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Note on a property of the Banach spaces

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Abstract. We show that we may consider a partial ordering $\leq$ in an infinite dimensional Banach space $(X, \|.\|)$, which we obtain through any normed Hamel base of the space, such that $(X, \|.\|, \leq)$ is a Banach lattice.

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1. Introduction

Why trying to see, concerning a Banach space X, whether there exists or not a partial ordering in X that is compatible with the topology? The particular geometric properties of Banach lattices and, the contrast concerning the continuity properties of the coodinate linear functionals associated either to a Schauder basis or to a Hamel base in a Banach space ([2], Chapter 4 and [3]), we decided to consider these matters altogether. We prove in Theorem 3.1 that $(X, \|.\|)$ being an infinite dimensional real Banach space and the normed vectors x_α $(\alpha \in \mathcal{A})$ detemining a Hamel base $\mathcal{H}$ of X, we may consider a partial order $\leq_{\mathcal{H}}$ in X such that the triple $(X, \|.\|_{\mathcal{H}}, \leq_{\mathcal{H}})$ is a Banach lattice where $\|.\|_{\mathcal{H}}$ is an equivalent norm to $\|.\|$ in X. In the Preliminaries, paragraph 2., we briefly set the notations. We consider real Banach spaces X and we say that a linear isomorphism which is a homeomorphism between two topological vector spaces is a linear homeomorphism ([4], II.1, p. 53 in a definition). Also in [4], we can find the algebraic Hamel base of a vector space X not reducing to $\{0\}$ namely (p. 42), $\mathcal{H} = \{x_\alpha : \alpha \in \mathcal{A}\}$ is Hamel base of X if $\mathcal{H}$ is an infinite linearly independent set which spans X, as we consider in paragraph 2.

2. Preliminaries

In what follows we consider a real Banach space $(X, \|.\|)$. Recall that $(X, \leq)$ is a Riesz space through a partial order $\leq$ in X if and only if $\leq$ is compatible with the linear stucture that is, $x+z \leq y+z$ whenever $x \leq y$, $x, y, z \in X$, we have that $\alpha x \geq 0$ for each $x \geq 0$, $\alpha \geq 0$ where $x \in X$ and α is a scalar and, further, there exist $x \vee y =$

$\sup\{x,y\}$, $x\wedge y=\inf\{x,y\}$ for each $x,y\in X$. .We write $(X,\|.\|,\leq)$ meaning that $(X,\|.\|)$ is a Banach space, $(X,\leq)$ is a Riesz space and $\|x\|\leq\|y\|$ whenever $|x|\leq|y|$ so that $(X,\|.\|,\leq)$ (or just X) is a Banach lattice. Here, we put $|x|=x\vee(-x)$ We write $x^+=x\vee 0$, $x^-=x\wedge 0$. We see easily that $x^-=(-x)\vee 0=-(x\wedge 0)$.More generally, $x\wedge y=-((-x)\vee(-y))$. We have that $x=x^+-x^-,|x|=x^++x^-$. Notice that $x\vee y=x+y-((-x)\vee(-y))=(x^+-x^-)+(y^+-y^-)-((-x)\vee(-y))+y-y=(x^++y^+-x^--y^-)-((y-x)\vee 0)-y$ ([2], Theorem 1.1.1. i), ii), p. 3) hence for $\leq$ a partial order compatible with the linear structure of X, X is a Riesz space provided that x^+ exists for each x in X.

Definition 2.1. *(Following* [4]*) For $\mathcal{A}$ a nonempty set of indices, we say that the family (λ_α) in $\mathbf{R}^{\mathcal{A}}$ is summable, $\sum_{\mathcal{A}}\lambda_\alpha=s$ if it holds that $\left|\sum_{\alpha\in A}\lambda_\alpha-s\right|\leq\varepsilon$ for each finite superset A of some set $A_\varepsilon\in\mathcal{F}(\mathcal{A})$, the class of all nonempty finite subsets of $\mathcal{A}$, $\varepsilon>0$ a priori given. The family (λ_α) is said to be absolutely summable if $(|\lambda_\alpha|)$ is a summable family.*

Notation 2.2. We let $l_{\mathcal{F}}(\mathcal{A})=\{(\lambda_\alpha)\in\mathbf{R}^{\mathcal{A}}:\lambda_\alpha=0$ for all $\alpha\notin A$ and some $A\in\mathcal{F}(\mathcal{A})\}$.

Notation 2.3. We write $l_1(\mathcal{A})$ for the Banach space determined by the absolutely summable families (λ_α) equipped with the norm $\|(\lambda_\alpha)\|_1=\sum_{\mathcal{A}}|\lambda_\alpha|$.

Remark 2.4. The space $l_1(\mathcal{A})$ is a Banach lattice when equipped with the partial ordering $(\lambda_\alpha)\leq(\mu_\alpha)$ if and only if $\lambda_\alpha\leq\mu_\alpha$ $(\alpha\in\mathcal{A})$. $l_1(\mathcal{A})$ is the completion of $((l_F(\mathcal{A}).\|.\|_1)$.

Proof. This follows from [4]. The partial ordering is extended the obvious way. ☐

Letting $\{x_\alpha:\alpha\in\mathcal{A}\}$ be a normed Hamel base of X, $\|x_\alpha\|=1$, $\alpha\in\mathcal{A}$, putting $\sum_A s_\alpha x_\alpha\prec_{\mathcal{H}}\sum_A t_\alpha x_\alpha$ if and only if $s_\alpha\leq t_\alpha$ ($\alpha\in\mathcal{A}$, the finite sms are understood)), we have that $(X,\prec_{\mathcal{H}})$ is a Riesz space. Notice that the linear operator $T(\lambda_\alpha)=\sum\lambda_\alpha x_\alpha$ on $l_{\mathcal{F}}(\mathcal{A})$ to $(X,\|.\|,\prec_{\mathcal{H}})$ is injective, continous of norm 1. We may consider the linear homeomorphism $(\widetilde{T}/K):(l_1(\mathcal{A})/K,\|.:l_1(\mathcal{A})/K\|)\to(X,\|.\|)$, $\widetilde{T}$ for the linear extension to $l_1(\mathcal{A})$ of T, where $K=Ker(\widetilde{T})$.

3. The results

Following [1], $(X,\|.\|,\leq)$ being a Banach lattice we say that a subspace Y of X has the solid property if $x\in Y$ whenever $|x|\leq|y|$ and $y\in Y$. Y being closed, we then may consider the partial ordering $[x]\preceq[y]$ in the quotient X/Y if and only if $y-x\in P$ where $P=\cup\{\pi(x):x\geq 0\},\pi(x)=[x]$, π for the canonical map. Clearly that $\preceq$ is compatible with the linear structure Also $[x]^+=[x^+],(X/Y,\preceq)$ is a Riesz space such that $[x]\vee[y]=[x\vee y],[x]\wedge[y]=[x\wedge y]$ and $[|x|]=|[x]|$ ([1], 14G, p. 13). We have that $[0]\preceq[x]$ if and only if for each $v\in[x]$ there is some $w\in[0],w\leq x$ hence also $[x]\preceq[y]$ if and only if for each $v\in[y]$, there is some $w\in[x]$ such that $w\leq v$. It follows that $|[x]|\preceq|[y]|$ imples that for each $v\in[y]$ there exists $w\in[x],|w|\leq|v|$ hence $\|[x]:X/Y\|\leq\|[y]:X/Y\|$ and $(Y/X,\preceq)$ is a Banach lattice. We see easily that $K = Ker(\widetilde{T})$ as above in the Preliminaries is a closed subspace of $l_1(A)$ having the solid property, hence $(l_1(\mathcal{A})/K, \leq)$ is a Banach lattice where we keep denoting the ordering in the quotient by the same symbol $\leq$.

Clearly that $\theta : (E, \|. : E\|, \leq_E) \rightarrow (F, \|. : F\|)$ being a linear homeomorphism between Banach spaces such that E is a Banach lattice, putting $\theta(a) \leq_\theta \theta(b)$ if and only if $a \leq_E b$ in E we obtain that $(F, \leq_\theta)$ is a Riesz space. We have that $\theta(a \vee b) = \theta(a) \vee \theta(b)$ and, more generally, θ preserves the lattice operations. Further, if we put $\|\theta(a)\|_\theta = \|a : E\|$ for $\theta(a) \in F$ we have that $(F, \|.\|_\theta)$ is a Banach space and it follows from the open mapping theorem that the norms $\|. : F\|, \|.\|_\theta$ are equivalent in F. Also for $\mid \theta(a) \mid \leq_\theta \mid \theta(b) \mid$ we find that $\mid a \mid \leq_E \mid b \mid$ hence $\|a : E\| \leq \|b : E\|$, $\|\theta(a)\|_\theta \leq \|\theta(b)\|_\theta$, we obtain that $(F, \|.\|_\theta, \leq_\theta)$ is a Banach lattice.

Denoting $\theta = \widetilde{T}/K : (l_1(\mathcal{A})/K, \|. : l_1(\mathcal{A})/K\|) \rightarrow (X, \|.\|)$ in the above sense (we have that each $x \in X$ is a unique image $\theta\left[(\lambda_\alpha(x))\right], (\lambda_\alpha(x)) \in l_1(\mathcal{A})$) we have

Theorem 3.1. *The elements $\theta\left[(\lambda_\alpha(x))\right] = x$ determine the Banach space $(X, \|.\|_\theta)$ where the norm $\|.\|_\theta$ is equivalent to the original norm of X.*

Proof. This follows from above. □

Corollary 3.2. *Given an infinite dimensional real Banach space $(X, \|.\|)$ and a normed Hamel base $\mathcal{H} = \{x_\alpha : \alpha \in A\}$ of X, there exist an equivalent norm $\|.\|_\mathcal{H}$ in X and a partial ordering $\leq_\mathcal{H}$ in X associated to $\mathcal{H}$ such that the triple $(X, \|.\|_\mathcal{H}, \leq_\mathcal{H})$ is a Banach lattice.*

Proof. This follows from above theorem where we denote $\|.\|_\mathcal{H} = \|.\|_\theta$, $\leq_\mathcal{H} = \leq_\theta$ following the above definition. □

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