., *Approximate Identities and Factorization in Banach Modules*, Berlin - Heidelberg - New York, Springer Verlag, **768**(1970).

[8] Feichtinger, H.G., *A characterization of Wiener's algebra on locally compact groups*, Arch. Math., **29**(1977), 136-140.

[9] Feichtinger, H.G., Gröchenig, K.H., *Banach spaces related to integrable group representations and their atomic decompositions I*, J. Funct. Anal., **86**(1989), no. 2, 307-340.

[10] Feichtinger, H.G., Gürkanlı, A.T., *On a family of weighted convolution algebras*, Internat. J. Math. Sci., **13**(1990), 517-525.

[11] Fournier, J.J., Stewart, J., *Amalgams of L^p and ℓ^q*, Bull. Amer. Math. Soc., **13**(1985), no. 1, 1-21.

[12] Gaudry, G.I., *Multipliers of weihted Lebesgue and measure spaces*, Proc. London Math. Soc., **19**(1969), no. 3, 327-340.

[13] Gürkanli, A.T., *Multipliers of some Banach ideals and Wiener-Ditkin sets*, Math. Slovaca, **55**(2005), no. 2, 237-248.

[14] Gürkanli, A.T., *Compact embeddings of the spaces $A^p_{w,\omega}(\mathbb{R}^d)$*, Taiwanesse J. Math., **12**(2008), no. 7, 1757-1767.

[15] Hewitt, E., Ross, K.A., *Abstract Harmonic Analysis I, II*, Berlin - Heidelberg - New York, Springer Verlag, 1979.

[16] Holland, F., *Harmonic analysis on amalgams of L^p and ℓ^q*, J. London Math. Soc., **10**(1975), no. 2, 295-305.

[17] Krogstad, H.E., *Multipliers of Segal algebras*, Math. Scand., **38**(1976), 285-303.

[18] Liu, T.S., Van Rooij, A., *Sums and intersections of normed linear spaces*, Math. Nach., **42**(1969), 29-42.

[19] Liu, T.S., Van Rooij, A., Wang, J.K., *On some group algebra of modules related to Wiener's algebra M_1*, Pacific J. Math., **55**(1974), no. 2, 507-520.

[20] Reiter, H., *Classical harmonic analysis and locally compact groups*, Oxford University Press, Oxford, 1968.

[21] Rieffel, M.A., *Induced Banach representation of Banach algebras and locally compact groups*, J. Funct. Anal., **1**(1967), 443-491.

[22] Sağır, B., *On functions with Fourier transforms in $W(B,Y)$*, Demonstratio Mathematica, **33**(2000), no. 2, 355-363.

[23] Squire, M., Torres de, L., *Amalgams of L^p and ℓ^q,* Ph.D. Thesis, McMaster University, 1984.

[24] Stewart, J., *Fourier transforms of unbounded measures*, Canad. J. Math., **31**(1979), no. 6, 1281-1292.

[25] Szeptycki, P., *On functions and measures whose Fourier transforms are functions*, Math. Ann., **179**(1968), 31-41.

[26] Wang, H.G., *Homogeneous Banach algebras*, Marcel Dekker, Inc., New York-Basel, 1972.

[27] Warner, C.R., *Closed ideals in the group algebra $L^1(G) \cap L^2(G)$*, Trans. Amer. Math. Soc., **121**(1966), 408-423.

[28] Wiener, N., *On the representation of functions by trigonometric integrals*, Math. Z., **24**(1926), 575-616.

[29] Wiener, N., *The Fourier integral and certain of its applications,* New York, Dover Pub. Inc., 1958.

[30] Yap, L.Y.H., *Ideals in subalgebras of the group algebras,* Studia Math., **35**(1970), 165-175.

[31] Yap, L.Y.H., *On two classes of subalgebras of $L^1(G)$*, Proc. Japan Acad., **48**(1972), 315-319.

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