Stud. Univ. Babeş-Bolyai Math. 56(2011), No. 4, 65–69

The univalence and the convexity properties for a new integral operator

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Abstract. For analytic functions f in the open unit disk $\mathcal{U}$, an integral operator $I_{\alpha,\beta}$ is introduced. The object of the paper is to obtain the conditions of the univalence and the convexity of the integral operator $I_{\alpha,\beta}$.

Mathematics Subject Classification (2010): 30C45.

Keywords: Integral operator, univalence, starlike, convexity.

1. Introduction

Let $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of the functions $f \in \mathcal{A}$, which are univalent in $\mathcal{U}$. We denote by $\mathcal{S}^*$ the subclass of $\mathcal{A}$ consisting of all starlike functions in $\mathcal{U}$. Also, we denote by $\mathcal{K}$ the subclass of $\mathcal{A}$ consisting of all convex functions in $\mathcal{U}$.

We consider $\mathcal{K}(\alpha)$ the subclass of $\mathcal{A}$ consisting of all the convex functions f of the order α, satisfying:

$$Re\left(\frac{zf''(z)}{f'(z)} + 1\right) > \alpha, \quad (z \in \mathcal{U}), \tag{1.1}$$

for some α $(0 \leq \alpha < 1)$. We have $\mathcal{K}(0) = \mathcal{K}$.

Note that $f \in \mathcal{K}$, if and only if $zf' \in \mathcal{S}^*$.

In this work, we introduce a new integral operator, which is defined by

$$I_{\alpha,\beta}(z) = \int_0^z \left(\frac{f(u)}{u}\right)^{\alpha} (f'(u))^{\beta}\, du, \tag{1.2}$$

for α, β be complex numbers, $f \in \mathcal{A}$.

For $\beta = 0$, α be a complex number, $f \in \mathcal{A}$, from (1.2) we have the integral operator Kim-Merkes [2],

$$H_\alpha(z) = \int_0^z \left(\frac{f(u)}{u}\right)^\alpha du. \tag{1.3}$$

From (1.2), for $\alpha = 0$, β be a complex number, $f \in \mathcal{A}$, we obtain the integral operator Pfaltzgraff [5],

$$G_\beta(z) = \int_0^z \left(f'(u)\right)^\beta du. \tag{1.4}$$

2. Preliminary results

We need the following lemmas.

Lemma 2.1. [1]. *If the function f is analytic in $\mathcal{U}$ and*

$$\left(1 - |z|^2\right) \left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \tag{2.1}$$

for all $z \in \mathcal{U}$, then the function f is univalent in $\mathcal{U}$.

Lemma 2.2. *(Schwarz [3]). Let f be the function regular in the disk $\mathcal{U}_R = \{z \in \mathbb{C} : |z| < R\}$ with $|f(z)| < M$, M fixed. If f has in $z = 0$ one zero with multiply $\geq m$, then*

$$|f(z)| \leq \frac{M}{R^m}|z|^m, \ \left(z \in \mathcal{U}_R\right), \tag{2.2}$$

the equality (in the inequality (2.2) for $z \neq 0$) can hold if

$$f(z) = e^{i\theta}\frac{M}{R^m}z^m,$$

where θ is constant.

3. Main results

Theorem 3.1. *Let α, β be complex numbers, M, L positive real numbers and $f \in \mathcal{A}$, $f(z) = z + a_2z^2 + \ldots$. If*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq M, \quad \left(z \in \mathcal{U}\right), \tag{3.1}$$

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq L, \quad \left(z \in \mathcal{U}\right), \tag{3.2}$$

and

$$|\alpha|M + |\beta|L \leq \frac{3\sqrt{3}}{2}, \tag{3.3}$$

then the function

$$I_{\alpha,\beta}(z) = \int_0^z \left(\frac{f(u)}{u}\right)^\alpha \left(f'(u)\right)^\beta du, \tag{3.4}$$

is in the class $\mathcal{S}$.

Proof. The function $I_{\alpha,\beta}(z)$ is regular in $\mathcal{U}$ and $I_{\alpha,\beta}(0) = I'_{\alpha,\beta}(0) - 1 = 0$. We have:

$$\frac{zI''_{\alpha,\beta}(z)}{I'_{\alpha,\beta}(z)} = \alpha\left(\frac{zf'(z)}{f(z)} - 1\right) + \beta\frac{zf''(z)}{f'(z)}, \tag{3.5}$$

for all $z \in \mathcal{U}$.

From (3.5) we obtain:

$$\left(1-|z|^2\right)\left|\frac{zI''_{\alpha,\beta}(z)}{I'_{\alpha,\beta}(z)}\right| \leq \left(1-|z|^2\right)\left[|\alpha|\left|\frac{zf'(z)}{f(z)} - 1\right| + |\beta|\left|\frac{zf''(z)}{f'(z)}\right|\right], \tag{3.6}$$

for all $z \in \mathcal{U}$. By Lemma 2.2, from (3.1) and (3.2) we get

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq M|z|, \quad (z \in \mathcal{U}), \tag{3.7}$$

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq L|z|, \quad (z \in \mathcal{U}) \tag{3.8}$$

and by (3.6) we obtain

$$\left(1-|z|^2\right)\left|\frac{zI''_{\alpha,\beta}(z)}{I'_{\alpha,\beta}(z)}\right| \leq \left(1-|z|^2\right)|z|\left(|\alpha|M + |\beta|L\right), \tag{3.9}$$

for all $z \in \mathcal{U}$. Since

$$\max_{|z|\leq 1}\left[\left(1-|z|^2\right)|z|\right] = \frac{2}{3\sqrt{3}},$$

by (3.3) and (3.9) we have

$$\left(1-|z|^2\right)\left|\frac{zI''_{\alpha,\beta}(z)}{I'_{\alpha,\beta}(z)}\right| \leq 1, \quad (z \in \mathcal{U}). \tag{3.10}$$

By Lemma 2.1, we obtain that the integral operator $I_{\alpha,\beta}$ is in the class $\mathcal{S}$. □

Theorem 3.2. *Let* α, β *be real numbers, with the properties* $\alpha \geq 0$, $\beta \geq 0$ *and*

$$0 < \alpha + \beta < 1 \tag{3.11}$$

We suppose that the functions $f \in \mathcal{S}^*$ *and* $g \in \mathcal{S}^*$, *where* $g(z) = zf'(z)$. *Then, the integral operator* $I_{\alpha,\beta}$ *defined by*

$$I_{\alpha,\beta}(z) = \int_0^z \left(\frac{f(u)}{u}\right)^{\alpha}\left(f'(u)\right)^{\beta} du, \tag{3.12}$$

is convex by the order $1-\alpha-\beta$.

Proof. From (3.5) we obtain that:

$$\frac{zI''_{\alpha,\beta}(z)}{I'_{\alpha,\beta}(z)}+1=\alpha\frac{zf'(z)}{f(z)}-\alpha+\beta\left(\frac{zf''(z)}{f'(z)}+1\right)-\beta+1 \tag{3.13}$$

and hence, we have

$$Re\left(\frac{zI''_{\alpha,\beta}(z)}{I'_{\alpha,\beta}(z)}+1\right)=\alpha Re\frac{zf'(z)}{f(z)}-\alpha+\beta Re\left(\frac{zf''(z)}{f'(z)}+1\right)-\beta+1, \tag{3.14}$$

for all $z\in\mathcal{U}$.

But $f\in\mathcal{S}^*$ and $g\in\mathcal{S}^*$, where $g(z)=zf'(z)$.

We apply this affirmation in (3.14), we obtain that:

$$Re\left(\frac{zI''_{\alpha,\beta}(z)}{I'_{\alpha,\beta}(z)}+1\right)>1-\alpha-\beta. \tag{3.15}$$

Using the hypothesis $\alpha+\beta<1$, in (3.15), we obtain that $I_{\alpha,\beta}$ is convex function by the order $1-\alpha-\beta$. □

4. Corollaries

Corollary 4.1. *Let α be a complex number, $\alpha\neq 0$ and $f\in\mathcal{A}$,*

$$f(z)=z+a_2z^2+a_3z^3+\dots$$

If

$$\left|\frac{zf'(z)}{f(z)}-1\right|\leq\frac{3\sqrt{3}}{2|\alpha|},\quad (z\in\mathcal{U}). \tag{4.1}$$

then the integral operator H_α, defined by (1.3), belongs to the class $\mathcal{S}$.

Proof. For $\beta=0$, from Theorem 3.1 we obtain Corollary 4.1. □

Corollary 4.2. *Let β be a complex number, $\beta\neq 0$ and $f\in\mathcal{A}$,*

$$f(z)=z+a_2z^2+a_3z^3+\dots$$

If

$$\left|\frac{zf''(z)}{f'(z)}\right|\leq\frac{3\sqrt{3}}{2|\beta|},\quad (z\in\mathcal{U}), \tag{4.2}$$

then the integral operator G_β, defined by (1.4), is in the class $\mathcal{S}$.

Proof. We take $\alpha=0$ in Theorem 3.1. □

Corollary 4.3. *If α is a real number, $0<\alpha<1$ and the function $f\in\mathcal{S}^*$, then the integral operator H_α defined in (1.3) is convex by the order $1-\alpha$.*

Proof. For $\beta=0$ in Theorem 3.2, we obtain Corollary 4.3. □

Corollary 4.4. *If β is a real number, $0<\beta<1$ and the function $f\in\mathcal{K}$, then the integral operator G_β, defined by (1.4), is convex by the order $1-\beta$.*

Proof. We take $\alpha=0$ in Theorem 3.2. □

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