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Fuzzy anti-bounded linear operators

Bivas Dinda, T.K. Samanta and Iqbal H. Jebril

Abstract. Various types of fuzzy anti-continuity and fuzzy anti-boundedness are defined. Few properties of them are established. The intra and inter relation between various types of fuzzy anti-continuity and fuzzy anti-boundedness are studied.

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1. Introduction

The concept of fuzzy set theory was first introduce by Zadeh[13] in 1965 and thereafter, the concept of fuzzy set theory applied on different branches of pure and applied mathematics in different ways by several authors. The concept of fuzzy norm was introduced by Katsaras [9] in 1984. In 1992, Felbin[7] introduced the idea of fuzzy norm on a linear space. Cheng-Moderson [4] introduced another idea of fuzzy norm on a linear space whose associated metric is same as the associated metric of Kramosil-Michalek [10]. In 2003, Bag and Samanta [1] modified the definition of fuzzy norm of Cheng-Moderson [4] and established the concept of continuity and boundednes of a linear operator with respect to their fuzzy norm in [2].

Later on Jebril and Samanta [8] introduced the concept of fuzzy anti-norm on a linear space depending on the idea of Bag and Samanta [3]. The motivation of introducing fuzzy anti-norm is to study fuzzy set theory with respect to the non-membership function. It is useful in the process of decision making.

In this paper various types of fuzzy anti-continuities and fuzzy anti-boundedness; namely, fuzzy anti-continuity, sequential fuzzy anti-continuity, strong fuzzy anti-continuity, weak fuzzy anti-continuity, strong fuzzy anti-boundedness and weak fuzzy anti-boundedness are defined. The intra-relations among fuzzy anti-continuities and intra-relation among strongly fuzzy anti-bounded and weakly fuzzy anti-bounded are studied. Also, the inter relation between fuzzy anti-continuities and fuzzy anti-boundedness are studied. Also it is established an important property for fuzzy anti-continuity; namely, any linear operator between fuzzy anti-normed linear spaces is strongly and weakly fuzzy anti-continuous if and only if it is strongly and weakly fuzzy anti-bounded respectively.

2. Preliminaries

This section contain some basic definition and preliminary results which will be needed in the sequel.

Definition 2.1. [12] *A binary operation* $\diamond : [0,1] \times [0,1] \to [0,1]$ *is continuous t-conorm if* $\diamond$ *satisfies the following conditions* :
(i) $\diamond$ *is commutative and associative* ,
(ii) $\diamond$ *is continuous* ,
(iii) $a \diamond 0 = a, \ \forall \ a \in [0,1]$,
(iv) $a \diamond b \leq c \diamond d$ *whenever* $a \leq c$, $b \leq d$ *and* $a, b, c, d \in [0,1]$.

A few examples of continuous t-conorm are $a \diamond b = a + b - ab, a \diamond b = \max\{a, b\}, a \diamond b = \min\{a + b, 1\}$.

Definition 2.2. [5] *Let* V *be linear space over the field* $F(= \mathbb{R} or \mathbb{C})$. *A fuzzy subset* ν *of* $V \times \mathbb{R}$ *is called a fuzzy antinorm on* V *with respect to a t-conorm* $\diamond$ *if and only if for all* $x, y \in V$
(i) $\forall \ t \in \mathbb{R}$ *with* $t \leq 0, \nu(x,t) = 1$;
(ii) $\forall \ t \in \mathbb{R}$ *with* $t > 0, \nu(x,t) = 0$ *if and only if* $x = \theta$;
(iii) $\forall \ t \in \mathbb{R}$ *with* $t > 0, \nu(cx,t) = \nu(x, \frac{t}{|c|})$ *if* $c \neq 0, \ c \in F$;
(iv) $\forall \ s, t \in \mathbb{R}$ *with* $\nu(x+y, s+t) \leq \nu(x,s) \diamond \nu(y,t)$;
(v) $\lim\limits_{t \to \infty} \nu(x,t) = 0$.

We further assume that for any fuzzy anti-normed linear space (V, A^*) with respect to a t-conorm $\diamond$,
(vi) $\nu(x,t) < 1, \ \forall \ t > 0 \Rightarrow x = \theta$.
(vii) $\nu(x, \cdot)$ is a continuous function of $\mathbb{R}$ and strictly decreasing on the subset $\{t : 0 < \nu(x,t) < 1\}$ of $\mathbb{R}$.
(viii) $a \diamond a = a, \ \forall \ a \in [0,1]$.

Theorem 2.3. [5] *Let* (V, A^*) *be a fuzzy antinormed linear space satisfying* (vi) *and* (vii) *and* $(viii)$. *Let* $\|x\|_\alpha^* = \wedge\{t : \nu(x,t) \leq 1 - \alpha\}, \alpha \in (0,1)$. *Also, let* $\nu' : V \times \mathbb{R} \to [0,1]$ *be defined by*

$$\nu'(x,t) = \begin{cases} \wedge\{1 - \alpha : \|x\|_\alpha^* \leq t\}, & if \quad (x,t) \neq (\theta, 0) \\ 1, & if \quad (x,t) = (\theta, 0) \end{cases}$$

Then $\nu' = \nu$.

Definition 2.4. [8]. *Let* (V, A^*) *be a fuzzy antinormed linear space. A sequence* $\{x_n\}_n$ *in* V *is said to be convergent to* $x \in V$ *if given* $t > 0, r \in (0,1)$ *there exist an integer* $n_0 \in \mathbb{N}$ *such that*

$$\nu(x_n - x, t) < r \ \forall \ n \geq n_0.$$

Definition 2.5. [8]. *Let (V, A^*) be a fuzzy antinormed linear space. A sequence $\{x_n\}_n$ in V is said to be Cauchy sequence to $x \in V$ if given $t > 0, r \in (0, 1)$ there exist an integer $n_0 \in \mathbb{N}$ such that*

$$\nu(x_{n+p} - x_n, t) < r \ \forall\ n \geq n_0, p = 1, 2, 3, \cdots .$$

Definition 2.6. [8]. *A subset A of a fuzzy antinormed linear space (V, A^*) is said to be bounded if and only if there exist $t > 0, r \in (0, 1)$ such that*

$$\nu(x, t) < r \ \forall\ x \in A$$

3. Fuzzy anti-continuity

Throughout this section unless otherwise stated (U, A^*) and (V, B^*) are any two fuzzy anti-normed linear spaces over the same field F.

Definition 3.1. *A mapping $T : (U, A^*) \to (V, B^*)$ is said to be* **fuzzy anti-continuous** *at $x_0 \in U$, if for any given $\epsilon > 0$,$\alpha \in (0, 1)$ there exist $\delta = \delta(\alpha, \epsilon) > 0$, $\beta = \beta(\alpha, \epsilon) \in (0, 1)$ such that for all $x \in U$*

$$\nu_U(x - x_0, \delta) < \beta \Rightarrow \nu_V(T(x) - T(x_0), \epsilon) < \alpha.$$

Definition 3.2. *A mapping $T : (U, A^*) \to (V, B^*)$ is said to be* **sequentially fuzzy anti-continuous** *at $x_0 \in U$, if for any sequence $\{x_n\}_n, x_n \in U$, $\forall\ n$ with $x_n \to x_0$ implies $T(x_n) \to T(x_0)$ in V, that is for all $t > 0$,*

$$\lim_{n\to\infty} \nu_U(x_n - x_0, t) = 0 \Rightarrow \lim_{n\to\infty} \nu_V(T(x_n) - T(x_0), t) = 0.$$

Definition 3.3. *A mapping $T : (U, A^*) \to (V, B^*)$ is said to be* **strongly fuzzy anti-continuous** *at $x_0 \in U$, if for any given $\epsilon > 0$ there exist $\delta = \delta(\alpha, \epsilon) > 0$ such that for all $x \in U$,*

$$\nu_V(T(x) - T(x_0), \epsilon) \leq \nu_U(x - x_0, \delta)$$

Definition 3.4. *A mapping $T : (U, A^*) \to (V, B^*)$ is said to be* **weakly fuzzy anti-continuous** *at $x_0 \in U$, if for any given $\epsilon > 0, \alpha \in (0, 1)$ there exist $\delta = \delta(\alpha, \epsilon) > 0$ such that for all $x \in U$,*

$$\nu_U(x - x_0, \delta) \leq 1 - \alpha \Rightarrow \nu_V(T(x) - T(x_0), \epsilon) \leq 1 - \alpha.$$

Theorem 3.5. *If a mapping T from a fuzzy anti-normed linear space (U, A^*) to a fuzzy anti-normed linear space (V, B^*) is strongly fuzzy anti-continuous then it is weakly fuzzy anti-continuous. But not conversely.*

Proof. From the definition it follows obviously. To show the converse result may not be true we consider the following example.

Example 3.6. As in the example of Note 3.3 of [6], we consider the fuzzy anti-normed linear spaces (X, ν_1) and (X, ν_2). Let $f(x) = \frac{x^4}{1+x^4}\ \forall\ x \in \mathbb{R}$. Now from Example 3 of [11] it directly follows that f is not strongly fuzzy anti-continuous. Here we now show that f is weakly fuzzy anti-continuous on X.
Let $x_0 \in X, \epsilon > 0$ and $\delta \in (0, 1)$. Now

$\nu_2(f(x)-f(x_0),\epsilon)<1-\alpha$ if $\frac{k|f(x)-f(x_0|}{\epsilon+k|f(x)-f(x_0|}<1-\alpha$
i.e., if

$$\frac{\epsilon}{\epsilon+k\mid\frac{x^4}{1+x^2}-\frac{x_0^4}{1+x_0^2}\mid}\geq\alpha$$

i.e., if

$$\frac{\frac{\epsilon(1+x^2)(1+x_0^2)}{k|x+x_0||x^2x_0^2+x^2+x_0^2|}}{\frac{\epsilon(1+x^2)(1+x_0^2)}{k|x+x_0||x^2x_0^2+x^2+x_0^2|}+\mid x-x_0\mid}\geq\alpha$$

i.e., if

$$\alpha\mid x-x_0\mid\leq(1-\alpha)\frac{\epsilon}{k}\frac{(1+x^2)(1+x_0^2)}{\mid x+x_0\mid\mid x^2x_0^2+x^2+x_0^2\mid}$$
$$\leq(1-\alpha)\frac{\epsilon}{k}$$

So, depending upon $(1-\alpha)\frac{\epsilon}{k}$ we may choose $\delta>0$ such that $\alpha(\delta+\mid x-x_0\mid)\leq\delta i.e.,\nu_1(x-x_0,\delta)<1-\alpha.$
Thus we see that for every $\epsilon>0,\alpha\in(0,1)\exists\delta>0$ such that

$$\nu_1(x-x_0,\delta)<1-\alpha\Rightarrow\nu_2(f(x)-f(x_0),\epsilon)<1-\alpha.$$

i.e., f is weakly fuzzy anti-continuous at x_0.

Theorem 3.7. *A mapping T from a fuzzy anti-normed linear space (U,A^*) to a fuzzy anti-normed linear space (V,B^*) is fuzzy anti-continuous if and only if it is sequentially fuzzy anti-continuous.*

Proof. The proof of the above theorem is directly follows from Theorem 13 of [11].

Theorem 3.8. *If a mapping T from a fuzzy anti-normed linear space (U,A^*) to a fuzzy anti-normed linear space (V,B^*) is strongly fuzzy anti-continuous then it is sequentially fuzzy anti-continuous.*

Proof. The proof of the above theorem is directly follows from Theorem 12 of [11].

Theorem 3.9. *Let $T:(U,A^*)\to(V,B^*)$ be a linear operator. If T is sequentially fuzzy anti-continuous at a point $x_0\in U$, then it is sequentially fuzzy anti-continuous on U.*

Proof. Let, $x\in U$ be an arbitrary point and let $\{x_n\}_n$ be a sequence in U such that $x_n\to x$. Then $\forall\ t>0$

$$\lim_{n\to\infty}\nu_U(x_n-x,t)=0$$
$$i.e., \lim_{n\to\infty}\nu_U((x_n-x+x_0)-x_0,t)=0$$

Since T is sequentially fuzzy anti-continuous at x_0 $\forall\ t>0$ we have

$$\lim_{n\to\infty}\nu_V((x_n-x+x_0)-x_0,t)=0$$
$$i.e., \lim_{n\to\infty}\nu_V(T(x_n)-T(x)+T(x_0)-T(x_0),t)=0$$

$$i.e., \lim_{n\to\infty} \nu_V(T(x_n) - T(x), t) = 0$$

Thus,

$$\lim_{n\to\infty} \nu_U(x_n - x, t) = 0, \ \forall\, t > 0 \Rightarrow \lim_{n\to\infty} \nu_V(T(x_n) - T(x), t) = 0, \ \forall\, t > 0.$$

Hence the proof.

4. Fuzzy anti-boundedness

Definition 4.1. *A mapping $T : (U, A^*) \to (V, B^*)$ is said to be strongly fuzzy anti-bounded on U if and only if there exist a positive real number M such that for all $x \in U$ and for all $t \in \mathbb{R}^+$,*

$$\nu_V(T(x), t) \le \nu_U(x, \frac{t}{M})$$

Example 4.2. The zero and identity operators are strongly fuzzy anti-bounded.

Example 4.3. It is an example of a strongly fuzzy anti-bounded linear operator other than the zero and the identity operator.

Let $(V, \|.\|)$ be a normed linear space over the field $K(= \mathbb{R} or \mathbb{C})$. Let, $\alpha_1, \alpha_2 \in R$ such that $\alpha_1 > \alpha_2 > 0$. Again, let $\nu_1, \nu_2 : V \times \mathbb{R}^+ \to [0, 1]$ be defined by

$$\nu_1(x, t) = \frac{\alpha_1\|x\|}{t + \alpha_1\|x\|} and \nu_2(x, t) = \frac{\alpha_2\|x\|}{t + \alpha_2\|x\|}$$

Also, define $a \diamond b, = \max\{a, b\}$ for all $a, b \in [0, 1]$.
Now we shall first show that (V, ν_1) and (V, ν_2) are fuzzy anti-normed linear space.
(i) The condition (i) is obvious.
(ii) $\nu_1(x, t) = 0 \Leftrightarrow \frac{\alpha_1\|x\|}{t+\alpha_1\|x\|} = 0 \Leftrightarrow \|x\| = 0 \Leftrightarrow x = \theta.$
(iii) Let $c \in K$ and $c \neq 0$

$$\nu_1(cx, t) = \frac{\alpha_1 \parallel cx \parallel}{t + \alpha_1 \parallel cx \parallel}$$

$$= \frac{\alpha_1 \parallel x \parallel}{\frac{t}{|c|} + \alpha_1 \parallel x \parallel} = \nu_1(x, \frac{t}{\mid c \mid})$$

(iv)

$$\nu_1(x + y, s + t) = \frac{\alpha_1 \parallel x + y \parallel}{s + t + \alpha_1 \parallel x + y \parallel}$$

$$= \frac{1}{\frac{s+t}{\alpha_1\|x+y\|} + 1}$$

$$\le \frac{1}{\frac{s+t}{\alpha_1\|x\|+\alpha_1\|y\|} + 1}$$

$$= \frac{\alpha_1 \parallel x \parallel + \alpha_1 \parallel y \parallel}{s + t + \alpha_1 \parallel x \parallel + \alpha_1 \parallel y \parallel}$$

Now if

$$\nu_1(x,s) \geq \nu_1(y,t) \Rightarrow \frac{\alpha_1 \parallel x \parallel}{s+\alpha_1 \parallel x \parallel} \geq \frac{\alpha_1 \parallel y \parallel}{t+\alpha_1 \parallel y \parallel};$$

$$\Rightarrow t \parallel x \parallel - s \parallel y \parallel$$

Therefore,

$$\frac{\alpha_1 \parallel x \parallel + \alpha_1 \parallel y \parallel}{s+t+\alpha_1 \parallel x \parallel + \alpha_1 \parallel y \parallel} - \frac{\alpha_1 \parallel x \parallel}{s+\alpha_1 \parallel x \parallel} \leq 0.$$

Thus

$$\nu_1(x+y,s+t) \leq \frac{\alpha_1 \parallel x \parallel + \alpha_1 \parallel y \parallel}{s+t+\alpha_1 \parallel x \parallel + \alpha_1 \parallel y \parallel}$$

$$\leq \frac{\alpha_1 \parallel x \parallel}{s+\alpha_1 \parallel x \parallel} = \nu_1(x,s) \diamond \nu_1(y,t)$$

Again if $\nu_1(y,t) \geq \nu_1(x,s)$ Similarly it can be shown that

$$\nu_1(x+y,s+t) \leq \frac{\alpha_1 \parallel y \parallel}{t+\alpha_1 \parallel y \parallel} = \nu_1(x,s) \diamond \nu_1(y,t)$$

Hence

$$\nu_1(x+y,s+t) \leq \nu_1(x,s) \diamond \nu_1(y,t)$$

(v)

$$\lim_{t\to\infty} \nu_1(x,t) = \lim_{t\to\infty} \frac{\alpha_1 \parallel x \parallel}{t+\alpha_1 \parallel x \parallel} = 0$$

Hence (V,ν_1) is a fuzzy anti-normed linear space. Similarly (V,ν_2) is also fuzzy anti-normed linear space.
We now define a mapping $T:(V,\nu_1) \to (V,\nu_2)$ by $T(x) = rx$ where $r(\neq 0) \in \mathbb{R}$ is fixed. Clearly T is a linear operator.
Let us choose an arbitrary but fixed $M > 0$ such that $M \geq \mid r \mid$ and $x \in V$. Now,

$$M \geq \mid r \mid \Rightarrow \alpha_1 M \parallel x \parallel \geq \alpha_2 \mid r \mid \parallel x \parallel$$

$$\Rightarrow t + \alpha_1 M \parallel x \parallel \geq t + \alpha_2 \mid r \mid \parallel x \parallel \;\; \forall\, t > 0.$$

$$\Rightarrow \frac{t}{t+\alpha_2 \mid r \mid \parallel x \parallel} \geq \frac{t}{t+\alpha_1 M \parallel x \parallel} \;\forall\, t > 0.$$

$$\Rightarrow \frac{t}{t+\alpha_2 \parallel rx \parallel} \geq \frac{\frac{t}{M}}{\frac{t}{M}+\alpha_1 \parallel x \parallel} \;\forall\, t > 0.$$

$$\Rightarrow 1 - \frac{t}{t+\alpha_2 \parallel rx \parallel} \leq 1 - \frac{\frac{t}{M}}{\frac{t}{M}+\alpha_1 \parallel x \parallel} \;\forall\, t > 0.$$

$$\Rightarrow \frac{\alpha_2 \parallel rx \parallel}{t+\alpha_2 \parallel rx \parallel} \leq \frac{\alpha_1 \parallel x \parallel}{\frac{t}{M}+\alpha_1 \parallel x \parallel} \;\forall\, t > 0.$$

i.e.,

$$\nu_2(T(x),t) \leq \nu_1(x, \frac{t}{M}) \;\forall\, t > 0 and \;\forall\, x \in V.$$

Hence T is strongly fuzzy anti-bounded on V.

Definition 4.4. *A mapping $T : (U, A^*) \to (V, B^*)$ is said to be weakly fuzzy anti-bounded on U if and only if for any $\alpha \in (0,1)$ there exist $M_\alpha(>0)$ such that for all $x \in U$ and for all $t \in \mathbb{R}^+$,*

$$\nu_U(x, \frac{t}{M_\alpha}) \leq 1-\alpha \Rightarrow \nu_V(T(x), t) \leq 1-\alpha$$

Theorem 4.5. *Let $T : (U, A^*) \to (V, B^*)$ be a linear operator. If T is strongly fuzzy anti-bounded then it is weakly fuzzy anti-bounded. But not conversely.*

Proof. First we suppose that T is strongly fuzzy anti-bounded. Then there exist $M > 0$ such that $\forall\ x \in U$ and $\forall\ t \in \mathbb{R}$,

$$\nu_V(T(x), t) \leq \nu_U(x, \frac{t}{M})$$

Thus for any $\alpha \in (0,1)$, there exists $M_\alpha(= M)$ such that

$$\nu_U(x, \frac{t}{M_\alpha}) \leq 1-\alpha \Rightarrow \nu_V(T(x), t) \leq 1-\alpha$$

Hence T is weakly fuzzy anti-bounded.

The converse of the above theorem is not necessarily true. For example

Example 4.6. Let $(V, \|\cdot\|)$ be a linear space over the field $K(= \mathbb{R} or \mathbb{C})$ and $\nu_1, \nu_2 : V \times \mathbb{R}^+ \to [0,1]$ be defined by

$$\nu_1(x,t) = \frac{2\|x\|^2}{t^2 + \|x\|^2}, if t > \|x\|$$
$$= 1, \text{if} 0 < t \leq \|x\|$$
$$and \nu_2(x,t) = \frac{\|x\|}{t + \|x\|}$$

Also define $a \diamond b = max\{a, b\}$

Already we have seen that (V, ν_2) is a fuzzy anti-normed linear space. Now we shall prove that (V, ν_1) is a fuzzy anti-normed linear space.

(i) Clearly follows from the definition of ν_1.

(ii) $\nu_1(x,t) = 0 \Leftrightarrow \frac{2\|x\|^2}{t^2+\|x\|^2} = 0 \Leftrightarrow \|x\| = 0 \Leftrightarrow x = \theta$.

(iii) Let, $c \in K$ and $c \neq 0$. If $t > \|cx\|$,

$$\nu_1(cx,t) = \frac{2\|cx\|^2}{t^2 + \|cx\|^2} = \frac{2|c|^2\|x\|^2}{t^2 + |c|^2\|x\|^2} = \frac{2\|x\|^2}{(\frac{t}{|c|})^2 + \|x\|^2} = \nu_1(x, \frac{t}{|c|})$$

Again if $0 < t \leq \|cx\|$ then $\nu_1(cx,t) = 1$
and $0 < t \leq \|cx\| \Rightarrow 0 < \frac{t}{|c|} \leq \|x\| \Rightarrow \nu_1(x, \frac{t}{|c|}) = 1$

(iv) Let $s, t \in \mathbb{R}^+, x, y \in V$
If $0 < s + t \leq \|x + y\|$, we have the following possibilities
(a) $0 < s \leq \|x\|$ and $0 < t \leq \|y\|$
(b) $0 < s \leq \|x\|$ and $t > \|y\|$
(c) $0 < t \leq \|y\|$ and $s > \|x\|$.

In each case $\nu_1(x+y, s+t) = 1 = \nu_1(x,s) \diamond \nu_1(y,t)$ Again, if $s + t > \|x + y\|$, we have the following four possibilities
(a) $s > \|x\|,\ t \leq \|y\|$

(b) $s \leq \|x\|, t > \|y\|$
(c) $s \leq \|x\|, t \leq \|y\|$
(d) $s > \|x\|, t > \|y\|$
In the cases (a), (b), (c)

$$\nu_1(x+y, s+t) = \frac{2\|x+y\|^2}{(s+t)^2 + \|x+y\|^2}$$
$$< 1 = \nu_1(x,s) \diamond \nu_1(y,t)$$

So, we now suppose that $s > \|x\| and t > \|y\|$ Now, $s+t > \|x\|+\|y\| \geq \|x+y\|$. Therefore,

$$\nu_1(x+y, s+t) = \frac{2\|x+y\|^2}{(s+t)^2 + \|x+y\|^2}$$
$$\leq \frac{2(\|x\|+\|y\|)^2}{(s+t)^2 + (\|x\|+\|y\|)^2}$$

Hence we have

$$\nu_1(x+y, s+t) \leq \frac{2(\|x\|+\|y\|)^2}{(s+t)^2 + (\|x\|+\|y\|)^2}$$
$$\leq \frac{2\|y\|^2}{t^2 + \|y\|^2} = \nu_1(y,t)$$

when $\nu_1(x,s) \leq \nu_1(y,t)$
Similarly,

$$\nu_1(x+y, s+t) \leq \frac{2(\|x\|+\|y\|)^2}{(s+t)^2 + (\|x\|+\|y\|)^2}$$
$$\leq \frac{2\|x\|^2}{t^2 + \|x\|^2} = \nu_1(x,s)$$

when $\nu_1(y,t) \leq \nu_1(x,s)$
Thus

$$\nu_1(x+y, s+t) \leq \nu_1(x,s) \diamond \nu_1(y,t)$$

$$(v) \lim_{t\to\infty} \nu_1(x,t) = \lim_{t\to\infty} \frac{2\|x\|^2}{t^2 + \|x\|^2} = 0$$

Thus we see that (V, ν_1) is a fuzzy anti-normad linear space.
Now we define a linear operator $T : (U, \nu_1) \to (V, \nu_2)$ by $T(x) = x,\ \forall\ x \in V$. Let, $\alpha \in (0,1), x \in V and t \in \mathbb{R}^+$ and choose $M_\alpha = \frac{1}{1-\alpha}$. We now prove that

$$\nu_1(x, \frac{t}{M_\alpha}) \leq 1-\alpha \Rightarrow \nu_2(T(x), t) \leq 1-\alpha.$$

$$\nu_1(x, \frac{t}{M_\alpha}) \leq 1-\alpha$$
$$\Rightarrow \frac{2\parallel x \parallel^2}{t^2(1-\alpha)^2 + \parallel x \parallel^2} \leq 1-\alpha$$
$$\Rightarrow 1 - \frac{2\parallel x \parallel^2}{t^2(1-\alpha)^2 + \parallel x \parallel^2} \geq 1-(1-\alpha) = \alpha$$

$$\Rightarrow \frac{t^2(1-\alpha)^2 - \| x \|^2}{t^2(1-\alpha)^2 + \| x \|^2} \geq \alpha$$
$$\Rightarrow t^2(1-\alpha)^3 \geq (1+\alpha)\| x \|^2$$
$$\Rightarrow \| x \| \leq \frac{t(1-\alpha)\sqrt{1-\alpha}}{\sqrt{1+\alpha}}$$
$$\Rightarrow t + \| x \| \leq t\frac{(1-\alpha)\sqrt{1-\alpha}+\sqrt{1+\alpha}}{\sqrt{1+\alpha}}$$
$$\Rightarrow \frac{t}{t+\| x \|} \geq \frac{\sqrt{1+\alpha}}{(1-\alpha)\sqrt{1-\alpha}+\sqrt{1+\alpha}}$$
$$\Rightarrow 1-\frac{t}{t+\| x \|} \leq 1-\frac{\sqrt{1+\alpha}}{(1-\alpha)\sqrt{1-\alpha}+\sqrt{1+\alpha}}$$
$$\Rightarrow \frac{\| x \|}{t+\| x \|} \leq \frac{(1-\alpha)\sqrt{1-\alpha}}{(1-\alpha)\sqrt{1-\alpha}+\sqrt{1+\alpha}}$$

Now

$$\frac{(1-\alpha)\sqrt{1-\alpha}}{(1-\alpha)\sqrt{1-\alpha}+\sqrt{1+\alpha}} \leq 1-\alpha \Leftrightarrow \sqrt{1-\alpha} \leq (1-\alpha)\sqrt{1-\alpha}+\sqrt{1+\alpha}$$
$$\Leftrightarrow \alpha\sqrt{1-\alpha} \leq \sqrt{1+\alpha}$$
$$\Leftrightarrow 1+\alpha+\alpha^3 \geq \alpha^2$$

which is true for all $\alpha \in (0,1)$.
Hence

$$\nu_1(x, \frac{t}{M_\alpha}) \leq 1-\alpha \Rightarrow \nu_2(T(x),t) \leq 1-\alpha.$$

Thus T is weakly fuzzy anti-bounded on V.
Now for $t > \|x\|, x \neq \theta$ we have

$$\nu_2(T(x),t) \leq \nu_1(x, \frac{t}{M}) \Leftrightarrow \frac{\|x\|}{t+\|x\|} \leq \frac{2M\|x\|^2}{t^2+M\|x\|^2}$$
$$\Leftrightarrow t^2\|x\| + M\|x\|^3 \leq 2tM\|x\|^2 + 2M\|x\|^3$$
$$\Leftrightarrow \left(2t\|x\|^2 + \|x\|^3\right) M \geq t^2\|x\|$$
$$M \to \infty \text{ as } t \to \infty$$

Hence T is not strongly fuzzy anti-bounded on V.

Definition 4.7. *A linear operator $T : (U, A^*) \to (V, B^*)$ is said to be uniformly fuzzy anti-bounded if and only if there exist $M > 0$ such that*

$$\|T(x)\|_\alpha^* \geq M \|x\|_\alpha^*, \alpha \in (0,1)$$

where $\{\|\cdot\|_\alpha^* : \alpha \in (0,1)\}$ is ascending family of fuzzy α-norms.

Theorem 4.8. *Let $T : (U, A^*) \to (V, B^*)$ be a linear operator and (U, A^*) and (V, B^*) satisfies (vi), (vii) and (viii). Then T is strongly fuzzy anti-bounded if and only if it is uniformly fuzzy anti-bounded with respect to fuzzy α-norms, $\alpha \in (0,1)$.*

Proof. Let $\{\|\cdot\|_\alpha^* : \alpha \in (0,1)\}$ be ascending family of α-norms. First suppose that T is strongly fuzzy anti-bounded. Then there exist $M > 0$ such that $\forall\ x \in U$ and $\forall\ s \in R$,

$$\nu_V(T(x), t) \leq \nu_U(x, \frac{s}{M}) i.e., \nu_V(T(x), t) \leq \nu_U(Mx, s)$$

$\|Mx\|_\alpha^* > t \Rightarrow \wedge\{s : \nu(Mx, s) \leq 1 - \alpha\} > t.$

$$\Rightarrow \exists s_0 > t \text{ such that } \nu(Mx, s_0) \leq 1 - \alpha$$

$$\Rightarrow \exists s_0 > t \text{ such that } \nu(T(x), s_0) \leq 1 - \alpha$$

$$\Rightarrow \|T(x)\|_\alpha^* \geq s_0 > t$$

Hence $\|T(x)\|_\alpha^* \geq \|Mx\|_\alpha^* = M\,\|x\|_\alpha^*$.
Thus T is uniformly fuzzy anti-bounded.
Conversely, suppose that there exist $M > 0$ such that $\forall\ x \in U$ and $\forall\ \alpha \in (0,1)$

$$\|T(x)\|_\alpha^* \geq M\,\|x\|_\alpha^*$$

Let $p > \nu_U(Mx, s) \Rightarrow p > \wedge\{\alpha \in (0,1) : \|Mx\|_\alpha^* \leq s\}$
$\Rightarrow$ there exist $\alpha_0 \in (0,1)$ such that $p > \alpha_0$ and $\|Mx\|_\alpha^* \leq s$
$\Rightarrow \|T(x)\|_\alpha^* \leq s$
$\Rightarrow \nu_V(T(x), s) \leq 1 - \alpha_0 < p.$

Hence, $\nu_V(T(x), s) \leq \nu_U(Mx, s) = \nu_U(x, \frac{s}{M})$.
Thus T is strongly fuzzy anti-bounded.

Theorem 4.9. *Let $T : (U, A^*) \to (V, B^*)$ be a linear operator. Then,*
(i) *T is strongly fuzzy anti-continuous on U if T is strongly fuzzy anti-continuous at a point $x_0 \in U$.*
(ii) *T is strongly fuzzy anti-continuous if and only if T is strongly fuzzy anti-bounded.*

Proof. **(i)** since, T is strongly fuzzy anti-continuous at $x_0 \in U$, for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\nu_V(T(x) - T(x_0), \epsilon) \leq \nu_U(x - x_0, \delta)$$

Taking $y \in U$ and replacing $x by x + x_0 - y$, we get,
$\nu_V(T(x) - T(x_0), \epsilon) \leq \nu_U(x - x_0, \delta)$
$\Rightarrow \nu_V(T(x + x_0 - y) - T(x_0), \epsilon) \leq \nu_U(x + x_0 - y - x_0, \delta)$
$\Rightarrow \nu_V(T(x) + T(x_0) - T(y) - T(x_0), \epsilon) \leq \nu_U(x - y, \delta)$
$\Rightarrow \nu_V(T(x) - T(y), \epsilon) \leq \nu_U(x - y, \delta)$
Since, $y \in U$ is arbitrary, T is strongly fuzzy anti-continuous on U.
(ii) First we suppose that T is strongly fuzzy anti-bounded. Thus there exist a positive real number M such that for all $x \in U$ and for all $\epsilon \in R^+$,

$$\nu_V(T(x), \epsilon) \leq \nu_U(x, \frac{\epsilon}{M})$$

$$i.e., \nu_V(T(x) - T(\theta), \epsilon) \leq \nu_U(x - \theta, \frac{\epsilon}{M})$$

$$i.e, \nu_V(T(x) - T(\theta), \epsilon) \leq \nu_U(x - \theta, \delta)$$

where $\delta = \frac{\epsilon}{M}$.
Thus T is strongly fuzzy anti-continuous at θ and hence T is strongly fuzzy anti-continuous on U.
Conversely, suppose that T is strongly fuzzy anti-continuous on U. Using fuzzy anti-continuity of T at $x = \theta$ for $\epsilon = 1$ there exist $\delta > 0$ such that for all $x \in U$,

$$\nu_V(T(x) - T(\theta), 1) \leq \nu_U(x - \theta, \delta).$$

If $x \neq \theta$and $t > 0$. Putting $x = ut$
$\nu_V(T(x), t) = \nu_V(uT(u), t) = \nu_V(T(u), 1) \leq \nu_U(u, \delta) = \nu_U(\frac{x}{t}, \delta) = \nu_U(x, \frac{t}{M})$,
where $M = \frac{1}{\delta}$. So, $\nu_V(T(x), t) \leq \nu_U(x, \frac{t}{M})$.
If $x \neq \theta$ and $t \leq 0$ then $\nu_V(T(x), t) = 1 = \nu_U(x, \frac{t}{M})$.
If $x = \theta$ and $t \in R$, then $T(\theta_U) = \theta_V$ and

$$\nu_V(\theta_V, t) = \nu_U(\theta_U, \frac{t}{M}) = 0, \text{ if } t > 0.$$

$$\nu_V(\theta_V, t) = \nu_U(\theta_U, \frac{t}{M}) = 1, \text{ if } t \leq 1.$$

Hence T is strongly fuzzy anti-bounded.

Theorem 4.10. *Let $T : (U, A^*) \to (V, B^*)$ be a linear operator. Then,*
(i) *T is weakly fuzzy anti-continuous on U if T is weakly fuzzy anti-continuous at a point $x_0 \in U$.*
(ii) *T is weakly fuzzy anti-continuous if and only if T is weakly fuzzy anti-bounded.*

Proof. **(i)** Since, T is weakly fuzzy anti-continuous at x_0 in U, for $\epsilon > 0$ and $\alpha \in (0, 1)$ there exist $\delta = \delta(\alpha, \epsilon) > 0$ such that $\forall\ x \in U$

$$\nu_U(x - x_0, \delta) \leq 1 - \alpha \Rightarrow \nu_V(T(x) - T(x_0), \epsilon) \leq 1 - \alpha.$$

Taking $y \in U$ and replacing x by $x + x_0 - y$ we get,

$$\nu_U(x + x_0 - y - x_0, \delta) \leq 1 - \alpha \Rightarrow \nu_V(T(x + x_0 - y) - T(x_0), \epsilon) \leq 1 - \alpha$$

$$i.e., \nu_U(x - y, \delta) \leq 1 - \alpha \Rightarrow \nu_V(T(x) + T(x_0) - T(y) - T(x_0), \epsilon) \leq 1 - \alpha$$

$$i.e., \nu_U(x - y, \delta) \leq 1 - \alpha \Rightarrow \nu_V(T(x) - T(y), \epsilon) \leq 1 - \alpha$$

Since, $y(\in U)$ is arbitrary it follows that T is weakly fuzzy anti-continuous on U.

(ii) First we suppose that T is fuzzy anti-bounded. Thus for any $\alpha \in (0, 1)$ there exist $M_\alpha > 0$ such that $\forall\ t \in R$ and $\forall\ x \in U$ we have

$$\nu_U(x, \frac{t}{M}) \leq 1 - \alpha \Rightarrow \nu_V(T(x), t) \leq 1 - \alpha$$

Therefore,

$$\nu_U(x - \theta, \frac{t}{M}) \leq 1 - \alpha \Rightarrow \nu_V(T(x) - T(\theta), t) \leq 1 - \alpha$$

$$i.e., \nu_U(x - \theta, \frac{\epsilon}{M_\alpha}) \leq 1 - \alpha \Rightarrow \nu_V(T(x) - T(\theta), \epsilon) \leq 1 - \alpha$$

$$i.e., \nu_U(x - \theta, \delta) \leq 1 - \alpha \Rightarrow \nu_V(T(x) - T(\theta), \epsilon) \leq 1 - \alpha$$

where $\delta = \frac{\epsilon}{M_\alpha}$
Thus, T is weakly fuzzy anti-continuous at x_0 and hence weakly fuzzy anti-continuous on U.
Conversely, suppose that T is weakly fuzzy anti-continuous on U. Using continuity of T at θ and taking $\epsilon = 1$ we have for all $\alpha \in (0,1)$ there exists $\delta(\alpha,1) > 0$ such that for all $x \in U$,

$$\nu_U(x-\theta,\delta) \leq 1-\alpha \Rightarrow \nu_V(T(x)-T(\theta),1) \leq 1-\alpha$$

$$i.e., \nu_U(x,\delta) \leq 1-\alpha \Rightarrow \nu_V(T(x),1) \leq 1-\alpha.$$

If $x \neq \theta$ and $t > 0$. Putting $x = \frac{u}{t}$ we have,

$$\nu_U(\frac{u}{t},\delta) \leq 1-\alpha \Rightarrow \nu_V(T(\frac{u}{t}),1) \leq 1-\alpha$$

$$i.e., \nu_U(u,t\delta) \leq 1-\alpha \Rightarrow \nu_V(T(u),t) \leq 1-\alpha$$

$$i.e., \nu_U(u,\frac{t}{M_\alpha}) \leq 1-\alpha \Rightarrow \nu_V(T(\frac{u}{t}),1) \leq 1-\alpha$$

where $M_\alpha = \frac{1}{\delta(\alpha,1)}$ If $x \neq \theta$ and $t \leq 0$, $\nu_U(x,\frac{t}{M_\alpha}) = \nu_V(T(x),t) = 1$ for any $M_\alpha > 0$.
If $x = \theta$ then for $M_\alpha > 0$,

$$\nu_U(x,\frac{t}{M_\alpha}) = \nu_V(T(x),t) = 0, ift > 0$$

$$\nu_U(x,\frac{t}{M_\alpha}) = \nu_V(T(x),t) = 1, ift \leq 0$$

Hence, T is weakly fuzzy anti-bounded.

Theorem 4.11. *Let $T : (U,A^*) \to (V,B^*)$ be a linear operator and (U,A^*) and (V,B^*) satisfies (vi), (vii) and (viii). Then T is weakly fuzzy anti-bounded if and only if T is fuzzy anti-bounded with respect to α-norms.*

Proof. First we suppose that T is weakly fuzzy anti-bounded. Then for all $\alpha \in (0,1)$ there exist $M_\alpha > 0$ such that $\forall\ x \in U, t \in R$ we have

$$\nu_U(x,\frac{t}{M_\alpha}) \leq 1-\alpha \Rightarrow \nu_V(T(x),t) \leq 1-\alpha$$

Hence we get, $\nu_U(M_\alpha x,t) \leq 1-\alpha \Rightarrow \nu_V(T(x),t) \leq 1-\alpha$
$i.e., \wedge\{\beta \in (0,1) : \|M_\alpha x\|_\beta^* \leq t\} \leq 1-\alpha \Rightarrow \wedge\{\beta \in (0,1) : \|T(x)\|_\beta^* \leq t\} \leq 1-\alpha$
Now we show that

$$\wedge\{\beta \in (0,1) : \|M_\alpha x\|_\beta^* \leq t\} \leq 1-\alpha \Leftrightarrow \|M_\alpha x\|_\alpha^* \leq t$$

If $x = \theta$ then the relation is obvious.
Suppose $x \neq \theta$.
Now, if

$$\wedge\{\beta \in (0,1) : \|M_\alpha x\|_\beta^* \leq t\} < 1-\alpha then \|M_\alpha x\|_\alpha^* \leq t \tag{4.1}$$

If $\wedge\{\beta \in (0,1) : \|M_\alpha x\|_\beta^* \leq t\} = 1-\alpha$ then there exists a decreasing sequence $\{\alpha_n\}_n$ in $(0,1)$ such that $\alpha_n \to \alpha$ and $\|M_\alpha x\|_\alpha^* \leq t$ Then by Theorem 3.7 we have

$$\|M_\alpha x\|_\alpha^* \leq t \tag{4.2}$$

From (4.1) and (4.2) we get

$$\wedge\{\beta \in (0,1) : \|M_\alpha x\|_\beta^* \leq t\} \leq 1-\alpha \Rightarrow \|M_\alpha x\|_\alpha^* \leq t \tag{4.3}$$

Next we suppose that $\|M_\alpha x\|_\alpha^* \leq t$.
If $\|M_\alpha x\|_\alpha^* < t$ then $\nu_U(M_\alpha x, t) \leq 1-\alpha$. i.e.,

$$\wedge\{\beta \in (0,1) : \|M_\alpha x\|_\beta^* \leq t\} \leq 1-\alpha \tag{4.4}$$

If $\|M_\alpha x\|_\alpha^* = t$ i.e.,$\wedge\{s : \nu_U(M_\alpha x, s) \leq 1-\alpha\} = t$ then there exist an increasing sequence $\{s_n\}_n$ in $\mathbb{R}^+$ such that $s_n \to t$ and

$$\begin{aligned}\nu_U(M_\alpha x, s_n) \leq 1-\alpha &\Rightarrow \lim_{n\to\infty} \nu_U(M_\alpha x, s_n) \leq 1-\alpha \\ &\Rightarrow \nu_U(M_\alpha x, \lim_{n\to\infty} s_n) \leq 1-\alpha \\ &\Rightarrow \nu_U(M_\alpha x, t) \leq 1-\alpha \\ &\Rightarrow \wedge\{\beta \in (0,1) : \|M_\alpha x\|_\beta^* \leq t\} \leq 1-\alpha\end{aligned}$$

Hence from (4.4) it follows that

$$\|M_\alpha x\|_\alpha^* \leq t \Rightarrow \wedge\{\beta \in (0,1) : \|M_\alpha x\|_\beta^* \leq t\} \leq 1-\alpha \tag{4.5}$$

From (4.3) and (4.5) we have

$$\wedge\{\beta \in (0,1) : \|M_\alpha x\|_\beta^* \leq t\} \leq 1-\alpha \Leftrightarrow \|M_\alpha x\|_\alpha^* \leq t \tag{4.6}$$

In the similar way we can show that

$$\wedge\{\beta \in (0,1) : \|T(x)\|_\beta^* \leq t\} \leq 1-\alpha \Leftrightarrow \|T(x)\|_\alpha^* \leq t \tag{4.7}$$

From (4.6) and (4.7) we have $\nu_U(M_\alpha x, t) \leq 1-\alpha \Rightarrow \nu_V(T(x), t) \leq 1-\alpha$
Then

$$\|M_\alpha x\|_\alpha^* \leq t \Rightarrow \|T(x)\|_\alpha^* \leq t$$

This implies that

$$\|T(x)\|_\alpha^* \geq \|M_\alpha x\|_\alpha^*$$

Conversely, suppose that $\forall\ \alpha \in (0,1), \exists M_\alpha > 0$ such that $\forall\ xInU$,

$$\|T(x)\|_\alpha^* \geq \|M_\alpha x\|_\alpha^*$$

Then for $x \neq \theta and \ \forall\ t > 0$,

$$\|M_\alpha x\|_\alpha^* \leq t \Rightarrow \|T(x)\|_\alpha^* \leq t$$

i.e.,

$$\wedge\{s : \nu_U(M_\alpha x, s) \leq 1-\alpha\} \leq t \Rightarrow \wedge\{s : \nu_V(T(x), s) \leq 1-\alpha\} \leq t$$

In the similar way as above we can show that

$$\wedge\{s : \nu_U(M_\alpha x, s) \leq 1-\alpha\} \leq t \Leftrightarrow \nu_U(M_\alpha x, t) \leq 1-\alpha$$

and

$$\wedge\{s : \nu_U(T(x), s) \leq 1-\alpha\} \leq t \Leftrightarrow \nu_U(T(x)x, t) \leq 1-\alpha$$

Thus we have

$$\nu_U(x, \frac{t}{M_\alpha}) \leq 1 - \alpha \Rightarrow \nu_V(T(x), t) \leq 1 - \alpha,\ \forall\ x \in U$$

If $x \neq \theta, t \leq 0$ and if $x = \theta, t > 0$ then the above relation is obvious. Hence the proof.

Theorem 4.12. *Let $T : (U, A^*) \to (V, B^*)$ be a linear operator and (U, A^*) and (V, B^*) satisfies (vi), (vii) and (viii). If U is finite dimensional then T is weakly fuzzy anti-bounded.*

Proof. Since, (U, A^*) and (V, B^*) satisfies (vi) and (viii) we may suppose that $\{\|\cdot\|_\alpha^* : \alpha \in (0,1)\}$ is ascending family of fuzzy α-anti-norms.
Since T is of finite dimension, $T : (U, A^*) \to (V, B^*)$ is bounded linear operator for each $\alpha \in (0,1)$. Thus by Theorem 4.11 it follows that T is weakly fuzzy anti-bounded.

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Bivas Dinda
Department of Mathematics
Mahishamuri Ramkrishna Vidyapith
West Bengal, India
e-mail: bvsdinda@gmail.com

T.K. Samanta
Department of Mathematics
Uluberia College, West Bengal, India
e-mail: mumpu_tapas5@yahoo.co.in

Iqbal H. Jebril
Department of Mathematics
Faculty of Science
Taibah University
Kingdom of Saudi Arabia
e-mail: iqbal501@hotmail.com