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On the existence of solutions for a class of fractional differential equations

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Abstract. In this paper, we study the existence and uniqueness of solutions to the Cauchy problem for nonautonomous fractional differential equations involving Caputo derivative in Banach spaces. Definition for the solution in the Carathéodory sense and fundamental lemma are introduced. Some sufficient conditions for the existence and uniqueness of solutions are established by means of fractional calculus, Hölder inequality via fixed point theorem under some weak conditions.

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1. Introduction

Fractional differential equations have gained considerable importance due to their application in various sciences, such as physics, mechanics, chemistry, engineering, etc. One can see the monographs of Diethelm [2], Miller and Ross [3], Kilbas et al. [4], Lakshmikantham et al. [5], Podlubny [6]. In survey, Agarwal et al. [7, 8] establish sufficient conditions for the existence and uniqueness of solutions for various classes of initial and boundary value problem for fractional differential equations and inclusions involving the Caputo derivative in finite and involving the Riemann-Liouville derivative in infinite dimensional spaces. Very recently, a lot of papers have been devoted to fractional differential equations and optimal controls in Banach spaces [9, 10, 11, 12, 13, 14, 15, 16].

In this paper, we reconsider the following Cauchy problem for nonautonomous fractional differential equations

$$\begin{cases} {}^cD^q u(t) = A(t)u(t) + f(t,u(t)),\ t \in J = [0,T],\ T > 0, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

in a Banach space X, where ${}^cD^q$ is the Caputo fractional derivative of order $q \in (0,1)$, $\{A(t), t \in J\}$ is a family of linear bounded operators in X, the function $t \to A(t)$ is continuous in the uniform operator topology, $f : J \times X \to X$ is Lebesgue measurable with respect to t and satisfies some assumptions that will be specified later.

A pioneering work on the existence of solutions for this kind of Cauchy problems has been studied by Balachandran and Park [9] in the case of $f : J \times X \to X$ is continuous and satisfies uniformly Lipschitz condition. In the present paper, we revisit this interesting problem and introduce a definition for solution of the system (1.1) in the Carathéodory sense and establish the existence and uniqueness of solutions for the system (1.1) under some weak conditions.

To prove our main results, we apply the classical fixed point theory including Krasnoselskii's fixed point theorem and Banach contraction principle via fractional calculus and Hölder inequality. Compared with the results appeared in [9], there are at least two differences: (i) assumptions on f are more general and easy to check; (ii) a definition for solution in the Carathéodory sense is established; (iii) two new existence results of solution in the Carathéodory sense are given.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts. Throughout this paper, $(X, \|\cdot\|)$ will be a Banach spaces. Let $C(J, X)$ be the Banach space of all continuous functions from J into X with the norm $\|u\|_C := \sup\{\|u(t)\| : t \in J\}$ for $u \in C(J, X)$.

Let us recall the following known definitions. For more details see [4].

Definition 2.1. *The fractional integral of order γ with the lower limit zero for a function f is defined as*

$$I^\gamma f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{f(s)}{(t-s)^{1-\gamma}} ds, \ t > 0, \ \gamma > 0,$$

provided the right side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2. *The Riemann-Liouville derivative of order γ with the lower limit zero for a function $f : [0, \infty) \to R$ can be written as*

$$^L D^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\gamma+1-n}} ds, \ t > 0, \ n-1 < \gamma < n.$$

Definition 2.3. *The Caputo derivative of order γ for a function $f : [0, \infty) \to R$ can be written as*

$$^c D^\gamma f(t) = \ ^L D^\gamma \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \ t > 0, \ n-1 < \gamma < n.$$

Remark 2.4. (i) If $f(t) \in C^n[0, \infty)$, then

$$^c D^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\gamma+1-n}} ds = I^{n-\gamma} f^{(n)}(t), \ t > 0, \ n-1 < \gamma < n.$$

(ii) The Caputo derivative of a constant is equal to zero.

(iii) If f is an abstract function with values in X, then integrals which appear in Definitions 2.1 and 2.2 are taken in Bochner's sense.

For measurable functions $m : J \to R$, define the norm

$$\|m\|_{L^p(J)} = \begin{cases} \left(\int_J |m(t)|^p dt \right)^{\frac{1}{p}}, \ 1 \leq p < \infty, \\ \inf\limits_{\mu(\bar{J})=0} \{ \sup\limits_{t \in J - \bar{J}} |m(t)| \}, \ p = \infty, \end{cases}$$

where $\mu(\bar{J})$ is the Lebesgue measure on $\bar{J}$. Let $L^p(J, R)$ be the Banach space of all Lebesgue measurable functions $m : J \to R$ with $\|m\|_{L^p(J)} < \infty$.

Lemma 2.5. *(Lemma 2.1, [17]) For all $\beta > 0$ and $\vartheta > -1$,*

$$\int_0^t (t-s)^{\beta-1} s^{\vartheta} ds = C(\beta, \vartheta) t^{\beta+\vartheta}$$

where $C(\beta, \vartheta) = \frac{\Gamma(\beta)\Gamma(\vartheta+1)}{\Gamma(\beta+\vartheta+1)}$.

Theorem 2.6. *(Krasnoselskii fixed point theorem) Let $\mathfrak{B}$ be a closed convex and nonempty subsets of X. Suppose that $\mathcal{L}$ and $\mathcal{N}$ are in general nonlinear operators which map $\mathfrak{B}$ into X such that*
(i) $\mathcal{L}x + \mathcal{N}y \in \mathfrak{B}$ whenever $x, y \in \mathfrak{B}$;
(ii) $\mathcal{L}$ is a contraction mapping;
(iii) $\mathcal{N}$ is compact and continuous.
Then there exists a $z \in \mathfrak{B}$ such that $z = \mathcal{L}z + \mathcal{N}z$.

3. Main results

In this section, we discuss the existence of solution for the system (1.1) by means of fixed point theorems.

We make the following assumptions:

[H1]: For any $u \in X$, $f(t, u)$ is Lebesgue measurable with respect to t on J.

[H2]: For any $t \in J$, $f(t, u)$ is continuous with respect to u on X.

[H3]: There exist a $q_1 \in (0, q)$ and a function $h(t) \in L^{\frac{1}{q_1}}(J, R^+) := L^{\frac{1}{q_1}}(J)$, such that $\|f(t, u)\| \leq h(t)$, for arbitrary $(t, u) \in J \times X$.

[H4]: For every $t \in J$, the set $K = \{(t-s)^{q-1} f(s, u(s)) : u \in C(J, X), s \in [0, t]\}$ is relatively compact.

Now, let us introduce the definition of a solution of the system (1.1).

Definition 3.1. *A function $u \in C(J, X)$ is called a solution of the system (1.1) on J if*
(i) the function $u(t)$ is absolutely continuous on J,
(ii) $u(0) = u_0$, and
(iii) u satisfies the equation in the system (1.1).

For brevity, let

$$H = \|h\|_{L^{\frac{1}{q_1}}(J)}, \ \|A(t)\| \leq M, \ \beta = \frac{q-1}{1-q_1} \in (-1, 0).$$

By Definition 2.1–2.3, using the same method in Theorem 3.2 of [1], we obtain the following lemma immediately.

Lemma 3.2. *Let the hypothesis [H1]–[H3] hold. A function $u \in C(J,X)$ is a solution of the fractional integral equation*

$$u(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} A(s)u(s)ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,u(s))ds, \quad (3.1)$$

if and only if u is a solution of the system (1.1).

Now, we are ready to present and prove our main results.

Theorem 3.3. *Assume that [H1]–[H4] hold. If the following condition*

$$\Omega_{M,T,q} = \frac{MT^q}{\Gamma(q+1)} < 1 \quad (3.2)$$

holds, then the system (1.1) has at least one solution.

Proof. Choose

$$r \geq \frac{\frac{HT^{(1+\beta)(1-q_1)}}{\Gamma(q)(1+\beta)^{1-q_1}} + \frac{M\|u_0\|T^q}{\Gamma(q+1)}}{1 - \frac{MT^q}{\Gamma(q+1)}}, \quad (3.3)$$

and define the set

$$C_r = \{u \in C(J,X) : \|u - u_0\| \leq r\}.$$

By Lemma 3.2, the system (1.1) is equivalent to the following fractional integral equation

$$u(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} A(s)u(s)ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,u(s))ds.$$

Now we define two operators P and Q on C_r as follows:

$$(Pu)(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,u(s))ds,$$

and

$$(Qu)(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} A(s)u(s)ds.$$

Therefore, the existence of a solution of the system (1.1) is equivalent to that the operator $P + Q$ has a fixed point on C_r. The proof is divided into three steps.

Step 1: For all $u, v \in C_r$, $Pu + Qv \in C_r$.

For every pair $u, v \in C_r$ and any $\delta > 0$, by using Hölder inequality, we get

$$
\begin{aligned}
&\|(Pu+Qv)(t+\delta)-(Pu+Qv)(t)\| \\
\leq\ & \frac{1}{\Gamma(q)}\int_0^t [(t-s)^{q-1}-(t+\delta-s)^{q-1}]h(s)ds \\
&+\frac{1}{\Gamma(q)}\int_t^{t+\delta}(t+\delta-s)^{q-1}h(s)ds \\
&+\frac{1}{\Gamma(q)}\int_0^t [(t-s)^{q-1}-(t+\delta-s)^{q-1}]M\|v(s)\|ds \\
&+\frac{1}{\Gamma(q)}\int_t^{t+\delta}(t+\delta-s)^{q-1}M\|v(s)\|ds \\
\leq\ & \frac{1}{\Gamma(q)}\left(\int_0^t [(t-s)^{q-1}-(t+\delta-s)^{q-1}]^{\frac{1}{1-q_1}}ds\right)^{1-q_1}\left(\int_0^t (h(s))^{\frac{1}{q_1}}ds\right)^{q_1} \\
&+\frac{1}{\Gamma(q)}\left(\int_t^{t+\delta}[(t+\delta-s)^{q-1}]^{\frac{1}{1-q_1}}ds\right)^{1-q_1}\left(\int_t^{t+\delta}(h(s))^{\frac{1}{q_1}}ds\right)^{q_1} \\
&+\frac{M(\|u_0\|+r)}{\Gamma(q)}\int_0^t (t-s)^{q-1}-(t+\delta-s)^{q-1}ds \\
&+\frac{M(\|u_0\|+r)}{\Gamma(q)}\int_t^{t+\delta}(t+\delta-s)^{q-1}ds \\
\leq\ & \frac{H}{\Gamma(q)}\left(\int_0^t (t-s)^{\beta}-(t+\delta-s)^{\beta}ds\right)^{1-q_1} \\
&+\frac{H}{\Gamma(q)}\left(\int_t^{t+\delta}(t+\delta-s)^{\beta}ds\right)^{1-q_1} \\
&+\frac{M(\|u_0\|+r)}{\Gamma(q)}\int_0^t (t-s)^{q-1}-(t+\delta-s)^{q-1}ds \\
&+\frac{M(\|u_0\|+r)}{\Gamma(q)}\int_t^{t+\delta}(t+\delta-s)^{q-1}ds \\
\leq\ & \frac{H}{\Gamma(q)(1+\beta)^{1-q_1}}\left(t^{1+\beta}-(t+\delta)^{1+\beta}+\delta^{1+\beta}\right)^{1-q_1} \\
&+\frac{H}{\Gamma(q)(1+\beta)^{1-q_1}}\delta^{(1+\beta)(1-q_1)} \\
&+\frac{M(\|u_0\|+r)}{\Gamma(q+1)}(t^q-(t+\delta)^q+\delta^q)+\frac{M(\|u_0\|+r)}{\Gamma(q+1)}\delta^q \\
\leq\ & \frac{2H}{\Gamma(q)(1+\beta)^{1-q_1}}\delta^{(1+\beta)(1-q_1)}+\frac{2M(\|u_0\|+r)}{\Gamma(q+1)}\delta^q.
\end{aligned}
$$

As $\delta \to 0$, the right-hand side of the above inequality tends to zero. Therefore $Pu + Qv \in C(J, X)$.

Moreover, for all $t \in J$, we get

$$\begin{aligned}
&\|(Pu)(t) + (Qv)(t) - u_0\| \\
\leq\ & \frac{H}{\Gamma(q)} \left(\int_0^t (t-s)^{\beta} ds \right)^{1-q_1} + \frac{M(\|u_0\| + r)}{\Gamma(q)} \int_0^t (t-s)^{q-1} ds \\
\leq\ & \frac{HT^{(1+\beta)(1-q_1)}}{\Gamma(q)(1+\beta)^{1-q_1}} + \frac{M(\|u_0\| + r)T^q}{\Gamma(q+1)} \\
\leq\ & \left[\frac{HT^{(1+\beta)(1-q_1)}}{\Gamma(q)(1+\beta)^{1-q_1}} + \frac{M\|u_0\|T^q}{\Gamma(q+1)} \right] + \frac{MT^q}{\Gamma(q+1)} r \\
\leq\ & r,
\end{aligned}$$

which implies that $Pu + Qv \in C_r$.

Step 2: Q is a contraction operator.

For arbitrary $u, v \in C_r$, we have

$$\begin{aligned}
\|Qu - Qv\| &\leq \frac{M}{\Gamma(q)} \left(\int_0^t (t-s)^{q-1} ds \right) \|u - v\|_C \\
&\leq \frac{MT^q}{\Gamma(q+1)} \|u - v\|_C,
\end{aligned}$$

which implies that

$$\|Qu - Qv\|_C \leq \Omega_{M,T,q} \|u - v\|_C.$$

From the condition (3.2), we know that Q is a contraction operator.

Step 3: We show that P is a complete continuous operator.

For that, let $\{u_n\}$ be a sequence of C_r such that $u_n \to u$ in C_r. Then, $f(s, u_n(s)) \to f(s, u(s))$ as $n \to \infty$ due to the hypotheses [H2].

Now, for all $t \in J$, we have

$$\|(Pu_n)(t) - (Pu)(t)\| \leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|f(s, u_n(s)) - f(s, u(s))\| ds.$$

On the one other hand using [H3], we get for each $t \in J$,

$$\|f(s, u_n(s)) - f(s, u(s))\| \leq 2h(s) \in L^{\frac{1}{q_1}}(J).$$

On the other hand, using the fact that the functions $s \to 2h(s)(t-s)^{q-1}$ is integrable on J, by means of the Lebesgue Dominated Convergence Theorem yields

$$\int_0^t (t-s)^{q-1} \|f(s, u_n(s)) - f(s, u(s))\| ds \to 0.$$

Thus, $Pu_n \to Pu$ as $n \to \infty$ which implies that P is continuous.

Let $\{u_n\}$ be a sequence on C_r then

$$(Pu_n)(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u_n(s)) ds,$$

for all $t \in J$, using Hölder inequality, we have

$$\begin{aligned}
\|(Pu_n)(t)\| &\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)ds \\
&\leq \frac{HT^{(1+\beta)(1-q_1)}}{\Gamma(q)(1+\beta)^{1-q_1}}.
\end{aligned}$$

This yields that the sequence $\{Pu_n\}$ is uniformly bounded.

Now, we need to prove that $\{Pu_n\}$ be equicontinuous.

For $0 \leq t_1 < t_2 \leq T$, we get

$$\begin{aligned}
&\|(Pu_n)(t_2) - (Pu_n)(t_1)\| \\
\leq\ & \frac{1}{\Gamma(q)}\int_0^{t_1}[(t_1-s)^{q-1} - (t_2-s)^{q-1}]h(s)ds \\
&+\frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}h(s)ds \\
\leq\ & \frac{1}{\Gamma(q)}\left(\int_0^{t_1}[(t_1-s)^{q-1} - (t_2-s)^{q-1}]^{\frac{1}{1-q_1}}ds\right)^{1-q_1}\left(\int_0^{t_1}(h(s))^{\frac{1}{q_1}}ds\right)^{q_1} \\
&+\frac{1}{\Gamma(q)}\left(\int_{t_1}^{t_2}[(t_2-s)^{q-1}]^{\frac{1}{1-q_1}}ds\right)^{1-q_1}\left(\int_{t_1}^{t_2}(h(s))^{\frac{1}{q_1}}ds\right)^{q_1} \\
\leq\ & \frac{H}{\Gamma(q)}\left(\int_0^{t_1}(t_1-s)^{\beta} - (t_2-s)^{\beta}ds\right)^{1-q_1} \\
&+\frac{H}{\Gamma(q)}\left(\int_{t_1}^{t_2}(t_2-s)^{\beta}ds\right)^{1-q_1} \\
\leq\ & \frac{H}{\Gamma(q)(1+\beta)^{1-q_1}}\left(t_1^{1+\beta} - t_2^{1+\beta} + (t_2-t_1)^{1+\beta}\right)^{1-q_1} \\
&+\frac{H}{\Gamma(q)(1+\beta)^{1-q_1}}(t_2-t_1)^{(1+\beta)(1-q_1)} \\
\leq\ & \frac{2H}{\Gamma(q)(1+\beta)^{1-q_1}}(t_2-t_1)^{(1+\beta)(1-q_1)}.
\end{aligned}$$

As $t_2 \to t_1$, the right-hand side of the above inequality tends to zero. Therefore $\{Pu_n\}$ is equicontinuous.

In view of the condition [H4] and Mazur Lemma, we know that $\overline{conv}K$ is compact.

For any $t^* \in J$,

$$\begin{aligned}
(Pu_n)(t^*) &= \frac{1}{\Gamma(q)}\lim_{k\to\infty}\sum_{i=1}^{k}\frac{t^*}{k}(t^* - \frac{it^*}{k})^{q-1}f(\frac{it^*}{k}, u_n(\frac{it^*}{k})) \\
&= \frac{t^*}{\Gamma(q)}\zeta_n,
\end{aligned}$$

where

$$\zeta_n = \lim_{k\to\infty}\sum_{i=1}^{k}\frac{1}{k}(t^*-\frac{it^*}{k})^{q-1}f(\frac{it^*}{k},u_n(\frac{it^*}{k})).$$

Since $\overline{conv}K$ is convex and compact, we know that $\zeta_n \in \overline{conv}K$. Hence, for any $t^* \in J$, the set $\{Pu_n\}$ $(n = 1, 2, \cdots)$ is relatively compact. From Ascoli-Arzela theorem every $\{Pu_n(t)\}$ contains a uniformly convergent subsequence $\{Pu_{n_k}(t)\}$ $(k = 1, 2, \cdots)$ on J. Thus, the set $\{Pu : u \in C_r\}$ is relatively compact.

Therefore, the continuity of P and relatively compactness of the set $\{Pu : u \in C_r\}$ implies that P is a completely continuous operator. By Krasnoselskii's fixed point theorem, we get that $P + Q$ has a fixed point in C_r. Then system (1.1) has a solution on $t \in J$, and this completes the proof. □

Now we assume the following hypotheses:

[H5]: There exist a $q_2 \in [0, q)$ and a real-valued function $\mu(t) \in L^{\frac{1}{q_2}}(J)$ such that

$$\|f(t, u) - f(t, v)\| \leq \mu(t)\|u - v\|, \text{ for all } u, v \in X,\ t \in J.$$

[H6]: Let

$$\Phi_{K,M,T,q,q_2} = \frac{KT^{(1+\beta')(1-q_2)}}{\Gamma(q)(1+\beta')^{1-q_2}} + \frac{MT^q}{\Gamma(q+1)} < 1$$

where $K = \|\mu\|_{L^{\frac{1}{q_2}}(J)}$, $\beta' = \frac{q-1}{1-q_2} \in (-1, 0)$.

Theorem 3.4. *Assume that [H1]–[H3], [H5]–[H6] hold. Then the system (*1.1*) has a unique solution.*

Proof. We define a operator F by

$$(Fu)(t) = u_0 + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}A(s)u(s)ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f(s, u(s))ds.$$

Therefore, the existence of a solution of the system (1.1) is equivalent to that the operator F has a fixed point in C_r, where r is given in (3.3).

We can show that $F(C_r) \subseteq C_r$. In fact, for any $u, v \in C_r$, by using Hölder inequality we get

$$\begin{aligned}\|(Fu)(t) - (Fv)(t)\| &\leq \frac{K\|u-v\|_C}{\Gamma(q)}\left(\int_0^t (t-s)^{\beta'}ds\right)^{1-q_2} \\ &\quad + \frac{M}{\Gamma(q)}\left(\int_0^t (t-s)^{q-1}ds\right)\|u-v\|_C \\ &\leq \left[\frac{KT^{(1+\beta')(1-q_2)}}{\Gamma(q)(1+\beta')^{1-q_2}} + \frac{MT^q}{\Gamma(q+1)}\right]\|u-v\|_C.\end{aligned}$$

Hence,

$$\|Fu - Fv\|_C \leq \Phi_{K,M,T,q,q_2}\|u - v\|_C.$$

In view of [H6], by applying the Banach contraction mapping principle we know that the operator F has a unique fixed point in C_r. Therefore, the system (1.1) has a unique solution. The proof is completed. □

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