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# Weighted composition operators on weighted Lorentz-Karamata spaces

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**Abstract.** In this paper, a characterization of the non-singular measurable transformations $T$ from $X$ into itself and complex-valued measurable functions $u$ on $X$ inducing weighted composition operators is obtained and subsequently their compactness and closedness of the range on the weighted Lorentz-Karamata spaces $L^w_{p,q;b}(X, \Sigma, \mu)$ are completely identified where $(X, \Sigma, \mu)$ is a $\sigma-$finite measure space and $1 < p \leq \infty$, $1 \leq q \leq \infty$.

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## 1. Introduction

A new generalization of Lebesgue, Lorentz, Zygmund, Lorentz-Zygmund and generalized Lorentz-Zygmund spaces was studied by J.S.Neves in [13]. By using the Karamata Theory, he introduced Lorentz-Karamata (simply $LK$) spaces and gave Bessel and Riesz potentials and emmedings of these spaces. In that paper, he studied the $LK$ spaces $L_{p,q;b}(R, \mu)$ where $p, q \in (0, \infty]$, $b$ is a slowly varying function on $(0, \infty)$ and $(R, \mu)$ is a measure space. These spaces give the generalized Lorentz-Zygmund spaces $L_{p,q;\alpha_1,\ldots\alpha_m}(R)$, Lorentz-Zygmund spaces $L^{p,q}(\log L)^{\alpha}(R)$, Zygmund spaces $L^p(\log L)^{\alpha}(R)$ (introduced in [3,16]), Lorentz spaces $L^{p,q}(R)$ and Lebesgue spaces $L^p(R)$ under convenient choices of slowly varying functions.

In [5,13], it is proved that $LK$ spaces $L_{p,q;b}(R, \mu)$ endowed with a convenient norm, is a rearrangment-invariant Banach function spaces with associate spaces $L_{p',q';b^{-1}}(R, \mu)$ if $(R, \mu)$ is a resonant measure space, $p \in (1, \infty)$ and $q \in [1, \infty]$. Also it is showed that when $p \in (1, \infty)$ and $q \in [1, \infty)$, $LK$ spaces have absolutely continuous norm.

## 2. Preliminaries

Throughout the paper $(X, \Sigma, \mu)$ will stand for a $\sigma$-finite measure space. We will use weight function $w$, i.e. a measurable, locally bounded function on $X$, satisfying $w(x) \geq 1$ for all $x \in X$ and $\chi_A$ for characteristic function of a set $A$. For any two non-negative expressions (i.e. functions or functionals), $A$ and $B$, the symbol $A \precsim B$ means that $A \leq cB$, for some positive constant $c$ independent of the variables in the expressions $A$ and $B$. If $A \precsim B$ and $B \precsim A$, we write $A \approx B$ and say that $A$ and $B$ are equivalent. Certain well-known terms such as Banach function space, rearrangement invariant Banach function space, associate space, absolutely continuous norm, etc. will be used frequently in the sequel without their definitions. However, the reader may be found their definitions e.g., in [3,5,8,13] and [16].

A positive measurable function $L$, defined on some neighborhood of infinity, is said to be slowly varying if, for every $s > 0$,

$$\frac{L(st)}{L(t)} \to 1 \qquad (t \to +\infty). \tag{2.1}$$

These functions were introduced by Karamata [10] (see also [14] for more information). Also another definition for slowly varying functions can be found in [13] such as:

**Definition 2.1.** *A positive and Lebesgue measurable function $b$ is said to be slowly varying (s.v.) on $(0, \infty)$ in the sense of Karamata if, for each $\varepsilon > 0$, $t^{\varepsilon} b(t)$ is equivalent to a non-decreasing function and $t^{-\varepsilon} b(t)$ is equivalent to a non-increasing function on $(0, \infty)$.*

The detailed study of Karamata Theory, properties and examples of slowly varying functions can be found in [5,10,14] and [16,Chap.V, p.186]. For example, let $m \in \mathbb{N}$ and $\alpha = (\alpha_1, ..., \alpha_m) \in \mathbb{R}^m$. If we denote by $\vartheta_{\alpha}^{m}$ the real function defined by

$$\vartheta_{\alpha}^{m}(t) = \prod_{i=1}^{m} l_i^{\alpha_i}(t) \quad \text{for all } t \in (0, \infty)$$

where $l_1, ..., l_m$ are positive functions defined on $(0, \infty)$ by

$$l_1(t) = 1 + |\log t|, \; l_i(t) = 1 + \log l_{i-1}(t), \; i \geq 2, \; m \geq 2,$$

then the following functions are s.v. on $[1, \infty)$:

1. $b(t) = \vartheta_{\alpha}^{m}(t)$ with $m \in \mathbb{N}$ and $\alpha \in \mathbb{R}^m$;
2. $b(t) = \exp(\log^{\alpha} t)$ with $0 < \alpha < 1$;
3. $b(t) = \exp(l_m^{\alpha}(t))$ with $0 < \alpha < 1$, $m \in \mathbb{N}$;
4. $b(t) = l_m(t)$ with $m \in \mathbb{N}$.

Given a s.v. function $b$ on $(0, \infty)$, we denote by $\gamma_b$ the positive function defined by

$$\gamma_b(t) = b\left(\max\left\{t, \frac{1}{t}\right\}\right) \qquad \text{for all } t > 0.$$

It is known that any slowly varying function $b$ on $(0, \infty)$ is equivalent to a slowly varying continuous function $\widetilde{b}$ on $(0, \infty)$. Consequently, without loss of generality, we assume that all slowly varying functions in question are continuous functions in $(0, \infty)$ [6]. We shall need the following property of s.v. functions, for which we refer to [13, Lemma 3.1].

**Lemma 2.2.** *Let $b$ be a slowly varying function on $(0,\infty)$.*
*(i) Let $r \in \mathbb{R}$. Then $b^r$ is a slowly varying function on $(0,\infty)$ and $\gamma_b^r(t) = \gamma_{b^r}(t)$ for all $t > 0$.*
*(ii) Given positive numbers $\varepsilon$ and $\kappa$, $\gamma_b(\kappa t) \approx \gamma_b(t)$,i.e., there are positive constants $c_\varepsilon$ and $C_\varepsilon$ such that*

$$c_\varepsilon \min\{\kappa^{-\varepsilon}, \kappa^{\varepsilon}\}\gamma_b(t) \leq \gamma_b(\kappa t) \leq C_\varepsilon \max\{\kappa^{-\varepsilon}, \kappa^{\varepsilon}\}\gamma_b(t) \tag{2.2}$$

*for all $t > 0$.*
*(iii) Let $\alpha > 0$. Then*

$$\int_0^t \tau^{\alpha-1}\gamma_b(\tau)d\tau \approx t^\alpha \gamma_b(t) \quad \text{and} \quad \int_t^\infty \tau^{-\alpha-1}\gamma_b(\tau)d\tau \approx t^{-\alpha}\gamma_b(t) \tag{2.3}$$

*for all $t > 0$.*

Now, let us take the measure as $wd\mu$. Let $f$ be a complex-valued measurable function defined on a $\sigma$-finite measure space $(X, \Sigma, wd\mu)$. Then the distribution function of $f$ is defined as

$$\mu_{f,w}(s) = w\{x \in X : |f(x)| > s\} = \int\limits_{\{x\in X:\ |f(x)|>s\}} w(x)\, d\mu(x)\,, \ s \geq 0. \tag{2.4}$$

The nonnegative rearrangement of $f$ is given by

$$f_w^*(t) = \inf\{s > 0 : \mu_{f,w}(s) \leq t\} = \sup\{s > 0 : \mu_{f,w}(s) > t\}\,, \ t \geq 0 \tag{2.5}$$

where we assume that $\inf \phi = \infty$ and $\sup \phi = 0$. Also the average(maximal) function of $f$ on $(0,\infty)$ is given by

$$f_w^{**}(t) = \frac{1}{t}\int_0^t f_w^*(s)\, ds. \tag{2.6}$$

Note that $\lambda_{f,w}(\cdot)\,, f_w^*(\cdot)$ and $f_w^{**}(\cdot)$ are nonincreasing and right continuous functions.

**Definition 2.3.** *Let $p, q \in (0,\infty]$ and let $b$ be a slowly varying function on $(0,\infty)$. The weighted Lorentz-Karamata (WLK) space $L^w_{p,q;b}(X, \Sigma, wd\mu)$ is defined to be the set of all functions such that*

$$\|f\|^w_{p,q;b} := \left\| t^{\frac{1}{p}-\frac{1}{q}}\gamma_b(t)\, f_w^{**}(t) \right\|_{q;(0,\infty)} \tag{2.7}$$

*is finite. Here $\|\cdot\|_{q;(0,\infty)}$ stands for the usual $L_q$ (quasi-) norm over the interval $(0,\infty)$.*

After this point, for the convenience, we will use $L^w_{p,q;b}(X)$ for $L^w_{p,q;b}(X, \Sigma, wd\mu)$. It is easy to show that (by the same arguments in [5,Theorem 3.4.41], [13]) the $WLK$ spaces $L^w_{p,q;b}(X)$ endowed with a convenient norm (2.7), is a rearrangment-invariant Banach function spaces and have absolutely continuous norm when $p \in (1,\infty)$ and $q \in [1,\infty)$. It is clear that, for $0 < p < \infty$, the $WLK$ space $L^w_{p,q;b}(X)$ contains the characteristic function of every measurable subset of $X$ with finite measure and hence, by linearity, every $wd\mu-$simple function. In this case, with a little thought, it is easy to see that the set of simple functions is dense in the $WLK$ space as the $WLK$ spaces have absolutely continuous norm for $p \in (1,\infty)$ and $q \in [1,\infty)$.

Let $T : X \to X$ be a measurable ($T^{-1}(E) \in \Sigma$, for any $E \in \Sigma$) and non-singular transformation ($w\left(T^{-1}(E)\right) = 0$ whenever $w(E) = 0$) and $u$ a complex-valued function defined on $X$. We define a linear transformation $W = W_{u,T}$ on the $WLK$ space $L^w_{p,q;b}(X)$ into the linear space of all complex-valued measurable functions by

$$W_{u,T}(f)(x) = u(T(x))\, f(T(x)) \tag{2.8}$$

for all $x \in X$ and $f \in L^w_{p,q;b}(X)$. If $W$ is bounded with range in $L^w_{p,q;b}(X)$, then it is called a *weighted composition operator* on $L^w_{p,q;b}(X)$. If $u \equiv 1$, then $W \equiv C_T : f \to f \circ T$ is called a *composition operator* induced by $T$. If $T$ is the identity mapping, then $W \equiv M_u : f \to u \cdot f$ is a *multiplication operator* induced by $u$. The study of these operators acting on Lebesgue and Lorentz spaces has been made in [4,9,15] and [1,2,11,12], respectively.

In the next part of this paper, we will characterize the boundedness, compactness and closedness of the range of the weighted composition operators on $WLK$ spaces $L^w_{p,q;b}(X)$ for $1 < p \leq \infty$, $1 \leq q \leq \infty$.

## 3. Results

**Theorem 3.1.** *Let $(X, \Sigma, wd\mu)$ be a $\sigma$-finite measure space and $u : X \to \mathbb{C}$ a measurable function. Let $T : X \to X$ be a non-singular measurable transformation such that the Radon-Nikodym derivative $f_T = wd\mu\left(T^{-1}\right)/wd\mu$ is in $L^\infty(\mu)$. Then*

$$W_{u,T} : f \to u \circ T \cdot f \circ T \tag{3.1}$$

*is bounded on $L^w_{p,q;b}(X)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$ if $u \in L^\infty(\mu)$.*

*Proof.* Suppose that $\|f_T\|_\infty = k$. The distribution function of

$$Wf = W_{u,T}(f) = u \circ T \cdot f \circ T$$

is found that

$$\begin{aligned}
\mu_{Wf,w}(s) &= w\{x \in X : \ |u(T(x))\, f(T(x))| > s\} \\
&= \int\limits_{\{x \in X:\ |u(T(x)) f(T(x))| > s\}} w(x)\, d\mu(x) \\
&= wT^{-1}\{x \in X : \ |u(x)\, f(x)| > s\} \\
&\leq wT^{-1}\{x \in X : \|u\|_\infty \ |f(x)| > s\} \qquad (3.2)\\
&\leq kw\{x \in X : \|u\|_\infty \ |f(x)| > s\} = k\mu_{\|u\|_\infty\ f,w}(s).
\end{aligned}$$

Hence for each $t \geq 0$, by (3.2) we get

$$\left\{s > 0 : \mu_{\|u\|_\infty\ f,w}(s) \leq \frac{t}{k}\right\} \subseteq \{s > 0 : \mu_{Wf,w}(s) \leq t\} \tag{3.3}$$

and

$$\begin{aligned}
(Wf)_w^*(t) &= \inf\{s>0:\mu_{Wf,w}(s)\le t\} \\
&\le \inf\left\{s>0:\mu_{\|u\|_\infty\, f,w}(s)\le \frac{t}{k}\right\} \\
&= \inf\left\{s>0:w\{x\in X:\|u\|_\infty\;|f(x)|>s\}\le \frac{t}{k}\right\} \qquad (3.4)\\
&= \|u\|_\infty f_w^*\left(\frac{t}{k}\right).
\end{aligned}$$

Also, we write that $(Wf)_w^{**}(t)\le \|u\|_\infty f_w^{**}\left(\frac{t}{k}\right)$ by (3.4). Therefore,

$$\begin{aligned}
\|Wf\|_{p,q;b}^w &= \left\|t^{\frac{1}{p}-\frac{1}{q}}\gamma_b(t)\,(Wf)_w^{**}(t)\right\|_{q;(0,\infty)} \\
&\le \left\|t^{\frac{1}{p}-\frac{1}{q}}\gamma_b(t)\,\|u\|_\infty f_w^{**}\left(\frac{t}{k}\right)\right\|_{q;(0,\infty)} \\
&\lesssim \|u\|_\infty k^{\frac{1}{p}}\left\|t^{\frac{1}{p}-\frac{1}{q}}\gamma_b(t)\,f_w^{**}(t)\right\|_{q;(0,\infty)} = k^{\frac{1}{p}}\|u\|_\infty\|f\|_{p,q;b}^w \qquad (3.5)
\end{aligned}$$

can be written by (2.2). Consequently, $W$ is a bounded operator on $L_{p,q;b}^w(X)$ with $1<p\le\infty$, $1\le q\le\infty$ and $\|W\|\lesssim k^{\frac{1}{p}}\|u\|_\infty$ by (3.5). □

**Remark 3.2.** The above theorem is also valid for $u\in L^\infty\left(w\left(T^{-1}\right)\right)$, i.e.

$$u\circ T\in L^\infty(\mu).$$

**Theorem 3.3.** *Let $u$ be a complex-valued measurable function and $T:X\to X$ be a non-singular measurable transformation such that $T(E_\varepsilon)\subseteq E_\varepsilon$ for all $\varepsilon>0$, where $E_\varepsilon=\{x\in X:|u(x)|>\varepsilon\}$. If $W_{u,T}$ is bounded on $L_{p,q;b}^w(X)$, $1<p\le\infty$, $1\le q\le\infty$, then $u\in L^\infty(\mu)$.*

*Proof.* Let us assume that $u\notin L^\infty(\mu)$. Then the set $E_n=\{x\in X:|u(x)|>n\}$ has a positive measure for all $n\in\mathbb{N}$. Since $T(E_n)\subseteq E_n$ or equivalently $\chi_{E_n}\le\chi_{T^{-1}(E_n)}$, we write that

$$\begin{aligned}
\{x\in X:|\chi_{E_n}(x)|>s\} &\subseteq \left\{x\in X:\left|\chi_{T^{-1}(E_n)}(x)\right|>s\right\} \\
&\subseteq \left\{x\in X:\left|u(T(x))\,\chi_{T^{-1}(E_n)}(x)\right|>ns\right\} \qquad (3.6)
\end{aligned}$$

and so

$$\begin{aligned}
(W\chi_{E_n})_w^*(t) &= \inf\left\{s>0:\mu_{W\chi_{E_n},w}(s)\le t\right\} \\
&= \inf\{s>0:w\{x\in X:\;|W\chi_{E_n}(x)|>s\}\le t\} \\
&= \inf\{s>0:w\{x\in X:\;|u(T(x))\,\chi_{E_n}(T(x))|>s\}\le t\} \qquad (3.7)\\
&= n\inf\left\{s>0:w\left\{x\in X:\;\left|u(T(x))\,\chi_{T^{-1}(E_n)}(x)\right|>ns\right\}\le t\right\} \\
&\ge n\inf\{s>0:w\{x\in X:\;|\chi_{E_n}(x)|>s\}\le t\} = n\,(\chi_{E_n})_w^*(t).
\end{aligned}$$

Thus we have $(W\chi_{E_n})_w^{**}(t)\ge n\,(\chi_{E_n})_w^{**}(t)$ for all $t>0$ by (3.7). This gives us the contradiction that $\|W\chi_{E_n}\|_{p,q;b}^w\ge n\,\|\chi_{E_n}\|_{p,q;b}^w$. □

If we combine Theorem 3.1 and Theorem 3.3, then we have the following theorem.

**Theorem 3.4.** *Let $u$ be a complex-valued measurable function and $T : X \to X$ be a non-singular measurable transformation such that the Radon-Nikodym derivative $f_T = wd\mu\left(T^{-1}\right)/wd\mu$ is in $L^\infty(\mu)$ and $T(E_\varepsilon) \subseteq E_\varepsilon$ for all $\varepsilon > 0$, where $E_\varepsilon = \{x \in X : |u(x)| > \varepsilon\}$. Then $W_{u,T}$ is bounded on $L^w_{p,q;b}(X)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$ if and only if $u \in L^\infty(\mu)$.*

Now, we are ready to discuss the compactness and the closed range of the weighted composition operator $W = W_{u,T} : f \to u \circ T \cdot f \circ T$ on the $WLK$ spaces $L^w_{p,q;b}(X)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$. Let $T : X \to X$ be a non-singular measurable transformation with the Radon-Nikodym derivative $f_T = wd\mu\left(T^{-1}\right)/wd\mu$. If $f_T \in L^\infty(\mu)$ with $\|f_T\|_\infty = k$, then we get

$$\begin{aligned}(Wf)^*_w(kt) &= \inf\{s > 0 : \mu_{Wf,w}(s) \leq kt\} \\ &= \inf\{s > 0 : w\{x \in X : |u(T(x))\, f(T(x))| > s\} \leq kt\} \\ &= \inf\left\{s > 0 : wT^{-1}\{x \in X : |(u \cdot f)(x)| > s\} \leq kt\right\} \\ &\leq \inf\{s > 0 : w\{x \in X : |(u \cdot f)(x)| > s\} \leq t\} = (M_u f)^*_w(t) \qquad (3.8)\end{aligned}$$

and similarly $(Wf)^{**}_w(kt) \leq (M_u f)^{**}_w(t)$ for all $f \in L^w_{p,q;b}(X)$ and $t > 0$. Therefore, by (2.2), we obtain

$$\begin{aligned}\|Wf\|^w_{p,q;b} &= \left\|z^{\frac{1}{p}-\frac{1}{q}}\gamma_b(z)\,(Wf)^{**}_w(z)\right\|_{q;(0,\infty)} \\ &= \left\|(kt)^{\frac{1}{p}-\frac{1}{q}}\gamma_b(kt)\,(Wf)^{**}_w(kt)\right\|_{q;(0,\infty)} \qquad (3.9) \\ &\lesssim k^{\frac{1}{p}}\left\|t^{\frac{1}{p}-\frac{1}{q}}\gamma_b(t)\,(M_u f)^{**}_w(t)\right\|_{q;(0,\infty)} = k^{\frac{1}{p}}\|M_u f\|^w_{p,q;b}.\end{aligned}$$

Now, if $f_T$ is bounded away from zero on $S$, i.e. $f_T > \delta$ almost everywhere for some $\delta > 0$, then

$$w\left(T^{-1}(E)\right) = \int_E f_T w d\mu \geq \delta w(E) \qquad (3.10)$$

for all $E \in \Sigma$, $E \subseteq S$, where $S = \{x : u(x) \neq 0\}$. Therefore, we have

$$\|Wf\|^w_{p,q;b} \geq \delta^{\frac{1}{p}}\|M_u f\|^w_{p,q;b}. \qquad (3.11)$$

Hence for each $f \in L^w_{p,q;b}(X)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$, we have

$$\|Wf\|^w_{p,q;b} \approx \|M_u f\|^w_{p,q;b} \qquad (3.12)$$

whenever $f_T \in L^\infty(\mu)$ and bounded away from zero. By [7, Theorem 2.4] and (3.12), we can write the following theorem:

**Theorem 3.5.** *Let $T : X \to X$ be a non-singular measurable transformation such that $f_T \in L^\infty(\mu)$ and is bounded away from zero. Let $u$ be a complex-valued measurable function and $W_{u,T}$ is bounded on the $WLK$ space $L^w_{p,q;b}(X)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$. Then the followings are equivalent:*

*(i) $W_{u,T}$ is compact,*

*(ii)* $M_u$ *is compact,*

*(iii)* $L^w_{p,q;b}(u,\varepsilon)$ *are finite dimensional for each* $\varepsilon > 0$*, where*

$$L^w_{p,q;b}(u,\varepsilon) = \left\{ f\chi_{(u,\varepsilon)} : f \in L^w_{p,q;b}(X) \right\} \text{ and } (u,\varepsilon) = \{x \in X : |u(x)| \geq \varepsilon\}.$$

We know that $W_{u,T} = C_T M_u$ and $wd\mu$ is atomic. Therefore, if we use [11, Theorem 3.1] for $W_{u,T}$ on the $WLK$ space $L^w_{p,q;b}(X)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$, then get the following theorem:

**Theorem 3.6.** *Let* $T : X \to X$ *be a non-singular measurable transformation such that* $f_T \in L^\infty(\mu)$ *and* $u$ *be a complex-valued measurable function with* $u \in L^\infty(\mu)$*. Let* $\{A_n\}_{n\in\mathbb{N}}$ *be all the atoms of* $X$ *with* $w(A_n) > 0$ *for all* $n \in \mathbb{N}$*. Then* $W_{u,T}$ *is compact on the* $WLK$ *space* $L^w_{p,q;b}(X)$*,* $1 < p \leq \infty$*,* $1 \leq q \leq \infty$ *if* $wd\mu$ *is purely atomic and*

$$c_n = \frac{w\left(T^{-1}(A_n)\right)}{w(A_n)} \to 0.$$

**Theorem 3.7.** *If* $wd\mu$ *is non-atomic and* $W_{u,T}$ *is bounded on the* $WLK$ *space* $L^w_{p,q;b}(X)$*,* $1 < p \leq \infty$*,* $1 \leq q \leq \infty$*, then* $W_{u,T}$ *is compact if and only if* $u \cdot f_T = 0$ *almost everywhere.*

*Proof.* Let us assume that $W = W_{u,T}$ is compact. If $u \cdot f_T \neq 0$ a.e., then there exist $c \geq 1$, such that the set

$$E = \left\{ x \in X : |u(x)| \text{ and } f_T(x) > \frac{1}{c} \right\} \tag{3.13}$$

has positive measure. Since $wd\mu$ is non- atomic, we can find a decreasing sequence $\{E_n\}_{n\in\mathbb{N}}$ of measurable subsets of $E$ such that $w(E_n) = \frac{a}{2^n}$, $0 < a < w(E)$. Now, if we construct a sequence such that $e_n = \frac{\chi_{E_n}}{\|\chi_{E_n}\|^w_{p,q;b}}$, then it is easy to see that $\{e_n\}_{n\in\mathbb{N}}$ is bounded in $L^w_{p,q;b}(X)$. For $m, n \in \mathbb{N}$, let $m = 2n$. Then we have

$$\begin{aligned}
&(We_n - We_m)^*_w\left(\frac{t}{c}\right) \\
&= \inf\left\{ s > 0 : \mu_{We_n - We_m, w}(s) \leq \frac{t}{c} \right\} \\
&= \inf\left\{ s > 0 : w\{x \in X : |u(T(x))\, e_n(T(x)) - u(T(x))\, e_m(T(x))| > s\} \leq \frac{t}{c} \right\} \\
&= \inf\left\{ s > 0 : wT^{-1}\{z \in E_n : |u(z)|\,|e_n(z) - e_m(z)| > s\} \leq \frac{t}{c} \right\} \\
&\geq \inf\{ s > 0 : w\{z \in E_n : |e_n(z) - e_m(z)| > sc\} \leq t\} \\
&= \frac{1}{c}\inf\{ s > 0 : w\{z \in E_n : |e_n(z) - e_m(z)| > s\} \leq t\} \\
&\geq \frac{1}{c}\inf\{ s > 0 : w\{z \in E_n \backslash E_m : |e_n(z) - e_m(z)| > s\} \leq t\}
\end{aligned}$$

for all $t \geq 0$. This gives us that

$$(We_n - We_m)^*_w\left(\frac{t}{c}\right) \geq \frac{\left(\chi_{E_n\backslash E_m}\right)^*_w(t)}{c\,\|\chi_{E_n}\|^w_{p,q;b}} \tag{3.14}$$

and so

$$\|We_n - We_m\|^w_{p,q;b} \gtrsim \frac{1}{c^2}\left(\frac{w(E_n \backslash E_m)}{w(E_n)}\right)^{\frac{1}{p}} \geq \varepsilon \tag{3.15}$$

for some $\varepsilon > 0$ and large values of $n$ by (ii) and (iii) of Lemma 2.2. Thus the sequence $\{We_n\}_{n\in\mathbb{N}}$ doesn't admit a convergent subsequence which conradicts the compactness of $W$. Hence $u \cdot f_T = 0$ a.e.
The converse of the proof is obvious. □

**Theorem 3.8.** *Let $T : X \to X$ be a non-singular measurable transformation with $f_T$ in $L^\infty(\mu)$ and bounded away from zero. Let $u$ be a complex-valued measurable function such that $W_{u,T}$ is bounded on the WLK space $L^w_{p,q;b}(X)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$. Then $W_{u,T}$ has closed range if and only if there exists a $\delta > 0$ such that $|u(x)| \geq \delta$ a.e. on the support of $u$.*

*Proof.* Suppose that $W = W_{u,T}$ has closed range. Therefore there exists an $\varepsilon > 0$ such that $\|Wf\|^w_{p,q;b} \geq \varepsilon \|f\|^w_{p,q;b}$ for all $f \in L^w_{p,q;b}(S)$ where $S$ is the support of $u$ and $L^w_{p,q;b}(S) = \left\{f\chi_S : f \in L^w_{p,q;b}(X)\right\}$. Now, let us choose $\delta > 0$ such that $k^{\frac{1}{p}}\delta < \varepsilon$ where $k = \|f_T\|_\infty$. Assume that the set $E = \{x \in X : |u(x)| < \delta\}$ has positive measure, i.e. $0 < w(E) < \infty$. Then $\chi_E \in L^w_{p,q;b}(S)$ and

$$\begin{aligned} \|W\chi_E\|^w_{p,q;b} &\lesssim k^{\frac{1}{p}} \|u \cdot \chi_E\|^w_{p,q;b} \leq k^{\frac{1}{p}}\delta \|\chi_E\|^w_{p,q;b} \\ &< \varepsilon \|\chi_E\|^w_{p,q;b} \end{aligned}$$

by (3.9). This conradiction says that $|u(x)| \geq \delta$ a.e. on the support of $u$.
Conversely, assume that there exists a $\delta > 0$ such that $|u(x)| \geq \delta$ a.e. on $S$. Since $f_T$ is bounded away from zero, we can write that $f_T > m$ for some $m > 0$. By using this fact and (3.11), we get

$$\|Wf\|^w_{p,q;b} \geq m^{\frac{1}{p}} \|u \cdot f\|^w_{p,q;b} \geq m^{\frac{1}{p}}\delta \|f\|^w_{p,q;b} \tag{3.16}$$

for all $f \in L^w_{p,q;b}(S)$. Therefore $W$ has closed range being $ker(W) = L^w_{p,q;b}(X \backslash S)$. □

**Corollary 3.9.** *If $T^{-1}(E_\varepsilon) \subseteq E_\varepsilon$ for each $\varepsilon > 0$ and $W_{u,T}$ has closed range, then $|u(x)| \geq \delta$ a.e. on $S$, the support of $u$ for some $\delta > 0$.*

Using the equivalence (3.12) and [1, Theorem 4.1], we can say the following theorem:

**Theorem 3.10.** *Let $T : X \to X$ be a non-singular measurable transformation such that $f_T \in L^\infty(\mu)$ and is bounded away from zero. Let $u$ be a complex-valued measurable function such that $W_{u,T}$ is bounded on the WLK space $L^w_{p,q;b}(X)$, $1 < p \leq \infty$, $1 \leq q \leq \infty$. Then the followings are equivalent:*
*(i) $W_{u,T}$ has closed range,*
*(ii) $M_u$ has closed range,*
*(iii) $|u(x)| \geq \delta$ a.e. for some $\delta > 0$ on $S$, the support of $u$.*

## References


[1] Arora, S.C., Datt, G., Verma, S., *Multiplication operators on Lorentz spaces*, Indian J. Math., **48(3)**(2006), 317-329.

[2] Arora, S.C., Datt, G., Verma, S., *Weighted composition operators on Lorentz spaces*, Bull. Korean Math. Soc., **44(4)**(2007), 701-708.

[3] Bennett, C., Sharpley, R., *Interpolation of operators*, Pure and Applied Math., **129**, Academic Press, New York, 1988.

[4] Chan, J.T., *A note on compact weighted composition operators on $L^p$ ()*, Acta Sci. Math. (Szeged), **56(2)**(1992), 165-168.

[5] Edmunds, D.E., Evans, W.D., *Hardy operators, Function spaces and Embeddings*, Springer Berlin, 2004.

[6] Gogatishvili, A., Opic, B., *Optimality of embeddings of Bessel-potential-type spaces into Lorentz-Karamata spaces*, Proc. Royal Soc. of Edinburgh, **134A**(2004), 1127-1147.

[7] Hudzik, H., Kumar, R., Kumar, R., *Matrix multiplication operators on Banach function spaces*, Proc. Indian Acad. Sci. Math. Sci., **116(1)**(2006), 71-81.

[8] Hunt, R.A., *On $L(p, q)$ spaces*, Extrait de L'enseignement mathematique, **XII**(1966), no. 4, 249-275.

[9] Jabbarzadeh, M.R., Pourreza, E., *A note on weighted composition operators on $L^p$-spaces*, Bull. Iranian Math. Soc., **29(1)**(2003), 47-54.

[10] Karamata, J., *Sur un mode de croissance reguliere des fonctions*, Mathematica (Cluj), **4**(1930), 38-53.

[11] Kumar, R., Kumar, R., *Composition operators on Banach function spaces*, Proc. Amer. Math. Soc., **133(7)**(2005), 2109-2118.

[12] Kumar, R., Kumar, R., *Compact composition operators on Lorentz spaces*, Mat. Vesnik, **57(3-4)**(2005), 109-112.

[13] Neves, J.S., *Lorentz-Karamata spaces, Bessel and Riesz potentials and embeddings*, Dissertationes Mathematicae, **405**(2002).

[14] Seneta, E., *Regularly Varying Functions*, Lecture Notes in Mathematics, **508**, Springer-Verlag, Berlin, 1976.

[15] Singh, R.K., Manhas, J.S., *Composition operators on function spaces,* North-Holland Math. Studies, 179, North-Holland Publishing Co., Amsterdam, 1971.

[16] Zygmund, A., *Trigonometric series*, **1**(1957), Cambridge Univ. Press, Cambridge.



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