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Transversality and separation of zeroes in second order differential equations

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Abstract. In this paper we consider some second order differential equations in a finite time interval. We give some conditions which ensure that the non-trivial solutions of these differential equations have a finite number of transverse zeroes.

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1. Introduction

The following second order non-autonomous and non-linear differential equation was considered in [1]:

$$(Lu)(t) := -(p(t)u'(t))' + q(t)u(t) = f(t, u(t)), \quad t \in (a, b). \tag{1.1}$$

Here $(a, b) \subseteq \mathbb{R}$, f is a non-linear continuous function, not necessarily Lipschitz continuous function in u, $f(t, 0) \equiv 0$, $p, q \in C^1[a, b]$ and $p(t) > 0$ for all $t \in [a, b]$.

Some sufficient conditions on the non-linearity of f were given which ensure that non-trivial solutions of the second order differential equations of the form (1.1) have a finite number of transverse zeroes ($u(0) = u'(0) = 0$) in a given finite time interval (a, b).

The solution of the equation (1.1) isn't unique when the function f is non-Lipschitz. For example the differential equation

$$-u'' = 24\sqrt{|u|}, \ t \in \mathbb{R}, \tag{1.2}$$

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has at least two solutions, $u_1 \equiv 0$ and u_2 given by

$$u_2(t) = \begin{cases} 0, & t \le 0 \\ -4t^4, & t > 0 \end{cases} . \tag{1.3}$$

Hence there exist non-unique, non-zero solutions possessing a non-transverse zero and, in particular, infinitely many zeroes on any open time interval containing $t = 0$.

In fact, Zeidler in [5] proved that there exist ordinary differential equations which have uncountable many solutions satisfying the conditions of transversality: $u(0) = u'(0) = 0$.

Laister and Beardmore in [1] give only locally conditions on function f, near $u = 0$, and independent of the sign of q which ensure that non-trivial solutions of (1.1) have a finite number of transverse zeroes in a finite time interval ([1], Theorem 2.1).

Let S a finite subset of $[a, b]$. and we denote by $[a, b]_S = [a, b] \setminus S$.

For the case when the equation (1.1) is written in the form

$$(Lu)(t) := -p(t)u''(t) + r(t)u'(t) + q(t)u(t) = f(t, u(t)), \quad t \in (a, b), \tag{1.4}$$

the condition $p \in C^1[a, b]$ can be replaced by $p \in C^1[a, b]_S$, and the situation described above remains true.

For example, with $S = \{0\}$, the differential equation

$$-(sgn\ t + 3)u''(t) = 144\sqrt{|u(t)|},\ t \in \mathbb{R}_S, \tag{1.5}$$

has at least two solutions, $u_1 \equiv 0$ and u_3 given by

$$u_3(t) = \begin{cases} -36(t+2)^4, & t < -2 \\ 0, & -2 \le t < 0 \\ -4t^4, & t > 0 \end{cases} . \tag{1.6}$$

Hence there exist non-unique, non-zero solutions possessing a non-transverse zero and, in particular, infinitely many zeroes on any open interval included in $(-2, 0)$.

2. Main results

We consider a second order differential equation of the form:

$$F(t, u, u', u'') = 0, \quad t \in (a, b) \subseteq \mathbb{R}. \tag{2.1}$$

For the convenience of the reader, following I.A. Rus ([3]), we present the proofs of the next two results:

Theorem 2.1. *We suppose that the following conditions are satisfied:*

1^o *the function F is homogeneous with respect to variables u, u', u'';*

2^o *for all $t_0 \in (a, b)$, $u_0', u_0'' \in \mathbb{R}$ there exists a unique solution of the equation (2.1) such that $u'(t_0) = u_0'$, $u''(t_0) = u_0''$.*

Then, if t_1 and t_2 are two successive zeroes of u_1', where u_1 is a solution of the equation (2.1), every other solution u_2 of the equation (2.1), for which $u_2'(t_1) \neq 0$, $u_2'(t_2) \neq 0$, has in (t_1, t_2) a unique zero.

Proof. We suppose that $u_2'(t) \neq 0$ for all $t \in [t_1, t_2]$. It is not a restriction to assume that

$$u_1'(t) > 0 \quad \text{for} \quad t \in (t_1, t_2) \quad \text{and}$$
$$u_2'(t) > 0 \quad \text{for} \quad t \in [t_1, t_2].$$

Then by Tonelli's Lemma (see [2]) it results that there exist $\lambda > 0$ and $t_0 \in (t_1, t_2)$ such that

$$u_2'(t_0) = \lambda u_1'(t_0) \quad \text{and}$$
$$u_2''(t_0) = \lambda u_1''(t_0).$$

From the conditions $1^o, 2^o$ we get that $u_2(t) \equiv \lambda u_1(t)$, i.e. a contradiction, which proves the theorem. □

Theorem 2.2. *We suppose that:*

1^o *the function F is homogeneous with respect to variables u, u', u'';*

2^o *for all $t_0 \in (a, b)$, $u_0, u_0' \in \mathbb{R}$ there exists a unique solution of the equation (2.1) such that $u(t_0) = u_0$, $u'(t_0) = u_0'$;*

3^o *the equation in t*

$$F(t, \gamma^2, \gamma, 1) = 0$$

hasn't any solution in the interval (a, b), for all $\gamma \in \mathbb{R}^*$.

Then for every solution u of the equation (2.1) the zeroes of u and u' separate each other on the interval $[a, b]$.

Proof. It is sufficient to prove that, if t_1, t_2 are two successive zeroes of u', then u has one zero in the interval (t_1, t_2).

We suppose that $u(t) \neq 0$, for all $t \in [t_1, t_2]$. By Tonelli's Lemma there exist $\lambda \in \mathbb{R}^*$ and $t_0 \in (t_1, t_2)$ such that

$$u(t_0) = \lambda u'(t_0) \quad \text{and} \quad u'(t_0) = \lambda u''(t_0).$$

We obtain that

$$u'(t_0) = \frac{1}{\lambda} u(t_0) \quad \text{and} \quad u''(t_0) = \frac{1}{\lambda^2} u(t_0).$$

Then, from the equation (2.1), we have that

$$F(t_0, u(t_0), u'(t_0), u''(t_0)) = 0$$

or

$$F(t_0, u(t_0), \frac{1}{\lambda} u(t_0), \frac{1}{\lambda^2} u(t_0)) = 0.$$

Because $u(t_0) \neq 0$ and $\lambda \neq 0$, by using the condition 1^o, we obtain that

$$F(t_0, \lambda^2, \lambda, 1) = 0$$

i.e. a contradiction with the condition 3^o, which proves the theorem. □

Corollary 2.3. *We suppose that the conditions of Theorem 2.1. are satisfied. If t_1 and t_2 are two successive transverse zeroes of u_1, where u_1 is a solution of the equation (2.1), then every other solution u_2 of the equation (2.1), for which $u_2'(t_1) \neq 0$, $u_2'(t_2) \neq 0$, has in (t_1, t_2) a unique zero.*

Remark 2.4. In the equation (1.1) we suppose that

1^o the function f is homogeneous in u

2^o for all $t_0 \in (a,b)$, $u_0', u_0'' \in \mathbb{R}$ there exists a unique solution of the equation (1.1) such that $u'(t_0) = u_0'$, $u''(t_0) = u_0''$.

Then, if t_1 and t_2 are two successive zeroes of u_1, where u_1 is a solution of the equation (1.1), every other solution u_2 of the equation (1.1), for which $u_2'(t_1) \neq 0$, $u_2'(t_2) \neq 0$, has in (t_1, t_2) a unique zero.

Remark 2.5. In the equation (1.1) we suppose that

1^o f is homogeneous in u;

2^o for all $t_0 \in (a,b)$, $u_0, u_0' \in \mathbb{R}$ there exists a unique solution of the equation (1.1) such that $u(t_0) = u_0$, $u'(t_0) = u_0'$;

3^o the equation in t

$$p(t) + p'(t)\gamma - q(t)\gamma^2 + f(t, \gamma^2) = 0$$

hasn't any solution in the interval (a,b), for all $\gamma \in \mathbb{R}^*$.

Then for every solution u of the equation (1.1) the zeroes of u and u' separate each other on the interval $[a,b]$.

Theorem 2.6. *We suppose that:*

1^o the function F is homogeneous with respect to variables u, u', u'';

2^o there exists a solution of the equation (2.1) that has a transverse zero in (a,b),

3^o the equation in t

$$F(t, \gamma^2, \gamma, 1) = 0$$

hasn't any solution in the interval (a,b), for all $\gamma \in \mathbb{R}^*$.

Then for every solution u of the equation (2.1) the non-transverse zeroes of u and u' separate each other on the interval $[a,b]$.

Proof. Let u be the solution of the equation (2.1) that has a transverse zero $t_* \in (a,b)$, i.e. $u(t_*) = u'(t_*) = 0$. It is sufficient to prove that if t_1, t_2 are two successive zeroes of u', which aren't transverse zeroes for u, then u has one zero in the interval (t_1, t_2).

We suppose that $u(t) \neq 0$, for all $t \in [t_1, t_2]$. By Tonelli's Lemma there exist $\lambda \in \mathbb{R}^*$ and $t_0 \in (t_1, t_2)$ such that

$$u(t_0) = \lambda u'(t_0) \text{ and } u'(t_0) = \lambda u''(t_0).$$

We obtain that

$$u'(t_0) = \frac{1}{\lambda} u(t_0) \text{ and } u''(t_0) = \frac{1}{\lambda^2} u(t_0).$$

Then, from the equation (2.1), we have that

$$F(t_0, u(t_0), u'(t_0), u''(t_0)) = 0$$

or

$$F(t_0, u(t_0), \frac{1}{\lambda} u(t_0), \frac{1}{\lambda^2} u(t_0)) = 0.$$

Because $u(t_0) \neq 0$ and $\lambda \neq 0$, by using the condition 1^o, we obtain that

$$F(t_0, \lambda^2, \lambda, 1) = 0$$

i.e. a contradiction with the condition 3^o, which proves the theorem. □

Let us consider the following second order non-autonomous differential equation

$$(Lu)(t) := -(p(t)u'(t))' + q(t)u(t) = 0, t \in (a,b), \tag{2.2}$$

where the p and q are such that

$$p, q \in C^1[a,b], p(t) > 0, t \in [a,b]. \tag{2.3}$$

It is well know the following result:

Theorem 2.7. *We suppose that the condition (2.3) holds. If u is any solution of (2.2) satisfying $u(t_0) = u'(t_0) = 0$, for some $t_0 \in [a,b]$, then $u \equiv 0$ on $[a,b]$.*

Corollary 2.8. *Let the hypotheses of Theorem 2.7 hold. If u is any non-trivial solution of (2.2), then u has a finite number of zeroes in $[a,b]$.*

Proof. Suppose that u has an infinite number of zeroes $t_n \in [a,b]$, $n \in \mathbb{N}$. Then by Bolzano-Weierstrass theorem and the continuity of u the exists a subsequence t_{n_j} such that $t_{n_j} \to t_0$ as $j \to \infty$ and $u(t_0) = 0$ for some $t_0 \in [a,b]$. By applying Rolle's theorem to u on $[t_0, t_{n_j}]$ (or $[t_{n_j}, t_0]$) and letting $j \to \infty$ shows that $u'(t_0) = 0$. Hence $u \equiv 0$ on $[a,b]$ by Theorem 2.7, as required. □

Remark 2.9. In the conditions of Theorem 2.7 any non-trivial solution of the equation (2.2) hasn't multiple zeroes.

Theorem 2.10. *Consider the following problem*

$$(Lu)(t) := -(p(t)u'(t))' + q(t)u(t) = f(t, u(t)), \quad t \in (a,b) \tag{2.4}$$

$$u(t_0) = u'(t_0) = 0. \tag{2.5}$$

If there exists $L_f > 0$ such that

$$|f(t,u) - f(t,v)| \le L_f|u - v|, t \in [a,b], \text{ and } u, v \in \mathbb{R}, \tag{2.6}$$

then there exists a unique solution of the problem (2.4)+(2.5).

Proof. The equation (2.4) with the conditions (2.5), $u(t_0) = u'(t_0) = 0$, is equivalent with the following fixed point equation:

$$u = A(u), \tag{2.7}$$

where $u \in C^2[a,b]$ and the operator $A : (C^2[a,b], ||.||_\tau) \to (C^2[a,b], ||.||_\tau)$ is defined by

$$(A(u))(t) = \int_{t_0}^{t} \frac{1}{p(r)} \left(\int_{t_0}^{r} [q(s)u(s) - f(s,u(s))]\, ds \right) dr. \tag{2.8}$$

Here

$$||u||_\tau = \max_{t \in [a,b]} |u(t)| e^{-\tau|t-a|}, \quad \tau > 0.$$

We have

$$|(A(u))(t) - (A(v))(t)| =$$

$$= \left| \int_{t_0}^{t} \frac{1}{p(r)} \left(\int_{t_0}^{r} [q(s)(u(s)-v(s)) - f(s,u(s)) + f(s,v(s))]\, ds \right) dr \right| \leq$$

$$\leq \quad \left| \int_{t_0}^{t} \frac{1}{p(r)} \left| \left(\int_{t_0}^{r} |q(s)|\, |u(s)-v(s)| e^{-\tau|s-t_0|} e^{\tau|s-t_0|} ds \right) \right| dr \right| \leq$$

$$\leq \quad \left| \int_{t_0}^{t} \frac{1}{p(r)} \left| \left(\int_{t_0}^{r} L_f |u(s)-v(s)| e^{-\tau|s-t_0|} e^{\tau|s-t_0|} ds \right) \right| dr \right| \leq$$

$$\leq M_p(M_q + L_f)||u-v||_\tau \left| \int_{t_0}^{t} \left| \int_{t_0}^{r} e^{\tau|s-t_0|} ds \right| dr \right|,$$

where $M_p = \max_{t\in[a,b]} \frac{1}{p(t)}$ and $M_q = \max_{t\in[a,b]} |q(t)|$.

But

$$\left| \int_{t_0}^{r} e^{\tau|s-t_0|} ds \right| \leq \frac{1}{\tau} e^{\tau|r-t_0|},$$

and so,

$$\left| \int_{t_0}^{t} \left| \int_{t_0}^{r} e^{\tau|s-t_0|} ds \right| dr \right| \leq \left| \int_{t_0}^{r} \frac{1}{\tau} e^{\tau|r-t_0|} dr \right| \leq \frac{1}{\tau^2} e^{\tau|t-t_0|}.$$

If follows that

$$|(A(u))(t) - (A(v))(t)| e^{-\tau|t-t_0|} \leq \frac{M_p(M_q+L_f)}{\tau^2} ||u-v||_\tau, \text{ for all } t \in [a,b].$$

Consequently

$$||A(u) - A(v)||_\tau \leq \frac{M_p(M_q+L_f)}{\tau^2} ||u-v||_\tau \text{ for all } u, v \in C^2[a,b].$$

By choosing τ large enough we have that the operator A is a contraction. By using Contraction mapping principle we obtain that the equation (2.4) has, in $C^2[a,b]$, a unique solution satisfying the conditions $u(t_0) = u'(t_0) = 0$. □

Corollary 2.11. *In the conditions of Theorem 2.10, if $f(t,0) = 0$ for all $t \in [a,b]$ then any non-trivial solution $u \in C^2[a,b]$ of the equation (2.4) hasn't transverse zeroes.*

Proof. Suppose that u is a non-trivial solution of the equation (2.4) that have a transverse zero $t_0 \in [a,b]$, i.e. $u(t_0) = u'(t_0) = 0$. From Theorem 2.10 the equation (2.4) with the conditions (2.5) has a unique solution. But, because $f(t,0) = 0$, the function $u(t) = 0$, $t \in [a,b]$, is a solution of the problem (2.4)+(2.5). This is a contradiction with the fact that u is a non-trivial solution of the equation (2.4). □

Remark 2.12. There exist equations of the form (2.4), with $f(t,0) \neq 0$, that have solutions with transverse zeroes and with zeroes with a degree of multiplicity greater than 2. See Example 2.13.

Example 2.13. Let us consider the equation (1.1) where

$$p(t) = t^2 + 1, \quad q(t) = 20, \quad f(t,u) = 11t^2 + \sqrt{|u|}, \quad t \in \mathbb{R}.$$

We have that all the conditions: f is a non-linear continuous function, not necessarily Lipschitz continuous function in u,$p, q \in C^1[a,b]$ and $p(t) > 0$ for all $t \in [a,b]$ are satisfied, except the condition $f(t,0) \equiv 0$. A solution u of this equation given by $u(t) = -t^4$ has a transverse zero $t_0 = 0$, which has degree of multiplicity equal to 4.

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