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A note on universally prestarlike functions

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Abstract. Universally prestarlike functions of order $\alpha \leq 1$ in the slit domain $\Lambda = \mathcal{C} \setminus [1, \infty)$ have been recently introduced by S. Ruscheweyh. This notion generalizes the corresponding one for functions in the unit disk Δ (and other circular domains in $\mathcal{C}$). In this paper, we discuss the universally prestarlike functions defined through fractional derivatives.

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1. Introduction

Let $H(\Omega)$ denote the set of all analytic functions defined in a domain Ω. For domain Ω containing the origin $H_0(\Omega)$ stands for the set of all function $f \in H(\Omega)$ with $f(0) = 1$. We also use the notation $H_1(\Omega) = \{zf : f \in H_0(\Omega)\}$. In the special case when Ω is the open unit disk $\Delta = \{z \in \mathcal{C} : |z| < 1\}$, we use the abbreviation H, H_0 and H_1 respectively for $H(\Omega), H_0(\Omega)$ and $H_1(\Omega)$. A function $f \in H_1$ is called starlike of order α with $(0 \leq \alpha < 1)$ satisfying the inequality

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (z \in \Delta) \tag{1.1}$$

and the set of all such functions is denoted by S_α. The convolution or Hadamard Product of two functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=0}^{\infty} b_n z^n$$

is defined as

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

A function $f \in H_1$ is called prestarlike of order α (with $\alpha \leq 1$) if

$$\frac{z}{(1-z)^{2-2\alpha}} * f(z) \in S_\alpha. \tag{1.2}$$

The set of all such functions is denoted by $\mathcal{R}_\alpha$. (see [4]) The notion of prestarlike functions has been extended from the unit disk to other disk and half planes containing the origin. Let Ω be one such disk or half plane.Then there are two unique parameters $\gamma \in \mathcal{C} \setminus \{0\}$ and $\rho \in [0,1]$ such that

$$\Omega_{\gamma,\rho} = \{w_{\gamma,\rho}(z) : z \in \Delta\} \tag{1.3}$$

where,

$$w_{\gamma,\rho}(z) = \frac{\gamma z}{1-\rho z}. \tag{1.4}$$

Note that $1 \notin \Omega_{\gamma,\rho}$ if and only if $|\gamma+\rho| \le 1$.

Definition 1.1. *(see [2], [3], [4]) Let $\alpha \le 1$, and $\Omega = \Omega_{\gamma,\rho}$ for some admissible pair (γ,ρ). A function $f \in H_1(\Omega_{\gamma,\rho})$ is called prestarlike of order α in $\Omega_{\gamma,\rho}$ if*

$$f_{\gamma,\rho}(z) = \frac{1}{\gamma} f(w_{\gamma,\rho}(z)) \in \mathcal{R}_\alpha \tag{1.5}$$

The set of all such functions f is denoted by $\mathcal{R}_\alpha(\Omega)$.

Let Λ be the slit domain $\mathcal{C} \setminus [1,\infty)$(the slit being along the positive real axis).

Definition 1.2. *(see [2], [3], [4]) Let $\alpha \le 1$. A function $f \in H_1(\Lambda)$ is called universally prestarlike of order α if and only if f is prestarlike of order α in all sets $\Omega_{\gamma,\rho}$ with $|\gamma+\rho| \le 1$. The set of all such functions is denoted by $\mathcal{R}^u_\alpha$.*

Definition 1.3. *(see [4]) Let $\phi(z)$ be an analytic function with positive real part on Δ, which satisfies $\phi(0) = 1$, $\phi'(0) > 0$ and which maps the unit disc Δ onto a region starlike with respect to 1 and symmetric with respect to the real axis. Then the class $\mathcal{R}^u_\alpha(\phi)$ consists of all analytic function $f \in H_1(\Lambda)$ satisfying*

$$\frac{D^{3-2\alpha} f}{D^{2-2\alpha} f} \prec \phi(z) \tag{1.6}$$

where, $(D^\beta f)(z) = \frac{z}{(1-z)^\beta} \star f$, for $\beta \ge 0$ and $\prec$ denotes the subordination.

In particular, for $\beta = n \in \mathrm{N}$, we have $D^{n+1} f = \frac{z}{n!}(z^{n-1} f)^{(n)}$.

Remark 1.4. We let $\mathcal{R}^u_\alpha(A,B)$ denote the class $\mathcal{R}^u_\alpha(\phi)$ where

$$\phi(z) = \frac{1+Az}{1+Bz} \ (-1 \le B < A \le 1).$$

For suitable choices of A,B,α the class $\mathcal{R}^u_\alpha(A,B)$ reduces to several well known classes of functions. $\mathcal{R}^u_{\frac{1}{2}}(1,-1)$ is the class S^* of starlike univalent functions.

Lemma 1.5. *(see [1]) If $P_1(z) = 1 + c_1 z + c_2 z^2 + \ldots$ is an analytic function with positive real part in Δ, then*

$$|c_2 - v c_1^2| \le \begin{cases} -4v+2, & v \le 0 \\ 2, & 0 \le v \le 1 \\ 4v+2, & v \ge 1 \end{cases}$$

when $v < 0$, or $v > 1$, the equality holds if and only if $P_1(z)$ is $\frac{1+z}{1-z}$ or one of its rotations. When $0 < v < 1$, then the equality holds if and only if $P_1(z)$is $\frac{1+z^2}{1-z^2}$ or one of its rotations. If $v = 0$, the equality holds if and only if $P_1(z) = \left(\frac{1}{2} + \frac{\lambda}{2}\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{\lambda}{2}\right)\frac{1-z}{1+z}$, $0 \leq \lambda \leq 1$ or one of its rotations. If $v = 1$, the equality holds if and only if $P_1(z)$ is the reciprocal of one of the function for which the equality holds in the case of $v = 0$. Also the above upper bound can be improved as follows when $0 < v < 1$

$$|c_2 - vc_1^2| + v|c_1|^2 \leq 2 \qquad (0 < v \leq \frac{1}{2}) \tag{1.7}$$

$$|c_2 - vc_1^2| + (1-v)|c_1|^2 \leq 2 \quad (\frac{1}{2} < v \leq 1). \tag{1.8}$$

Lemma 1.6. *(see [5]) If $P_1(z) = 1 + c_1 z + c_2 z^2 + \ldots$ is an analytic function with positive real part in Δ, then $|c_2 - vc_1^2| \leq 2max\{1, |2v-1|\}$ the inequality is sharp for the function $P_1(z) = \frac{1+z}{1-z}$.*

Remark 1.7. Let

$$F(z) = \sum_{k=0}^{\infty} a_k z^k = \int_0^1 \frac{d\mu(t)}{1-tz}$$

where

$$a_k = \int_0^1 t^k d\mu(t),$$

$\mu(t)$ is a probability measure on $[0,1]$. Let T denote the set of all such functions F which are analytic in the slit domain Λ.

To Prove our main result we need the following definition.

Definition 1.8. *Let f be analytic in a simply connected region of the z-plane containing the origin. The fractional derivative of f of order λ is defined by*

$$D_z^{\lambda} f(z) := \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta \qquad (0 < \lambda < 1) \tag{1.9}$$

where the multiplicity of $(z-\zeta)^{\lambda}$ is removed by requiring that $log(z-\zeta)$ is real for $z - \zeta > 0$. Using the above definition and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava introduced the operator $\Omega^{\lambda} : \mathcal{A} \to \mathcal{A}$ for λ any positive real number $\neq 2, 3, 4, \ldots$ defined by

$$(\Omega^{\lambda} f)(z) = \Gamma(2-\lambda) z^{\lambda} D_z^{\lambda} f(z) \tag{1.10}$$

and $\mathcal{A} = H_1(\Delta)$. The class $(\mathcal{R}_{\alpha}^{u})^{\lambda}(\phi)$ consists of function $f \in \mathcal{A}$ for which $\Omega^{\lambda} f \in (\mathcal{R}_{\alpha}^{u})(\phi)$. Note that $(\mathcal{R}_{\alpha}^{u})^{\lambda}(\phi)$ is the special case of the class $(\mathcal{R}_{\alpha}^{u})^{g}(\phi)$ when

$$g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} z^n \tag{1.11}$$

Let

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n \ (g_n > 0),$$

g be analytic in Δ *and* $f * g \neq 0$*. Since*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in (\mathcal{R}_\alpha^u)^g(\phi)$$

if and only if

$$(f * g)(z) = z + \sum_{n=2}^{\infty} g_n a_n z^n \in (\mathcal{R}_\alpha^u)(\phi), \tag{1.12}$$

we obtain the coefficient estimate for functions in the class $(\mathcal{R}_\alpha^u)^g(\phi)$*, from the corresponding estimate for functions in the class* $(\mathcal{R}_\alpha^u)(\phi)$

2. Main Result

Theorem 2.1. *Let the function* ϕ *given by* $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots$*. If*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in (\mathcal{R}_\alpha^u)^g(\phi),$$

then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{g_3(3-2\alpha)} \left(B_2 + B_1^2(2-2\alpha) + \frac{(3-2\alpha)\mu g_3 B_1^2}{g_2^2} \right), & \mu \leq \sigma_1 \\ \frac{B_1}{g_3(3-2\alpha)}, & \sigma_1 \leq \mu \leq \sigma_2 \\ \frac{1}{g_3(3-2\alpha)} \left(-B_2 - B_1^2(2-2\alpha) + \frac{(3-2\alpha)\mu g_3 B_1^2}{g_2^2} \right), & \mu \geq \sigma_2, \end{cases}$$

where

$$\sigma_1 = \frac{g_2^2}{g_3} \left[\frac{(B_2 - B_1) + (2-2\alpha)B_1^2}{(3-2\alpha)B_1^2} \right], \tag{2.1}$$

$$\sigma_2 = \frac{g_2^2}{g_3} \left[\frac{(B_2 + B_1) + (2-2\alpha)B_1^2}{(3-2\alpha)B_1^2} \right] \tag{2.2}$$

the result is sharp.

Proof. If $f * g \in \mathcal{R}_\alpha^u$, then there is a schwartz function $w(z)$, analytic in Δ with $w(0) = 0$ and $|w(z)| < 1$ in Δ such that $\frac{D^{3-2\alpha}(f*g)}{D^{2-2\alpha}(f*g)} = \phi(w(z))$. Define the function $P_1(z)$ by,

$$P_1(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \ldots$$

Since $w(z)$ is a schwartz function, we see that $Re P_1(z) > 0$ and $P_1(0) = 1$. Define the function

$$P(z) = \frac{D^{3-2\alpha}(f * g)}{D^{2-2\alpha}(f * g)} = 1 + b_1 z + b_2 z^2 + \ldots \tag{2.3}$$

Therefore,

$$P(z) = \phi\left(\frac{P_1(z) - 1}{P_1(z) + 1} \right).$$

Now,

$$\frac{P_1(z)-1}{P_1(z)+1} = \frac{c_1z+c_2z^2+\dots}{2+c_1z+c_2z^2+\dots}$$
$$= \frac{1}{2}\left[c_1z+[c_2-\frac{c_1^2}{2}]z^2+[c_3-c_1c_2+\frac{c_1^3}{4}z^3]+\dots\right]$$

Hence upon simplification, we get,

$$P(z) = 1+\frac{B_1c_1z}{2}+\left[\frac{B_1}{2}\left(c_2-\frac{c_1^2}{2}\right)+\frac{B_2c_1^2}{4}\right]z^2+\dots \tag{2.4}$$

Therefore,

$$1+b_1z+b_2z^2+\dots = 1+\frac{B_1c_1z}{2}+\left[\frac{B_1}{2}\left(c_2-\frac{c_1^2}{2}\right)+\frac{B_2c_1^2}{4}\right]z^2+\dots \tag{2.5}$$

Equating the like coefficients we get,

$$b_1 = \frac{B_1c_1}{2} \tag{2.6}$$

$$b_2 = \frac{B_1}{2}\left(c_2-\frac{c_1^2}{2}\right)+\frac{B_2c_1^2}{4} \tag{2.7}$$

Therefore, from the equation (2.3) we have

$$1+A_1z+A_2z^2+\dots = 1+b_1z+b_2z^2+\dots \tag{2.8}$$

where,

$$A_1 = [\mathcal{C}'(\alpha,2)a_2g_2-\mathcal{C}(\alpha,2)a_2g_2]$$

$$A_2 = \left[\mathcal{C}'(\alpha,3)a_3g-\mathcal{C}(\alpha,2)\mathcal{C}'(\alpha,2)a_2^2-\mathcal{C}(\alpha,3)a_3+(\mathcal{C}(\alpha,2)a_2)^2\right],$$

$$\mathcal{C}(\alpha,n) = \frac{\prod_{k=2}^{n}(k-2\alpha)}{(n-1)!},\ \mathcal{C}'(\alpha,n) = \frac{\prod_{k=2}^{n}(k+1-2\alpha)}{(n-1)!},$$

$$b_n = \int_0^1 t^n d\mu(t)$$

for $n=2,3,\dots$ and $\mu(t)$ a probability measure on $[0,1]$.
Equating the coefficients of z and z^2 respectively and simplifying we get,

$$a_2 = \frac{b_1}{g_2};\quad a_3 = \frac{b_2+(2-2\alpha)b_1^2}{g_3(3-2\alpha)}. \tag{2.9}$$

Applying the equations(2.6) and (2.7) in(2.9) , we get,

$$a_2 = \frac{B_1c_1}{2g_2};\quad a_3 = \frac{1}{g_3(3-2\alpha)}\left[\frac{B_1}{2}\left(c_2-\frac{c_1^2}{2}\right)+\frac{B_2c_1^2}{4}+(2-2\alpha)\frac{B_1^2c_1^2}{4}\right].$$

Now,

$$\begin{aligned}
a_3 - \mu a_2^2 &= \frac{1}{g_3(3-2\alpha)}\left[\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2 c_1^2}{4} + (2-2\alpha)\frac{B_1^2 c_1^2}{4}\right] - \mu\frac{B_1^2 c_1^2}{4g_2^2} \\
&= \frac{1}{g_3(3-2\alpha)}\frac{B_1}{2}\left[c_2 - c_1^2\left(\frac{1}{2} - \frac{B_2}{2B_1} - (2-2\alpha)\frac{B_1}{2} + (3-2\alpha)\mu\frac{g_3 B_1}{2g_2^2}\right)\right] \\
&= \frac{B_1}{2g_3(3-2\alpha)}\left[c_2 - c_1^2 v\right]
\end{aligned}$$

where,

$$v = \left[\frac{1}{2} - \frac{B_2}{2B_1} - (2-2\alpha)\frac{B_1}{2} + (3-2\alpha)\mu\frac{g_3 B_1}{2g_2^2}\right] \tag{2.10}$$

Now an application of lemma (1.5) (see [1]) yields the inequalities stated in the theorem under the respective conditions. For the sharpness of the results in the above theorem we have the following:

1. If $\mu = \sigma_1$, then the equality holds in the lemma (1.1) if and only if
$$P_1(z) = \left(\frac{1}{2} + \frac{\lambda}{2}\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{\lambda}{2}\right)\frac{1-z}{1+z} \quad 0 \le \lambda \le 1$$
or one of its rotations.
2. If $\mu = \sigma_2$, then
$$\frac{1}{P_1(z)} = \frac{1}{\left(\frac{1}{2} + \frac{\lambda}{2}\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{\lambda}{2}\right)\frac{1-z}{1+z}}.$$
3. If $\sigma_1 < \mu < \sigma_2$ $P_1(z) = \frac{1+\lambda z^2}{1-\lambda z^2}$.

To show that the bounds are sharp, we define the function $K_\alpha^{\phi_n}$ $(n = 2, 3, \ldots)$ by

$$\frac{D^{3-2\alpha} K_\alpha^{\phi_n}}{D^{3-2\alpha} K_\alpha^{\phi_n}} = \phi(z^{n-1}) \tag{2.11}$$

$K_\alpha^{\phi_n}(0) = 0$, $(K_\alpha^{\phi_n})'(0) = 1$ and function F_α^λ and G_α^λ $(0 \le \lambda \le 1)$ by

$$\frac{\left(D^{3-2\alpha} F_\alpha^\lambda\right)(z)}{\left(D^{2-2\alpha} F_\alpha^\lambda\right)(z)} = \phi\left(\frac{z(z+\lambda)}{1+\lambda z}\right) \tag{2.12}$$

$F_\alpha^\lambda(0) = 0$, $(F_\alpha^\lambda)'(0) = 1$ and similarly

$$\frac{\left(D^{3-2\alpha} G_\alpha^\lambda\right)(z)}{\left(D^{2-2\alpha} G_\alpha^\lambda\right)(z)} = \phi\left(\frac{z(z+\lambda)}{1+\lambda z}\right) \tag{2.13}$$

$G_\alpha^\lambda(0) = 0$, $(G_\alpha^\lambda)'(0) = 1$. Clearly, the functions $K_\alpha^{\phi_n}, F_\alpha^\lambda, G_\alpha^\lambda \in \mathcal{R}_\alpha^u$. Also we write $K_\alpha^\phi := K_\alpha^{\phi_2}$. If $\mu < \sigma_1$ or $\mu < \sigma_2$, then the equality holds if and only if f is K_α^ϕ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then the equality holds if and only if f is $K_\alpha^{\phi_3}$ or one of its rotations. If $\mu = \sigma_1$, then the equality holds if and only if f is F_α^λ or one of its rotations If $\mu = \sigma_2$ then the equality holds if and only if f is G_α^λ or one of its rotations. Hence the result. □

Corollary 2.2. *If $g(z) = \frac{z}{1-z} \in \mathcal{R}_0^u$ in Theorem 2.1 we get our earlier result viz., Theorem 3.1 of (see* [7]*).*

Corollary 2.3. *Taking*

$$g(z) = (\Omega^{\lambda} f)(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n z^n,$$

$(f * g)$ denotes the fractional derivative of f and hence if

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in (\mathcal{R}_{\alpha}^{u})^{g}(\phi) \tag{2.14}$$

then,

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{(3-\lambda)(2-\lambda)}{6(3-2\alpha)} \left(B_2 + B_1^2(2-2\alpha) + \frac{(3-2\alpha)\mu g_3 B_1^2}{g_2^2} \right), & \mu \leq \sigma_1 \\ \frac{(3-\lambda)(2-\lambda)B_1}{6(3-2\alpha)}, & \sigma_1 \leq \mu \leq \sigma_2 \\ \frac{(3-\lambda)(2-\lambda)}{6(3-2\alpha)} \left(-B_2 - B_1^2(2-2\alpha) + \frac{(3-2\alpha)\mu g_3 B_1^2}{g_2^2} \right), & \mu \geq \sigma_2, \end{cases}$$

where,

$$\sigma_1 = \frac{2(3-\lambda)}{3(2-\lambda)} \left[\frac{(B_2 - B_1) + (2-2\alpha)B_1^2}{(3-2\alpha)B_1^2} \right], \tag{2.15}$$

$$\sigma_2 = \frac{2(3-\lambda)}{3(2-\lambda)} \left[\frac{(B_2 + B_1) + (2-2\alpha)B_1^2}{(3-2\alpha)B_1^2} \right] \tag{2.16}$$

the result is sharp.

Proof. This corollary follows from the observations

$$g_2 := \frac{\Gamma(3)\Gamma(2-\lambda)}{\Gamma(3-\lambda)} = \frac{2}{2-\lambda} \tag{2.17}$$

and

$$g_3 := \frac{\Gamma(4)\Gamma(2-\lambda)}{\Gamma(4-\lambda)} = \frac{6}{(2-\lambda)(3-\lambda)}. \tag{2.18}$$

□

Corollary 2.4. *Taking*

$$g(z) = z + \sum_{n=2}^{\infty} n^m z^n, \quad m \in \mathcal{N}_o = \{0\} \cup \mathcal{N},$$

$(f * g)$ denotes the Sălăgean derivative of f (see [6]*) and hence if*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in (\mathcal{R}_{\alpha}^{u})^{g}(\phi)$$

then,

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{3^m(3-2\alpha)} \left(B_2 + B_1^2(2-2\alpha) + \frac{3^m(3-2\alpha)\mu B_1^2}{2^{2m}} \right), & \mu \leq \sigma_1 \\ \frac{B_1}{3^m(3-2\alpha)}, & \sigma_1 \leq \mu \leq \sigma_2 \\ \frac{1}{3^m(3-2\alpha)} \left(-B_2 - B_1^2(2-2\alpha) + \frac{3^m(3-2\alpha)\mu B_1^2}{2^{2m}} \right), & \mu \geq \sigma_2, \end{cases}$$

where,

$$\sigma_1 = \frac{2^{2m}}{3^m}\left[\frac{(B_2 - B_1) + (2-2\alpha)B_1^2}{(3-2\alpha)B_1^2}\right], \tag{2.19}$$

$$\sigma_2 = \frac{2^{2m}}{3^m}\left[\frac{(B_2 + B_1) + (2-2\alpha)B_1^2}{(3-2\alpha)B_1^2}\right] \tag{2.20}$$

the result is sharp.

Proof. This corollary follows from the observations $g_2 = 2^m$ and $g_3 = 3^m$. □

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