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Univariate inequalities based on Sobolev representations

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Abstract. Here we derive very general univariate tight integral inequalities of Chebyshev-Grüss, Ostrowski types, for comparison of integral means and Information theory. These are based on well-known Sobolev integral representations of a function. Our inequalities engage ordinary and weak derivatives of the involved functions. We give also applications. On the way to prove our main results we derive important estimates for the averaged Taylor polynomials and remainders of Sobolev integral representations. Our results expand to all possible directions.

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1. Introduction

This article is greatly motivated by the following theorems:

Theorem 1.1. *(Chebychev, 1882, [6]) Let $f, g : [a, b] \to \mathbb{R}$ absolutely continuous functions. If $f', g' \in L_\infty([a, b])$, then*

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) \, dx - \frac{1}{(b-a)^2} \left(\int_a^b f(x) \, dx \right) \left(\int_a^b g(x) \, dx \right) \right| \quad (1.1)$$

$$\leq \frac{1}{12} (b-a)^2 \|f'\|_\infty \|g'\|_\infty .$$

Theorem 1.2. *(G. Grüss, 1935, [10]) Let f, g integrable functions from $[a, b] \to \mathbb{R}$, such that $m \leq f(x) \leq M$, $\rho \leq g(x) \leq \sigma$, for all $x \in [a, b]$, where $m, M, \rho, \sigma \in \mathbb{R}$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) \, dx - \frac{1}{(b-a)^2} \left(\int_a^b f(x) \, dx \right) \left(\int_a^b g(x) \, dx \right) \right| \quad (1.2)$$

$$\leq \frac{1}{4}(M-m)(\sigma-\rho).$$

In 1938, A. Ostrowski [13] proved

Theorem 1.3. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty = \sup_{t\in(a,b)}|f'(t)| < +\infty$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f(x)\right| \leq \left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right]\cdot(b-a)\|f'\|_\infty, \tag{1.3}$$

for any $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible.

See also [1], [2], [3] for related works that inspired as well this article.

In this work using the univariate Sobolev type representation formulae, see Theorems 10, 14 and also Corollaries 11, 12, we estimate first their remainders and then the involved averaged Taylor polynomials.

Based on these estimates we derive lots of very tight inequalities on $\mathbb{R}$: of Chebyshev-Grüss type, Ostrowski type, for Comparison of integral means and Csiszar's f-Divergence with applications. Our results involve ordinary and weak derivatives and they go to all possible directions using various norms. All of our tools come from the excellent monograph by V. Burenkov, [5].

2. Basics

Here we follow [5].

For a measurable non empty set $\Omega\subset\mathbb{R}^n$, $n\in\mathbb{N}$ we shall denote by $L_p^{loc}(\Omega)$ $(1\leq p\leq\infty)$ - the set of funtions defined on Ω such that for each compact $K\subset\Omega$ $f\in L_p(K)$.

Definition 2.1. *Let $\Omega\subset\mathbb{R}^n$ be an open set, $\alpha\in\mathbb{Z}_+^n$, $\alpha\neq 0$ and $f,g\in L_1^{loc}(\Omega)$. The function g is a weak derivative of the function f of order α on Ω (briefly $g=D_w^\alpha f$) if $\forall\ \varphi\in C_0^\infty(\Omega)$ (i.e. $\varphi\in C^\infty(\Omega)$ compactly supported in Ω)*

$$\int_\Omega fD^\alpha\varphi dx = (-1)^{|\alpha|}\int_\Omega g\varphi dx. \tag{2.1}$$

Definition 2.2. *$W_p^l(\Omega)$ $(l\in\mathbb{N},\ 1\leq p\leq\infty)$ - Sobolev space, which is the Banach space of functions $f\in L_p(\Omega)$ such that $\forall\ \alpha\in\mathbb{Z}_+^n$ where $|\alpha|\leq l$ the weak derivatives $D_w^\alpha f$ exist on Ω and $D_w^\alpha f\in L_p(\Omega)$, with the norm*

$$\|f\|_{W_p^l(\Omega)} = \sum_{|\alpha|\leq l}\|D_w^\alpha f\|_{L_p(\Omega)}. \tag{2.2}$$

Definition 2.3. *For $l\in\mathbb{N}$, we define the Sobolev type local space*

$$\left(W_1^l\right)^{(loc)}(\Omega) := \{f:\Omega\to\mathbb{R} : f\in L_{loc}^1(\Omega)$$

and all f-distributional partials of orders $\leq l$ belong to

$L^1_{loc}(\Omega)\} = \{f \in L_1^{loc}(\Omega)$: *for each open set* G *compactly embedded into* Ω, $f \in W_1^l(G)\}$.

We use Definitions 2.1, 2.2, 2.3 on $\mathbb{R}$. Next we mention Sobolev's integral representation from [5].

Definition 2.4. *([5], p. 82) Let* $-\infty < a < b < \infty$,

$$\omega \in L_1(a,b), \qquad \int_a^b \omega(x)\,dx = 1. \tag{2.3}$$

Define

$$\Lambda(x,y) := \begin{cases} \int_a^y \omega(u)\,du, & a \le y \le x \le b, \\ -\int_y^b \omega(u)\,du, & a \le x < y \le b. \end{cases} \tag{2.4}$$

Proposition 2.5. *([5], p. 82) Let* f *be absolutely continuous on* $[a,b]$. *Then* $\forall$ $x \in (a,b)$

$$f(x) = \int_a^b f(y)\,\omega(y)\,dy + \int_a^b \Lambda(x,y)\,f'(y)\,dy, \tag{2.5}$$

the simplest case of Sobolev's integral representation.

Remark 2.6. ([5], pp. 82-83) We have that Λ is bounded:

$$\forall\ x,y \in [a,b], \quad |\Lambda(x,y)| \le \|\omega\|_{L_1(a,b)}, \tag{2.6}$$

and if $\omega \ge 0$, then

$$\forall\ x,y \in [a,b], \quad |\Lambda(x,y)| \le \Lambda(b,b) = 1.$$

If ω is symmetric with respect to $\frac{a+b}{2}$, then $\forall\ y \in [a,b]$ we have

$$\left|\Lambda\left(\frac{a+b}{2}, y\right)\right| \le \frac{1}{2}.$$

Examples of ω:

$\omega(x) = \frac{1}{b-a}$, $\forall\ x \in (a,b)$,

also

$\omega(x) = \frac{1}{2m}\left(\chi_{\left(a,a+\frac{1}{m}\right)} + \chi_{\left(b-\frac{1}{m},b\right)}\right)$, where $\chi_{(\alpha,\beta)}$ denotes the characteristic function of (α,β), $m \in \mathbb{N}$ and $m \ge 2(b-a)^{-1}$.

If $f \in \left(W_1^l\right)^{loc}(a,b)$, then f is equivalent to a function, which is locally absolutely continuous on (a,b) (its ordinary derivative, which exists almost everywhere on (a,b), is a weak derivative f'_w of f). Thus (2.5) holds for almost every $x \in (a,b)$ if f' is replaced by f'_w.

In this article sums of the form $\sum_{k=1}^{0} \cdot = 0$.

We mention

Theorem 2.7. *([5], p. 83) Let* $l \in \mathbb{N}$, $-\infty \le a < \alpha < \beta < b \le \infty$ *and*

$$\begin{cases} \omega \in L_1(\mathbb{R}), & \textit{(support)}\ \operatorname{supp}\omega \subset [\alpha,\beta], \\ \int_{\mathbb{R}} \omega(x)\,dx = 1 \end{cases} \tag{2.7}$$

Moreover, suppose that the derivative $f^{(l-1)}$ *exists and is locally absolutely continuous on* (a,b). *Then* $\forall\ x \in (a,b)$

$$f(x) = \sum_{k=0}^{l-1} \frac{1}{k!} \int_a^b f^{(k)}(y)(x-y)^k \omega(y)\,dy$$

$$+\frac{1}{(l-1)!} \int_a^b (x-y)^{l-1} \Lambda(x,y) f^{(l)}(y)\,dy, \tag{2.8}$$

and

$$f(x) = \sum_{k=0}^{l-1} \frac{1}{k!} \int_\alpha^\beta f^{(k)}(y)(x-y)^k \omega(y)\,dy$$

$$+\frac{1}{(l-1)!} \int_{a_x}^{b_x} (x-y)^{l-1} \Lambda(x,y) f^{(l)}(y)\,dy, \tag{2.9}$$

where $a_x = x$, $b_x = \beta$ *for* $x \in (a,\alpha]$; $a_x = \alpha$, $b_x = \beta$ *for* $x \in (\alpha,\beta)$; $a_x = \alpha$, $b_x = x$ *for* $x \in [\beta, b)$.

If, in particular, $-\infty < a < b < \infty$, $f^{(l-1)}$ *exists and is absolutely continuous on* $[a,b]$, *then (2.8), (2.9) hold* $\forall\ x \in [a,b]$ *and for any interval* $(\alpha,\beta) \subset (a,b)$.

Corollary 2.8. *([5], p. 85) Suppose that* $l > 1$, *condition (2.7) is replaced by*

$$\begin{cases} \omega \in C^{(l-2)}(\mathbb{R}), & \operatorname{supp}\omega \subset [\alpha,\beta], \\ \int_{\mathbb{R}} \omega(x)\,dx = 1, \end{cases} \tag{2.10}$$

and the derivative $\omega^{(l-2)}$ *is absolutely continuous on* $[a,b]$.

Then for the same f *as in Theorem 2.7,* $\forall\ x \in (a,b)$

$$f(x) = \int_\alpha^\beta \left(\sum_{k=0}^{l-1} \frac{(-1)^k}{k!} \left[(x-y)^k \omega(y) \right]_y^{(k)} \right) f(y)\,dy$$

$$+\frac{1}{(l-1)!} \int_{a_x}^{b_x} (x-y)^{l-1} \Lambda(x,y) f^{(l)}(y)\,dy. \tag{2.11}$$

In particular here

$$\omega(\alpha) = ... = \omega^{(l-2)}(\alpha) = \omega(\beta) = ... = \omega^{(l-2)}(\beta) = 0. \tag{2.12}$$

Corollary 2.9. *([5], p. 86) Suppose that* $l, m \in \mathbb{N}$, $m < l$. *Then for the same* f *and* ω *as in Corollary 2.8,* $\forall\ x \in (a,b)$

$$f^{(m)}(x) = \int_\alpha^\beta \left(\sum_{k=0}^{l-m-1} \frac{(-1)^{k+m}}{k!} \left[(x-y)^k \omega(y) \right]_y^{(k+m)} \right) f(y)\,dy \tag{2.13}$$

$$+\frac{1}{(l-m-1)!} \int_{a_x}^{b_x} (x-y)^{l-m-1} \Lambda(x,y) f^{(l)}(y)\,dy.$$

Remark 2.10. ([5], p. 86) The first summand in (2.11) can take the form:

$$\begin{cases} \int_\alpha^\beta \left(\sum_{s=0}^{l-1} \sigma_s (x-y)^s \omega^{(s)}(y)\right) f(y)\, dy, \\ \text{where } \sigma_s := \frac{(-1)^s}{s!} \sum_{k=s}^{l-s-1} \begin{pmatrix} s+k \\ k \end{pmatrix}. \end{cases} \tag{2.14}$$

Similarly we have for the first summand of (2.13) the following form

$$\begin{cases} \int_\alpha^\beta \left(\sum_{s=m}^{l-1} \sigma_{s,m} (x-y)^{s-m} \omega^{(s)}(y)\right) f(y)\, dy, \\ \text{where } \sigma_{s,m} := \frac{(-1)^s}{(s-m)!} \sum_{k=s}^{l-s-1} \begin{pmatrix} s+k \\ k \end{pmatrix}. \end{cases} \tag{2.15}$$

We need

Theorem 2.11. *([5], p. 91) Let $l \in N$, $-\infty \le a < \alpha < \beta < b \le \infty$, ω satisfy condition*

$$\omega \in L_1(\mathbb{R}), \operatorname{supp} \omega \subset [\alpha, \beta], \int_{\mathbb{R}} \omega(x)\, dx = 1, \tag{2.16}$$

and $f \in \left(W_1^l\right)^{loc}(a,b)$. Then for almost every $x \in (a,b)$

$$f(x) = \sum_{k=0}^{l-1} \frac{1}{k!} \int_\alpha^\beta f_w^{(k)}(y)(x-y)^k \omega(y)\, dy$$

$$+\frac{1}{(l-1)!} \int_{a_x}^{b_x} (x-y)^{l-1} \Lambda(x,y) f_w^{(l)}(y)\, dy, \tag{2.17}$$

where a_x, b_x as in Theorem 2.7.

We denote $f_w^{(0)} := f$.

Remark 2.12. ([5], p. 92) By Theorem 2.11 it follows that if in Corollaries 2.8, 2.9 $f \in \left(W_1^l\right)^{loc}(a,b)$ then equalities (2.11) and (2.13) hold almost everywhere on (a,b), if we replace $f^{(l)}$, $f^{(m)}$ by the weak derivatives $f_w^{(l)}$, $f_w^{(m)}$; respectively.

Next we estimate the remainders of the above mentioned Sobolev representations.

We make

Remark 2.13. Denote by $\overline{f}^{(k)}$ either $f^{(k)}$ or $f_w^{(k)}$, where $k \in \mathbb{N}$. Let $0 \le m < l$, $m \in \mathbb{Z}_+$. We estimate

$$R_{m,l} f(x) := \frac{1}{(l-m-1)!} \int_\alpha^\beta (x-y)^{l-m-1} \Lambda(x,y) \overline{f}^{(l)}(y)\, dy, \tag{2.18}$$

for $x \in (\alpha, \beta)$, where Λ as in (2.4), see also (2.6).

So we have

$$R_{0,l} f(x) := \frac{1}{(l-1)!} \int_\alpha^\beta (x-y)^{l-1} \Lambda(x,y) \overline{f}^{(l)}(y)\, dy. \tag{2.19}$$

Thus we obtain

$$|R_{m,l}f(x)| \leq \frac{\|\omega\|_{L_1(a,b)} \cdot (\beta-\alpha)^{l-m-1}}{(l-m-1)!} \int_\alpha^\beta \left|\overline{f}^{(l)}(y)\right| dy \tag{2.20}$$

$$= \frac{\|\omega\|_{L_1(a,b)} \cdot \left\|\overline{f}^{(l)}\right\|_{L_1(\alpha,\beta)} \cdot (\beta-\alpha)^{l-m-1}}{(l-m-1)!},$$

$x \in (\alpha,\beta)$.

We also have

$$|R_{m,l}f(x)| \leq \frac{\|\omega\|_{L_1(a,b)}}{(l-m-1)!} \int_\alpha^\beta |x-y|^{l-m-1} \left|\overline{f}^{(l)}(y)\right| dy =: I_1.$$

If $\overline{f}^{(l)} \in L_\infty(\alpha,\beta)$, then

$$I_1 \leq \frac{\|\omega\|_{L_1(a,b)} \left\|\overline{f}^{(l)}\right\|_{L_\infty(\alpha,\beta)}}{(l-m-1)!} \left(\int_\alpha^\beta |x-y|^{l-m-1} dy\right).$$

But

$$\int_\alpha^\beta |x-y|^{l-m-1} dy = \int_\alpha^x (x-y)^{l-m-1} dy + \int_x^\beta (y-x)^{l-m-1} dy$$

$$= \frac{(\beta-x)^{l-m} + (x-\alpha)^{l-m}}{l-m}.$$

Therefore if $\overline{f}^{(l)} \in L_\infty(\alpha,\beta)$, then

$$|R_{m,l}f(x)| \leq \frac{\|\omega\|_{L_1(a,b)} \left\|\overline{f}^{(l)}\right\|_{L_\infty(\alpha,\beta)}}{(l-m)!} \left((\beta-x)^{l-m} + (x-\alpha)^{l-m}\right), \tag{2.21}$$

$x \in (\alpha,\beta)$.

Let now $p,q>1 : \frac{1}{p}+\frac{1}{q}=1$. If $\overline{f}^{(l)} \in L_p(\alpha,\beta)$, then

$$I_1 \leq \frac{\|\omega\|_{L_1(a,b)}}{(l-m-1)!} \left(\int_\alpha^\beta |x-y|^{q(l-m-1)} dy\right)^{\frac{1}{q}} \left\|\overline{f}^{(l)}\right\|_{L_p(\alpha,\beta)}.$$

But

$$\int_\alpha^\beta |x-y|^{q(l-m-1)} dy = \int_\alpha^x (x-y)^{q(l-m-1)} dy + \int_x^\beta (y-x)^{q(l-m-1)} dy$$

$$= \frac{(x-\alpha)^{q(l-m-1)+1} + (\beta-x)^{q(l-m-1)+1}}{q(l-m-1)+1}.$$

Hence if $\overline{f}^{(l)} \in L_p(\alpha,\beta)$, then

$$|R_{m,l}f(x)| \leq \frac{\|\omega\|_{L_1(a,b)} \left\|\overline{f}^{(l)}\right\|_{L_p(\alpha,\beta)}}{(l-m-1)!}$$

$$\times \left(\frac{(\beta - x)^{q(l-m-1)+1} + (x - \alpha)^{q(l-m-1)+1}}{q(l-m-1)+1} \right)^{\frac{1}{q}}, \tag{2.22}$$

$x \in (\alpha, \beta)$.

If $\sup p\omega \subset [\alpha, \beta]$, then

$$\|\omega\|_{L_1(a,b)} = \|\omega\|_{L_1(\alpha,\beta)} . \tag{2.23}$$

If $\omega \in C(\mathbb{R})$ and $\sup p\omega \subset [\alpha, \beta]$, then

$$\|\omega\|_{L_1(\alpha,\beta)} \leq \|\omega\|_{\infty,[\alpha,\beta]} \cdot (\beta - \alpha) . \tag{2.24}$$

We make

Remark 2.14. Here we estimate from the Taylor's averaged polynomial, see (2.9) and (2.17), tha part

$$Q^{l-1} f(x) := \sum_{k=1}^{l-1} \frac{1}{k!} \int_\alpha^\beta \overline{f}^{(k)}(y) (x - y)^k \omega(y) \, dy, \tag{2.25}$$

called also quasi-averaged Taylor polynomial. When $l = 1$, then $Q^0 f(x) = 0$.

We see that

$$\left| Q^{l-1} f(x) \right| \leq \sum_{k=1}^{l-1} \frac{1}{k!} \int_\alpha^\beta \left| \overline{f}^{(k)}(y) \right| |x - y|^k |\omega(y)| \, dy$$

$$\leq \left(\sum_{k=1}^{l-1} \frac{(\beta - \alpha)^k}{k!} \left\| \overline{f}^{(k)} \right\|_{L_1(\alpha,\beta)} \right) \|\omega\|_{L_\infty(\mathbb{R})} , \tag{2.26}$$

given that $\|\omega\|_{L_\infty(\mathbb{R})} < \infty$, $x \in (\alpha, \beta)$.

Similarly, when $\overline{f}^{(k)} \in L_\infty(\alpha, \beta)$, $k = 1, ..., l-1$, and $\|\omega\|_{L_\infty(\mathbb{R})} < \infty$ we derive

$$\left| Q^{l-1} f(x) \right| \leq \sum_{k=1}^{l-1} \frac{\left\| \overline{f}^{(k)} \right\|_{L_\infty(\alpha,\beta)} \|\omega\|_{L_\infty(\mathbb{R})}}{k!} \int_\alpha^\beta |x - y|^k \, dy$$

$$= \left(\sum_{k=1}^{l-1} \left(\frac{(\beta - x)^{k+1} + (x - \alpha)^{k+1}}{(k+1)!} \right) \left\| \overline{f}^{(k)} \right\|_{L_\infty(\alpha,\beta)} \right) \|\omega\|_{L_\infty(\mathbb{R})} , \tag{2.27}$$

$x \in (\alpha, \beta)$.

Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\overline{f}^{(k)} \in L_p(\alpha, \beta)$, $k = 1, ..., l-1$, and again $\|\omega\|_{L_\infty(\mathbb{R})} < \infty$. Then

$$\left| Q^{l-1} f(x) \right| \leq \sum_{k=1}^{l-1} \frac{\left\| \overline{f}^{(k)} \right\|_{L_p(\alpha,\beta)}}{k!} \left(\int_\alpha^\beta |x - y|^{kq} \, dy \right)^{\frac{1}{q}} \|\omega\|_{L_\infty(\mathbb{R})}$$

$$= \left(\sum_{k=1}^{l-1} \left(\frac{(\beta - x)^{(kq+1)} + (x-\alpha)^{(kq+1)}}{kq+1} \right)^{\frac{1}{q}} \frac{\left\| \overline{f}^{(k)} \right\|_{L_p(\alpha,\beta)}}{k!} \right) \|\omega\|_{L_\infty(\mathbb{R})}, \tag{2.28}$$

$x \in (\alpha, \beta)$.

Assume $\omega \in L_1(\mathbb{R})$ and $\overline{f}^{(k)} \in L_\infty(\alpha,\beta)$, $k = 1, ..., l-1$, then

$$\left| Q^{l-1} f(x) \right| \leq \left(\sum_{k=1}^{l-1} \frac{\left\| \overline{f}^{(k)} \right\|_{L_\infty(a,\beta)} (\beta-\alpha)^k}{k!} \right) \|\omega\|_{L_1(\mathbb{R})}, \tag{2.29}$$

$x \in (\alpha, \beta)$.

Assume $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\overline{f}^{(k)} \in L_p(\alpha,\beta)$, $k = 1, ..., l-1$; $\omega \in L_q(\alpha,\beta)$, then

$$\left| Q^{l-1} f(x) \right| \leq \left(\sum_{k=1}^{l-1} \frac{(\beta-\alpha)^k}{k!} \left\| \overline{f}^{(k)} \right\|_{L_p(\alpha,\beta)} \right) \|\omega\|_{L_q(\alpha,\beta)}, \tag{2.30}$$

$x \in (\alpha, \beta)$.

Assume $p, q, r > 1 : \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, $\overline{f}^{(k)} \in L_p(\alpha,\beta)$, $k = 1, ..., l-1$; $\omega \in L_q(\alpha,\beta)$, then

$$\left| Q^{l-1} f(x) \right| \leq$$

$$\left(\sum_{k=1}^{l-1} \frac{\left\| \overline{f}^{(k)} \right\|_{L_p(\alpha,\beta)}}{k!} \left(\frac{(\beta - x)^{(kr+1)} + (x-\alpha)^{(kr+1)}}{(kr+1)} \right)^{\frac{1}{r}} \right) \|\omega\|_{L_q(\alpha,\beta)}, \tag{2.31}$$

$x \in (\alpha, \beta)$.

We also make

Remark 2.15. Here $l > 1$, $\omega \in C^{(l-2)}(\mathbb{R})$, $\sup p\omega \subset [\alpha, \beta]$, $\int_{\mathbb{R}} \omega(x)\, dx = 1$, and the derivative $\omega^{(l-2)}$ is absolutely continuous on $[a, b]$. Hence we have that

$$Q^{l-1} f(x) = \sum_{k=1}^{l-1} \frac{(-1)^k}{k!} \int_\alpha^\beta \left(\left[(x-y)^k \omega(y) \right]_y^{(k)} \right) f(y)\, dy, \tag{2.32}$$

$\forall\ x \in (\alpha, \beta)$.

And it holds

$$\left| Q^{l-1} f(x) \right| \leq \sum_{k=1}^{l-1} \frac{1}{k!} \int_\alpha^\beta \left| \left[(x-y)^k \omega(y) \right]_y^{(k)} \right| |f(y)|\, dy, \tag{2.33}$$

$\forall\ x \in (\alpha, \beta)$.

Consequently, $\forall\ x \in (\alpha, \beta)$,

$$\left| Q^{l-1} f(x) \right| \leq$$

$$\begin{cases} \left(\sum_{k=1}^{l-1}\frac{1}{k!}\left\|\left[(x-y)^k\,\omega(y)\right]_y^{(k)}\right\|_\infty\right)\|f\|_{L_1(\alpha,\beta)}\,, & \text{if } f\in L_1(\alpha,\beta)\,, \\ \left(\sum_{k=1}^{l-1}\frac{1}{k!}\left\|\left[(x-y)^k\,\omega(y)\right]_y^{(k)}\right\|_{L_1(\alpha,\beta)}\right)\|f\|_{L_\infty(\alpha,\beta)}\,, & \text{if } f\in L_\infty(\alpha,\beta)\,, \\ \text{when } p,q>1:\frac{1}{p}+\frac{1}{q}=1, \text{ we have} \\ \left(\sum_{k=1}^{l-1}\frac{1}{k!}\left\|\left[(x-y)^k\,\omega(y)\right]_y^{(k)}\right\|_{L_q(\alpha,\beta)}\right)\|f\|_{L_p(\alpha,\beta)}\,, & \text{if } f\in L_p(\alpha,\beta)\,. \end{cases} \tag{2.34}$$

Let $l,m\in\mathbb{N}$, $m<l$, and f,ω as above, $x\in(\alpha,\beta)$.

We consider here

$$Q_m^{l-1}f(x):=\sum_{k=1}^{l-m-1}\frac{(-1)^{k+m}}{k!}\int_\alpha^\beta\left(\left[(x-y)^k\,\omega(y)\right]_y^{(k+m)}\right)f(y)\,dy. \tag{2.35}$$

When $l=m+1$, then $Q_m^{l-1}f(x):=0$.

Hence it holds

$$\left|Q_m^{l-1}f(x)\right|\le\sum_{k=1}^{l-m-1}\frac{1}{k!}\int_\alpha^\beta\left|\left(\left[(x-y)^k\,\omega(y)\right]_y^{(k+m)}\right)\right||f(y)|\,dy, \tag{2.36}$$

$\forall\ x\in(\alpha,\beta)$.

Consequently, $\forall\ x\in(\alpha,\beta)$,

$$\left|Q_m^{l-1}f(x)\right|\le$$

$$\begin{cases} \left(\sum_{k=1}^{l-m-1}\frac{1}{k!}\left\|\left[(x-y)^k\,\omega(y)\right]_y^{(k+m)}\right\|_\infty\right)\|f\|_{L_1(\alpha,\beta)}\,, & \text{if } f\in L_1(\alpha,\beta)\,, \\ \left(\sum_{k=1}^{l-m-1}\frac{1}{k!}\left\|\left[(x-y)^k\,\omega(y)\right]_y^{(k+m)}\right\|_{L_1(\alpha,\beta)}\right)\|f\|_{L_\infty(\alpha,\beta)}\,, & \text{if } f\in L_\infty(\alpha,\beta), \\ \text{when } p,q>1:\frac{1}{p}+\frac{1}{q}=1, \text{ we have} \\ \left(\sum_{k=1}^{l-m-1}\frac{1}{k!}\left\|\left[(x-y)^k\,\omega(y)\right]_y^{(k+m)}\right\|_{L_q(\alpha,\beta)}\right)\|f\|_{L_p(\alpha,\beta)}\,, & \text{if } f\in L_p(\alpha,\beta)\,. \end{cases} \tag{2.37}$$

We also need

Remark 2.16. Here again $\overline{f}^{(k)}$ means either $f^{(k)}$ or $f_w^{(k)}$, $k\in\mathbb{N}$. We rewrite (2.9), (2.11) and (2.17). For $x\in(\alpha,\beta)$ we get

$$f(x)=\int_\alpha^\beta f(y)\,\omega(y)\,dy+Q^{l-1}f(x)+R_{0,l}f(x)\,. \tag{2.38}$$

Also for $x\in(\alpha,\beta)$ we rewrite (2.13) (see also Remark 2.12) as follows:

$$\overline{f}^{(m)}(x)=(-1)^m\int_\alpha^\beta f(y)\,\omega^{(m)}(y)\,dy+Q_m^{l-1}f(x)+R_{m,l}f(x)\,. \tag{2.39}$$

3. Main results

On our way to prove the general Chebyshev-Grüss type inequalities we establish the general

Theorem 3.1. *For f, g under the assumptions of any of Theorem 2.7, Corollary 2.8 and Theorem 2.11 we obtain that*

$$\Delta(f,g) := \left| \int_\alpha^\beta \omega(x) f(x) g(x) dx - \left(\int_\alpha^\beta \omega(x) f(x) dx \right) \left(\int_\alpha^\beta \omega(x) g(x) dx \right) \right|$$

$$\leq \frac{1}{2} \left[\left(\int_\alpha^\beta |\omega(x)| |g(x)| \left| Q^{l-1} f(x) \right| dx + \int_\alpha^\beta |\omega(x)| |f(x)| \left| Q^{l-1} g(x) \right| dx \right) \right.$$

$$\left. + \left(\int_\alpha^\beta |\omega(x)| |g(x)| |R_{0,l} f(x)| dx + \int_\alpha^\beta |\omega(x)| |f(x)| |R_{0,l} g(x)| dx \right) \right]. \tag{3.1}$$

Proof. For $x \in (\alpha, \beta)$ we have

$$f(x) = \int_\alpha^\beta f(y) \omega(y) dy + Q^{l-1} f(x) + R_{0,l} f(x),$$

and

$$g(x) = \int_\alpha^\beta g(y) \omega(y) dy + Q^{l-1} g(x) + R_{0,l} g(x).$$

Hence

$$\omega(x) f(x) g(x)$$

$$= \omega(x) g(x) \int_\alpha^\beta f(y) \omega(y) dy + \omega(x) g(x) Q^{l-1} f(x) + \omega(x) g(x) R_{0,l} f(x),$$

and

$$\omega(x) f(x) g(x)$$

$$= \omega(x) f(x) \int_\alpha^\beta g(y) \omega(y) dy + \omega(x) f(x) Q^{l-1} g(x) + \omega(x) f(x) R_{0,l} g(x).$$

Therefore

$$\int_\alpha^\beta \omega(x) f(x) g(x) dx = \left(\int_\alpha^\beta \omega(x) g(x) dx \right) \left(\int_\alpha^\beta f(x) \omega(x) dx \right)$$

$$+ \int_\alpha^\beta \omega(x) g(x) Q^{l-1} f(x) dx + \int_\alpha^\beta \omega(x) g(x) R_{0,l} f(x) dx,$$

and

$$\int_\alpha^\beta \omega(x) f(x) g(x) dx = \left(\int_\alpha^\beta \omega(x) f(x) dx \right) \left(\int_\alpha^\beta g(x) \omega(x) dx \right)$$

$$+ \int_\alpha^\beta \omega(x) f(x) Q^{l-1} g(x) dx + \int_\alpha^\beta \omega(x) f(x) R_{0,l} g(x) dx.$$

Consequently there hold

$$\int_{\alpha}^{\beta}\omega(x)f(x)g(x)\,dx-\left(\int_{\alpha}^{\beta}\omega(x)f(x)\,dx\right)\left(\int_{\alpha}^{\beta}g(x)\omega(x)\,dx\right)$$
$$=\int_{\alpha}^{\beta}\omega(x)g(x)Q^{l-1}f(x)\,dx+\int_{a}^{\beta}\omega(x)g(x)R_{0,l}f(x)\,dx,$$

and

$$\int_{\alpha}^{\beta}\omega(x)f(x)g(x)\,dx-\left(\int_{\alpha}^{\beta}\omega(x)f(x)\,dx\right)\left(\int_{\alpha}^{\beta}g(x)\omega(x)\,dx\right)$$
$$=\int_{\alpha}^{\beta}\omega(x)f(x)Q^{l-1}g(x)\,dx+\int_{\alpha}^{\beta}\omega(x)f(x)R_{0,l}g(x)\,dx.$$

Adding the last two equalities and divide by two, we get

$$\int_{\alpha}^{\beta}\omega(x)f(x)g(x)\,dx-\left(\int_{\alpha}^{\beta}\omega(x)f(x)\,dx\right)\left(\int_{\alpha}^{\beta}g(x)\omega(x)\,dx\right)$$
$$=\frac{1}{2}\left[\left(\int_{\alpha}^{\beta}\omega(x)g(x)Q^{l-1}f(x)\,dx+\int_{\alpha}^{\beta}\omega(x)f(x)Q^{l-1}g(x)\,dx\right)\right.$$
$$\left.+\left(\int_{\alpha}^{\beta}\omega(x)g(x)R_{0,l}f(x)\,dx+\int_{\alpha}^{\beta}\omega(x)f(x)R_{0,l}g(x)\,dx\right)\right],$$

hence proving the claim. □

General Chebyshev-Grüss inequalities follow.

We give

Theorem 3.2. *Let f,g with $f^{(l-1)}$, $g^{(l-1)}$ absolutely continuous on $[a,b]\subset\mathbb{R}$, $l\in\mathbb{N}$; $(\alpha,\beta)\subset(a,b)$. Let also $\omega\in L_1(\mathbb{R})$, $\sup p\omega\subset[\alpha,\beta]$, $\int_{\mathbb{R}}\omega(x)\,dx=1$. Then*

$$\left|\int_{\alpha}^{\beta}\omega(x)f(x)g(x)\,dx-\left(\int_{\alpha}^{\beta}\omega(x)f(x)\,dx\right)\left(\int_{\alpha}^{\beta}\omega(x)g(x)\,dx\right)\right|\leq$$
$$\frac{\|\omega\|_{L_1(\mathbb{R})}^2}{2}\left[\left[\|g\|_{\infty,(\alpha,\beta)}\left(\sum_{k=1}^{l-1}\frac{\|f^{(k)}\|_{\infty,(\alpha,\beta)}(\beta-\alpha)^k}{k!!}\right)+\right.\right.$$
$$\left.\|f\|_{\infty,(\alpha,\beta)}\left(\sum_{k=1}^{l-1}\frac{\|g^{(k)}\|_{\infty,(\alpha,\beta)}(\beta-\alpha)^k}{k!}\right)\right]+$$
$$\left.\left[\left(\|g\|_{\infty,(\alpha,\beta)}\left\|f^{(l)}\right\|_{L_1(a,\beta)}+\|f\|_{\infty,(\alpha,\beta)}\left\|g^{(l)}\right\|_{L_1(\alpha,\beta)}\right)\frac{(\beta-\alpha)^{l-1}}{(l-1)!}\right]\right].\quad(3.2)$$

Proof. By (2.20) and (2.29). □

Theorem 3.3. *Let* $f, g \in C^l([a,b])$, $[a,b] \subset \mathbb{R}$, $l \in \mathbb{N}$, $(\alpha, \beta) \subset (a,b)$. *Let also* $\omega \in L_\infty(\mathbb{R})$, $\operatorname{supp}\omega \subset [\alpha, \beta]$, $\int_{\mathbb{R}} \omega(x)\,dx = 1$. *Then*

$$\left| \int_\alpha^\beta \omega(x) f(x) g(x)\,dx - \left(\int_\alpha^\beta \omega(x) f(x)\,dx\right)\left(\int_\alpha^\beta \omega(x) g(x)\,dx\right)\right| \leq$$

$$\|\omega\|_{L_1(\mathbb{R})} \left[\frac{\|\omega\|_{L_1(\mathbb{R})}}{2} \left\{ \|g\|_{\infty,(\alpha,\beta)} \left(\sum_{k=1}^{l-1} \left(\frac{\|f^{(k)}\|_{\infty,(\alpha,\beta)} (\beta-\alpha)^k}{k!} \right)\right) + \right.\right.$$

$$\left. \|f\|_{\infty,(\alpha,\beta)} \left(\sum_{k=1}^{l-1} \left(\frac{\|g^{(k)}\|_{\infty,(\alpha,\beta)} (\beta-\alpha)^k}{k!} \right)\right) \right\} +$$

$$\left.\left[\|\omega\|_{\infty,(\alpha,\beta)} \frac{(\beta-\alpha)^{l+1}}{(l+1)!} \left(\|g\|_{\infty,(\alpha,\beta)} \left\| f^{(l)} \right\|_{\infty,(\alpha,\beta)} + \|f\|_{\infty,(\alpha,\beta)} \left\| g^{(l)} \right\|_{\infty,(\alpha,\beta)} \right)\right]\right]. \tag{3.3}$$

Proof. By (2.21) and (2.29). □

We further present

Theorem 3.4. *Let* $f, g \in \left(W_1^l\right)^{loc}(a,b)$; $a, b \in \mathbb{R}$; $(\alpha,\beta) \subset (a,b)$, $l \in \mathbb{N}$; $\omega \in L_\infty(\mathbb{R})$, $\operatorname{supp}\omega \subset [\alpha, \beta]$, $\int_{\mathbb{R}} \omega(x)\,dx = 1$. *Then*

$$\left| \int_\alpha^\beta \omega(x) f(x) g(x)\,dx - \left(\int_\alpha^\beta \omega(x) f(x)\,dx\right)\left(\int_\alpha^\beta \omega(x) g(x)\,dx\right)\right| \leq$$

$$\frac{\|\omega\|^2_{L_\infty(\mathbb{R})}}{2} \left\{ \left[\|g\|_{L_1(\alpha,\beta)} \left(\sum_{k=1}^{l-1} \left(\frac{(\beta-\alpha)^k}{k!} \left\| f_w^{(k)} \right\|_{L_1(\alpha,\beta)} \right)\right) + \right.\right.$$

$$\left. \|f\|_{L_1(\alpha,\beta)} \left(\sum_{k=1}^{l-1} \left(\frac{(\beta-\alpha)^k}{k!} \left\| g_w^{(k)} \right\|_{L_1(\alpha,\beta)} \right)\right) \right] +$$

$$\left.\left[\left(\|g\|_{L_1(\alpha,\beta)} \left\| f_w^{(l)} \right\|_{L_1(\alpha,\beta)} + \|f\|_{L_1(\alpha,\beta)} \left\| g_w^{(l)} \right\|_{L_1(\alpha,\beta)} \right) \frac{(\beta-\alpha)^l}{(l-1)!} \right]\right\}. \tag{3.4}$$

Proof. By (2.20) and (2.26). □

Theorem 3.5. *Let* $f, g \in \left(W_1^l\right)^{loc}(a,b)$; $a, b \in \mathbb{R}$; $(\alpha,\beta) \subset (a,b)$, $l \in \mathbb{N}$; $\omega \in L_\infty(\mathbb{R})$, $\operatorname{supp}\omega \subset [\alpha, \beta]$, $\int_{\mathbb{R}} \omega(x)\,dx = 1$. *Furthermore assume* $f_w^{(k)}, g_w^{(k)} \in L_\infty(\alpha,\beta)$, $k = 1, ..., l$. *Then*

$$\left| \int_\alpha^\beta \omega(x) f(x) g(x)\,dx - \left(\int_\alpha^\beta \omega(x) f(x)\,dx\right)\left(\int_\alpha^\beta \omega(x) g(x)\,dx\right)\right| \leq$$

$$\frac{\|\omega\|^2_{L_\infty(\mathbb{R})}}{2} \left\{ \left[\|g\|_{L_1(\alpha,\beta)} \left(\sum_{k=1}^{l-1} \frac{\left\| f_w^{(k)} \right\|_{L_\infty(\alpha,\beta)} (\beta-\alpha)^{k+1}}{k!} \right) + \right.\right.$$

$$\|f\|_{L_1(\alpha,\beta)}\left(\sum_{k=1}^{l-1}\frac{\left\|g_w^{(k)}\right\|_{L_\infty(\alpha,\beta)}(\beta-\alpha)^{k+1}}{k!}\right)\Bigg]+$$

$$\left[\left(\|g\|_{L_1(\alpha,\beta)}\left\|f_w^{(l)}\right\|_{L_\infty(\alpha,\beta)}+\|f\|_{L_1(\alpha,\beta)}\left\|g_w^{(l)}\right\|_{L_\infty(\alpha,\beta)}\right)\frac{(\beta-\alpha)^{l+1}}{(l-1)!}\right]\Bigg\}. \tag{3.5}$$

Proof. As in (2.21) and by (2.27). □

Theorem 3.6. *Let* $f,g\in\left(W_1^l\right)^{loc}(a,b)$; $a,b\in\mathbb{R}$; $(\alpha,\beta)\subset(a,b)$, $l\in\mathbb{N}$; $\omega\in L_\infty(\mathbb{R})$, $\operatorname{supp}\omega\subset[\alpha,\beta]$, $\int_{\mathbb{R}}\omega(x)\,dx=1$. *Furthermore assume for* $p>1$ *that* $f_w^{(k)},g_w^{(k)}\in L_p(\alpha,\beta)$, $k=1,...,l$. *Then*

$$\Delta(f,g):=\left|\int_\alpha^\beta\omega(x)f(x)g(x)dx-\left(\int_\alpha^\beta\omega(x)f(x)dx\right)\left(\int_\alpha^\beta\omega(x)g(x)dx\right)\right|$$

$$\leq\frac{\|\omega\|_{L_\infty(\mathbb{R})}^2}{2}\left[\left\{\|g\|_{L_1(\alpha,\beta)}\left(\sum_{k=1}^{l-1}\frac{(\beta-\alpha)^{k+1-\frac{1}{p}}}{k!}\left\|f_w^{(k)}\right\|_{L_p(\alpha,\beta)}\right)+\right.\right.$$

$$\left.\|f\|_{L_1(\alpha,\beta)}\left(\sum_{k=1}^{l-1}\frac{(\beta-\alpha)^{k+1-\frac{1}{p}}}{k!}\left\|g_w^{(k)}\right\|_{L_p(\alpha,\beta)}\right)\right\}+$$

$$\left.\left\{\left(\|g\|_{L_1(\alpha,\beta)}\left\|f_w^{(l)}\right\|_{L_p(\alpha,\beta)}+\|f\|_{L_1(\alpha,\beta)}\left\|g_w^{(l)}\right\|_{L_p(\alpha,\beta)}\right)\frac{(\beta-\alpha)^{l+1-\frac{1}{p}}}{(l-1)!}\right\}\right]. \tag{3.6}$$

Proof. Working as in (2.22) and from (2.30). □

Remark 3.7. When $f,g\in C^l([a,b])$, $l\in\mathbb{N}$, the Theorems 3.4, 3.5, 3.6 are again valid. In this case we replace $f_w^{(k)}$, $g_w^{(k)}$ by $f^{(k)}$, $g^{(k)}$ in all inequalities (3.4), (3.5) and (3.6); $k=1,...,l$.

We continue with

Theorem 3.8. *Let* $l\in\mathbb{N}-\{1\}$, $\omega\in C^{(l-2)}(\mathbb{R})$, $\operatorname{supp}\omega\subset[\alpha,\beta]$, $\int_{\mathbb{R}}\omega(x)\,dx=1$, *and the derivative* $\omega^{(l-2)}$ *is absolutely continuous on* $[a,b]\subset\mathbb{R}$; $(\alpha,\beta)\subset(a,b)$. *Here assume* $f,g\in\left(W_1^l\right)^{loc}(a,b)$, *or* $f,g\in C^l([a,b])$. *Here* $\overline{f}^{(l)}$ *denotes either* $f_w^{(l)}$ *or* $f^{(l)}$, *and* $\Delta(f,g)$ *as in (*3.1*).*

We have the following cases:

i) It holds

$$\Delta(f,g)\leq\frac{\|\omega\|_\infty}{2}\left[2\|g\|_{L_1(\alpha,\beta)}\|f\|_{L_1(\alpha,\beta)}\left(\sum_{k=1}^{l-1}\frac{1}{k!}\sup_{x\in(\alpha,\beta)}\left\|\left[(x-y)^k\omega(y)\right]_y^{(k)}\right\|_\infty\right)\right.$$

$$\left.+\left(\|g\|_{L_1(\alpha,\beta)}\left\|\overline{f}^{(l)}\right\|_{L_1(\alpha,\beta)}+\|f\|_{L_1(\alpha,\beta)}\left\|\overline{g}^{(l)}\right\|_{L_1(\alpha,\beta)}\right)\frac{(\beta-\alpha)^l}{(l-1)!}\|\omega\|_\infty\right]. \tag{3.7}$$

ii) Assume further that $f, g, \overline{f}^{(l)}, \overline{g}^{(l)} \in L_\infty(\alpha, \beta)$. *Then*

$$\Delta(f,g) \leq \frac{\|\omega\|_\infty}{2} \Bigg[\Bigg\{ \left(\|f\|_{L_\infty(\alpha,\beta)} \|g\|_{L_1(\alpha,\beta)} + \|g\|_{L_\infty(\alpha,\beta)} \|f\|_{L_1(\alpha,\beta)} \right) \cdot \left(\sum_{k=1}^{l-1} \frac{1}{k!} \sup_{x \in (\alpha,\beta)} \left\| \left[(x-y)^k \omega(y) \right]_y^{(k)} \right\|_{L_1(\alpha,\beta)} \right) \Bigg\} + \Bigg\{ \left(\|g\|_{L_1(\alpha,\beta)} \left\| \overline{f}^{(l)} \right\|_{L_\infty(\alpha,\beta)} + \|f\|_{L_1(\alpha,\beta)} \left\| \overline{g}^{(l)} \right\|_{L_\infty(\alpha,\beta)} \right) \frac{(\beta-\alpha)^{l+1}}{(l-1)!} \|\omega\|_\infty \Bigg\} \Bigg]. \tag{3.8}$$

iii) Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$; *assume further that* $f, g, \overline{f}^{(l)}, \overline{g}^{(l)} \in L_p(\alpha, \beta)$. *Then*

$$\Delta(f,g) \leq \frac{\|\omega\|_\infty}{2} \Bigg[\Bigg\{ \left(\|g\|_{L_1(\alpha,\beta)} \|f\|_{L_p(\alpha,\beta)} + \|f\|_{L_1(\alpha,\beta)} \|g\|_{L_p(\alpha,\beta)} \right) \cdot \left(\sum_{k=1}^{l-1} \frac{1}{k!} \sup_{x \in (\alpha,\beta)} \left\| \left[(x-y)^k \omega(y) \right]_y^{(k)} \right\|_{L_q(\alpha,\beta)} \right) \Bigg\} + \Bigg\{ \left(\|g\|_{L_1(\alpha,\beta)} \left\| \overline{f}^{(l)} \right\|_{L_p(\alpha,\beta)} + \|f\|_{L_1(\alpha,\beta)} \left\| \overline{g}^{(l)} \right\|_{L_p(\alpha,\beta)} \right) \frac{(\beta-\alpha)^{l+1-\frac{1}{p}}}{(l-1)!} \|\omega\|_\infty \Bigg\} \Bigg]. \tag{3.9}$$

Proof. By (3.1), (2.34) and by Theorems 3.4, 3.5, 3.6. ☐

Next we give a series of Ostrowski type inequalities.

Theorem 3.9. *Let* $l \in \mathbb{N}, [a,b] \subset \mathbb{R}$, $a < \alpha < \beta < b$ *and* $\omega \in L_1(\mathbb{R})$, $\operatorname{supp}\omega \subset [\alpha, \beta]$, $\int_{\mathbb{R}} \omega(x)\,dx = 1$. *Assume* f *on* $[a,b] : f^{(l-1)}$ *exists and is absolutely continuous on* $[a,b]$. *Then for any* $x \in (\alpha, \beta)$ *we get*

$$\left| f(x) - \int_\alpha^\beta f(y)\,\omega(y)\,dy - Q^{l-1} f(x) \right| \leq \frac{\|\omega\|_{L_1(\mathbb{R})} \left\| f^{(l)} \right\|_{L_1(\alpha,\beta)} (\beta-\alpha)^{l-1}}{(l-1)!} := A_1. \tag{3.10}$$

If additionally we assume $\|\omega\|_{L_\infty(\mathbb{R})} < \infty$, *then* $\forall\ x \in (\alpha, \beta)$, *we get*

$$\left| f(x) - \int_\alpha^\beta f(y)\,\omega(y)\,dy \right| \leq \left(\sum_{k=1}^{l-1} \left(\frac{(\beta-x)^{k+1} + (x-\alpha)^{k+1}}{(k+1)!} \right) \left\| f^{(k)} \right\|_\infty \right) \|\omega\|_{L_\infty(\mathbb{R})} + \frac{\|\omega\|_{L_1(\mathbb{R})} \left\| f^{(l)} \right\|_{L_1(\alpha,\beta)} (\beta-\alpha)^{l-1}}{(l-1)!} := B_1(x). \tag{3.11}$$

Proof. By (2.38), (2.20) and (2.27). ☐

Theorem 3.10. *All as in Theorem* 3.9. *Assume* $f \in C^l([a,b])$. *Then* $\forall\ x \in (\alpha,\beta)$,

$$\left| f(x) - \int_\alpha^\beta f(y)\,\omega(y)\,dy - Q^{l-1}f(x) \right| \leq$$

$$\frac{\|\omega\|_{L_1(\mathbb{R})} \left\|f^{(l)}\right\|_\infty \left((\beta - x)^l + (x-\alpha)^l \right)}{l!} =: A_2(x). \tag{3.12}$$

If additionally we assume $\|\omega\|_{L_\infty(\mathbb{R})} < \infty$, *then* $\forall\ x \in (\alpha,\beta)$,

$$\left| f(x) - \int_\alpha^\beta f(y)\,\omega(y)\,dy \right| \leq$$

$$\left(\sum_{k=1}^{l-1} \left(\frac{(\beta-x)^{k+1} + (x-\alpha)^{k+1}}{(k+1)!} \right) \left\| f^{(k)} \right\|_\infty \right) \|\omega\|_{L_\infty(\mathbb{R})} +$$

$$\frac{\|\omega\|_{L_1(\mathbb{R})} \left\|f^{(l)}\right\|_\infty \left((\beta - x)^l + (x-\alpha)^l \right)}{l!} =: B_2(x). \tag{3.13}$$

Proof. By (2.38), (2.21) and (2.27). □

We continue with

Theorem 3.11. *Let all as in Theorem* 3.9 *or* $f \in \left(W_1^l\right)^{loc}(a,b)$ *and rest as in Theorem* 3.9. *Then* $\forall\ x \in (\alpha,\beta)$ *(or almost every* $x \in (\alpha,\beta)$*, respectively), we get*

$$E(f)(x) := \left| f(x) - \int_\alpha^\beta f(y)\,\omega(y)\,dy - Q^{l-1}f(x) \right| \leq$$

$$\frac{\|\omega\|_{L_1(a,b)} \left\|\overline{f}^{(l)}\right\|_{L_1(\alpha,\beta)} (\beta-\alpha)^{l-1}}{(l-1)!} =: A_3 \tag{3.14}$$

Additionaly, if $\|\omega\|_{L_\infty(\mathbb{R})} < \infty$, $\forall\ x \in (\alpha,\beta)$ *(or almost every* $x \in (\alpha,\beta)$*, respectively), we get*

$$\Delta(f)(x) := \left| f(x) - \int_\alpha^\beta f(y)\,\omega(y)\,dy \right| \leq$$

$$\left(\sum_{k=1}^{l-1} \frac{(\beta-\alpha)^k}{k!} \left\| \overline{f}^{(k)} \right\|_{L_1(\alpha,\beta)} \right) \|\omega\|_{L_\infty(\mathbb{R})}$$

$$+ \frac{\|\omega\|_{L_1(a,b)} \left\|\overline{f}^{(l)}\right\|_{L_1(\alpha,\beta)} (\beta-\alpha)^{l-1}}{(l-1)!} =: B_3. \tag{3.15}$$

Proof. By (2.38), (2.20) and (2.26). □

Theorem 3.12. *Let all as in Theorem* 3.9 *or* $f \in \left(W_1^l\right)^{loc}(a,b)$ *and rest as in Theorem* 3.9. *Assume further* $\overline{f}^{(l)} \in L_\infty(\alpha,\beta)$. *Then* $\forall\ x \in (\alpha,\beta)$ *(or almost every* $x \in (\alpha,\beta)$*, respectively), we get*

$$E(f)(x) \leq \frac{\|\omega\|_{L_1(\mathbb{R})} \left\|\overline{f}^{(l)}\right\|_{L_\infty(\alpha,\beta)} \left((\beta-x)^l + (x-\alpha)^l\right)}{l!} =: A_4(x). \quad (3.16)$$

Additionaly $\overline{f}^{(k)} \in L_\infty(\alpha,\beta)$, $k = 1,\ldots,l-1$ *and if* $\|\omega\|_{L_\infty(\mathbb{R})} < \infty$, *then* $\forall$ $x \in (\alpha,\beta)$ *(or almost every* $x \in (\alpha,\beta)$*, respectively), we get*

$$\Delta(f)(x) \leq \left(\sum_{k=1}^{l-1}\left(\frac{\left((\beta-x)^{k+1}+(x-\alpha)^{k+1}\right)}{(k+1)!}\right)\left\|\overline{f}^{(k)}\right\|_{L_\infty(\alpha,\beta)}\right)\|\omega\|_{L_\infty(\mathbb{R})}$$

$$+\frac{\|\omega\|_{L_1(\mathbb{R})}\left\|\overline{f}^{(l)}\right\|_{L_\infty(\alpha,\beta)}\left((\beta-x)^l+(x-\alpha)^l\right)}{l!} =: B_4(x). \quad (3.17)$$

Proof. By (2.38), (2.21) and (2.27). □

Theorem 3.13. *Let all as in Theorem* 3.9 *or* $f \in \left(W_1^l\right)^{loc}(a,b)$ *and rest as in Theorem* 3.9. *Let* $p,q > 1 : \frac{1}{p}+\frac{1}{q} = 1$. *Assume further* $\overline{f}^{(l)} \in L_p(\alpha,\beta)$. *Then* $\forall\ x \in (\alpha,\beta)$ *(or almost every* $x \in (\alpha,\beta)$*, respectively), we get*

$$E(f)(x) \leq \frac{\|\omega\|_{L_1(\mathbb{R})}\left\|\overline{f}^{(l)}\right\|_{L_p(\alpha,\beta)}}{(l-1)!}\cdot$$

$$\left(\frac{(\beta-x)^{q(l-1)+1}+(x-\alpha)^{q(l-1)+1}}{q(l-1)+1}\right)^{\frac{1}{q}} =: A_5(x). \quad (3.18)$$

Additionally, if $\overline{f}^{(k)} \in L_p(\alpha,\beta)$, $k = 1,\ldots,l-1$ *and* $\|\omega\|_{L_\infty(\mathbb{R})} < \infty$, *then* $\forall\ x \in (\alpha,\beta)$ *(or almost every* $x \in (\alpha,\beta)$*, respectively), we get*

$$\Delta(f)(x) \leq$$

$$\left(\sum_{k=1}^{l-1}\left(\frac{\left((\beta-x)^{(kq+1)}+(x-\alpha)^{(kq+1)}\right)}{kq+1}\right)^{\frac{1}{q}}\frac{\left\|\overline{f}^{(k)}\right\|_{L_p(\alpha,\beta)}}{k!}\right)\|\omega\|_{L_\infty(\mathbb{R})}$$

$$+A_5(x) =: B_5(x). \quad (3.19)$$

Proof. By (2.38), (2.22) and (2.28). □

We further give

Theorem 3.14. *Let all as in Theorem* 3.12. *Here assume* $\omega \in L_1(\mathbb{R})$. *Then* $\forall$ $x \in (\alpha,\beta)$ *(or almost every* $x \in (\alpha,\beta)$*, respectively), we get*

$$\Delta(f)(x) := \left|f(x) - \int_\alpha^\beta f(y)\,\omega(y)\,dy\right| \leq$$

$$\left(\sum_{k=1}^{l-1} \frac{\left\| \overline{f}^{(k)} \right\|_{L_\infty(\alpha,\beta)} (\beta-\alpha)^k}{k!} \right) \|\omega\|_{L_1(\mathbb{R})} + A_4(x) =: B_6(x). \quad (3.20)$$

Proof. By (2.29) and (3.16). □

Theorem 3.15. *Let all be as in Theorem* 3.13. *Here assume* $\omega \in L_q(\alpha,\beta)$, $q > 1$. *Then* $\forall\ x \in (\alpha,\beta)$ *(or almost every* $x \in (\alpha,\beta)$*, respectively), we get*

$$\Delta(f)(x) \leq \left(\sum_{k=1}^{l-1} \frac{(\beta-\alpha)^k}{k!} \left\| \overline{f}^{(k)} \right\|_{L_p(\alpha,\beta)} \right) \|\omega\|_{L_q(\alpha,\beta)}$$
$$+A_5(x) =: B_7(x). \quad (3.21)$$

Proof. By (2.30) and (3.18). □

Theorem 3.16. *Let all as in Theorem* 3.9 *or* $f \in \left(W_1^l\right)^{loc}(a,b)$ *and rest as in Theorem* 3.9. *Let* $p,q,r > 1 : \frac{1}{p}+\frac{1}{q}+\frac{1}{r} = 1$, $\overline{f}^{(k)} \in L_p(\alpha,\beta)$, $k = 1,...,l-1$; $\omega \in L_q(\alpha,\beta)$. *Then* $\forall\ x \in (\alpha,\beta)$ *(or almost every* $x \in (\alpha,\beta)$*, respectively), we get*

$$\left| f(x) - \int_\alpha^\beta f(y)\,\omega(y)\,dy - R_{0,l}f(x) \right| \leq$$
$$\left(\sum_{k=1}^{l-1} \frac{\left\| \overline{f}^{(k)} \right\|_{L_p(\alpha,\beta)}}{k!} \left(\frac{(\beta-x)^{(kr+1)} + (x-\alpha)^{(kr+1)}}{(kr+1)} \right)^{\frac{1}{r}} \right) \|\omega\|_{L_q(\alpha,\beta)} =: \Phi(x).$$
$$(3.22)$$

Proof. By (2.31) and (2.38). □

We also present

Theorem 3.17. *Let* $\mathbb{N} \ni l > 1$ *and* $\omega \in C^{(l-2)}(\mathbb{R})$, $\operatorname{supp}\omega \subset [\alpha,\beta]$, $\int_{\mathbb{R}} \omega(x)\,dx = 1$, $\omega^{(l-2)}$ *is absolutely continuous on* $[a,b]$, $[\alpha,\beta] \subset (a,b)$; $a,b \in \mathbb{R}$. *Here* $f \in C^l([a,b])$ *or* $f \in \left(W_1^l\right)^{loc}(a,b)$. *For every* $x \in (\alpha,\beta)$ *(or almost every* $x \in (\alpha,\beta)$*, respectively), we get for*

$$\Delta(f)(x) := \left| f(x) - \int_\alpha^\beta f(y)\,\omega(y)\,dy \right|$$

that

i) It holds

$$\Delta(f)(x) \leq \left(\sum_{k=1}^{l-1} \frac{1}{k!} \left\| \left[(x-y)^k \omega(y) \right]_y^{(k)} \right\|_\infty \right) \|f\|_{L_1(\alpha,\beta)} +$$
$$\frac{\|\omega\|_{L_1(\mathbb{R})} \left\| \overline{f}^{(l)} \right\|_{L_1(\alpha,\beta)} (\beta-\alpha)^{l-1}}{(l-1)!} =: C_1(x). \quad (3.23)$$

ii) If $f, \overline{f}^{(l)} \in L_\infty(\alpha,\beta)$, *then*

$$\Delta(f)(x) \le \left(\sum_{k=1}^{l-1} \frac{1}{k!} \left\| \left[(x-y)^k \omega(y) \right]_y^{(k)} \right\|_{L_1(\alpha,\beta)} \right) \|f\|_{L_\infty(\alpha,\beta)} +$$

$$\frac{\|\omega\|_{L_1(\mathbb{R})} \left\| \overline{f}^{(l)} \right\|_{L_\infty(\alpha,\beta)} \left((\beta-x)^l + (x-\alpha)^l \right)}{l!} =: C_2(x). \tag{3.24}$$

iii) Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Assume further that* $f, \overline{f}^{(l)} \in L_p(\alpha,\beta)$. *Then*

$$\Delta(f)(x) \le \left(\sum_{k=1}^{l-1} \frac{1}{k!} \left\| \left[(x-y)^k \omega(y) \right]_y^{(k)} \right\|_{L_q(\alpha,\beta)} \right) \|f\|_{L_p(\alpha,\beta)} +$$

$$\frac{\|\omega\|_{L_1(\mathbb{R})} \left\| \overline{f}^{(l)} \right\|_{L_p(\alpha,\beta)}}{(l-1)!} \left(\frac{(\beta-x)^{q(l-1)+1} + (x-\alpha)^{q(l-1)+1}}{q(l-1)+1} \right)^{\frac{1}{q}} =: C_3(x). \tag{3.25}$$

Proof. By (2.34) and Theorems 3.11, 3.12, 3.13. □

We finish Ostrowski type inequalities with

Theorem 3.18. *Let* $l, m \in \mathbb{N}$, $m < l$; $\omega \in C^{(l-2)}(\mathbb{R})$, $\operatorname{supp} \omega \subset [\alpha,\beta]$, $\int_{\mathbb{R}} \omega(x)\,dx = 1$, $\omega^{(l-2)}$ *is absolutely continuous on* $[a,b]$, $[\alpha,\beta] \subset (a,b)$; $a, b \in \mathbb{R}$. *Here* $f \in C^l([a,b])$ *or* $f \in \left(W_1^l\right)^{loc}(a,b)$. *For every* $x \in (\alpha,\beta)$ *(or almost every* $x \in (\alpha,\beta)$*, respectively), we get for*

$$E_\beta(f)(x) := \left| \overline{f}^{(m)}(x) - (-1)^m \int_\alpha^\beta f(y)\,\omega^{(m)}(y)\,dy - Q_m^{l-1} f(x) \right|, \tag{3.26}$$

and

$$\Delta_\beta(f)(x) := \left| \overline{f}^{(m)}(x) - (-1)^m \int_\alpha^\beta f(y)\,\omega^{(m)}(y)\,dy \right| \tag{3.27}$$

that

i) it holds

$$E_\beta(f)(x) \le \frac{\|\omega\|_{L_1(\mathbb{R})} \left\| \overline{f}^{(l)} \right\|_{L_1(\alpha,\beta)} (\beta-\alpha)^{l-m-1}}{(l-m-1)!} =: E_1, \tag{3.28}$$

and

$$\Delta_\beta(f)(x) \le \left(\sum_{k=1}^{l-m-1} \frac{1}{k!} \left\| \left[(x-y)^k \omega(y) \right]_y^{(k+m)} \right\|_\infty \right) \|f\|_{L_1(\alpha,\beta)}$$

$$+E_1 =: G_1(x), \tag{3.29}$$

ii) if $\overline{f}^{(l)} \in L_\infty(\alpha,\beta)$, *then*

$$E_\beta(f)(x) \leq \frac{\|\omega\|_{L_1(\mathbb{R})} \left\|\overline{f}^{(l)}\right\|_{L_\infty(\alpha,\beta)}}{(l-m)!} \left((\beta-x)^{l-m} + (x-\alpha)^{l-m}\right) =: E_2(x), \tag{3.30}$$

if additionally we assume $f \in L_\infty(\alpha,\beta)$, *then*

$$\Delta_\beta(f)(x) \leq$$

$$\left(\sum_{k=1}^{l-m-1} \frac{1}{k!} \left\| \left[(x-y)^k \omega(y)\right]_y^{(k)} \right\|_{L_1(\alpha,\beta)}\right) \|f\|_{L_\infty(\alpha,\beta)} + E_2(x) =: G_2(x), \tag{3.31}$$

iii) let $p,q>1 : \frac{1}{p}+\frac{1}{q}=1$, *assume further that* $\overline{f}^{(l)} \in L_p(\alpha,\beta)$, *then*

$$E_\beta(f)(x) \leq \frac{\|\omega\|_{L_1(\mathbb{R})} \left\|\overline{f}^{(l)}\right\|_{L_p(\alpha,\beta)}}{(l-m-1)!}\cdot$$

$$\left(\frac{(\beta-x)^{q(l-m-1)+1} + (x-\alpha)^{q(l-m-1)+1}}{q(l-m-1)+1}\right)^{\frac{1}{q}} =: E_3(x), \tag{3.32}$$

and if additionally $f \in L_p(\alpha,\beta)$, *then*

$$\Delta_\beta(f)(x) \leq \left(\sum_{k=1}^{l-m-1} \frac{1}{k!} \left\| \left[(x-y)^k \omega(y)\right]_y^{(k+m)} \right\|_{L_q(\alpha,\beta)}\right) \|f\|_{L_p(\alpha,\beta)}$$

$$+E_3(x) =: G_3(x). \tag{3.33}$$

Proof. By (2.20), (2.21), (2.22), (2.37) and (2.39). □

We make

Remark 3.19. In preparation to present comparison of integral means inequalities we consider $(\alpha_1,\beta_1) \subseteq (\alpha,\beta)$. We consider also a weight function $\psi \geq 0$ which is Lebesgue integrable on $\mathbb{R}$ with $\sup p\psi \subset [\alpha_1,\beta_1] \subset [a,b]$, and $\int_{\mathbb{R}} \psi(x)\,dx = 1$. Clearly here $\int_{\alpha_1}^{\beta_1} \psi(x)\,dx = 1$.

E.g. for $x \in (\alpha_1,\beta_1)$, $\psi(x) := \frac{1}{\beta_1-\alpha_1}$, zero elsewhere, etc.

We will apply the following principle: In general a constraint of the form $|F(x)-G| \leq \varepsilon$, where F is a function and G, ε real numbers so that all make sense, implies that

$$\left|\int_{\mathbb{R}} F(x)\psi(x)\,dx - G\right| \leq \varepsilon. \tag{3.34}$$

Next we give a series of comparison of integral means inequalities based on Ostrowski type inequalities presented in this article. We use Remark 3.19.

Theorem 3.20. *All as in Theorem* 3.9. *Then*

$$u(f) := \left| \int_{\alpha_1}^{\beta_1} f(x)\psi(x)dx - \int_{\alpha}^{\beta} f(y)\omega(y)dy - \int_{\alpha_1}^{\beta_1} Q^{l-1} f(x)\psi(x)dx \right| \leq A_1, \tag{3.35}$$

and

$$m(f) := \left| \int_{\alpha_1}^{\beta_1} f(x)\psi(x)\,dx - \int_{\alpha}^{\beta} f(y)\,\omega(y)\,dy \right| \leq$$
$$\left(\sum_{k=1}^{l-1} \frac{(\beta-\alpha)^{k+1}}{(k+1)!} \left\| f^{(k)} \right\|_{\infty} \right) \|\omega\|_{L_\infty(\mathbb{R})} +$$
$$\frac{\|\omega\|_{L_1(\mathbb{R})} \left\| f^{(l)} \right\|_{L_1(\alpha,\beta)} (\beta-\alpha)^{l-1}}{(l-1)!}. \tag{3.36}$$

Proof. By Remark 3.19, Theorem 3.9, and the fact that the functions $(\beta-x)^{k+1} + (x-\alpha)^{k+1}$, $k = 1, ..., l-1$ are positive and convex with maximum $(\beta-\alpha)^{k+1}$. □

Theorem 3.21. *All as in Theorem* 3.10. *Then*

$$u(f) \leq \frac{\|\omega\|_{L_1(\mathbb{R})} \left\| f^{(l)} \right\|_{\infty} (\beta-\alpha)^{l}}{l!}, \tag{3.37}$$

and

$$m(f) \leq \left(\sum_{k=1}^{l-1} \frac{(\beta-\alpha)^{k+1}}{(k+1)!} \left\| f^{(k)} \right\|_{\infty} \right) \|\omega\|_{L_\infty(\mathbb{R})} +$$
$$\frac{\|\omega\|_{L_1(\mathbb{R})} \left\| f^{(l)} \right\|_{\infty} (\beta-\alpha)^{l}}{l!}. \tag{3.38}$$

Proof. Just maximize $A_2(x)$ of (3.12) and $B_2(x)$ of (3.13), etc. □

Theorem 3.22. *All as in Theorem* 3.11. *Then*

$$u(f) \leq A_3, \tag{3.39}$$

and

$$m(f) \leq B_3. \tag{3.40}$$

Theorem 3.23. *All as in Theorem* 3.12. *Then*

$$u(f) \leq \frac{\|\omega\|_{L_1(\mathbb{R})} \left\| \overline{f}^{(l)} \right\|_{L_\infty(\alpha,\beta)} (\beta-\alpha)^{l}}{l!}, \tag{3.41}$$

and

$$m(f) \leq \left(\sum_{k=1}^{l-1} \frac{(\beta-\alpha)^{k+1}}{(k+1)!} \left\| \overline{f}^{(k)} \right\|_{L_\infty(\alpha,\beta)} \right) \|\omega\|_{L_\infty(\mathbb{R})} +$$
$$\frac{\|\omega\|_{L_1(\mathbb{R})} \left\| \overline{f}^{(l)} \right\|_{L_\infty(\alpha,\beta)} (\beta-\alpha)^{l}}{l!}. \tag{3.42}$$

Theorem 3.24. *All as in Theorem* 3.13. *Then*

$$u(f) \leq \int_{\alpha_1}^{\beta_1} A_5(x)\,\psi(x)\,dx, \tag{3.43}$$

and

$$m(f) \leq \int_{\alpha_1}^{\beta_1} B_5(x)\,\psi(x)\,dx. \tag{3.44}$$

Proof. By the principle: if $|F(x) - G| \leq \varepsilon(x)$, then $\left|\int F(x)\,\psi(x)\,dx - G\right| \leq \int \varepsilon(x)\,\psi(x)\,dx$, etc. Here $A_5(x)$ as in (3.18) and $B_5(x)$ as in (3.19). □

Theorem 3.25. *All as in Theorem* 3.14. *Then*

$$m(f) \leq \int_{\alpha_1}^{\beta_1} B_6(x)\,\psi(x)\,dx, \tag{3.45}$$

where $B_6(x)$ *as in (*3.20*).*

Theorem 3.26. *All as in Theorem* 3.15. *Then*

$$m(f) \leq \int_{\alpha_1}^{\beta_1} B_7(x)\,\psi(x)\,dx, \tag{3.46}$$

where $B_7(x)$ *as in (*3.21*).*

Theorem 3.27. *All as in Theorem* 3.16. *Then*

$$\left|\int_{\alpha_1}^{\beta_1} f(x)\,\psi(x)\,dx - \int_{\alpha}^{\beta} f(y)\,\omega(y)\,dy - \int_{\alpha_1}^{\beta_1} (R_{0,l} f(x))\,\psi(x)\,dx\right| \leq$$

$$\int_{\alpha_1}^{\beta_1} \Phi(x)\,\psi(x)\,dx, \tag{3.47}$$

where $\Phi(x)$ *as in (*3.22*).*

We continue with

Theorem 3.28. *All as in Theorem* 3.17. *Then*

i)

$$m(f) \leq \left(\sum_{k=1}^{l-1} \frac{1}{k!} \sup_{x\in[\alpha_1,\beta_1]} \left\| \left[(x-y)^k\,\omega(y)\right]_y^{(k)} \right\|_\infty \right) \|f\|_{L_1(\alpha,\beta)} + \frac{\|\omega\|_{L_1(\mathbb{R})} \left\|\overline{f}^{(l)}\right\|_{L_1(\alpha,\beta)} (\beta-\alpha)^{l-1}}{(l-1)!}, \tag{3.48}$$

ii) if $f, \overline{f}^{(l)} \in L_\infty(\alpha,\beta)$, *then*

$$m(f) \leq \int_{\alpha_1}^{\beta_1} C_2(x)\,\psi(x)\,dx, \tag{3.49}$$

iii) let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$; *assume further* $f, \overline{f}^{(l)} \in L_p(\alpha,\beta)$, *then*

$$m(f) \leq \int_{\alpha_1}^{\beta_1} C_3(x)\,\psi(x)\,dx. \tag{3.50}$$

Here $C_2(x)$ *as in (*3.24*) and* $C_3(x)$ *as in (*3.25*).*

We finish the results about comparison of integral means with

Theorem 3.29. *All as in Theorem* 3.18*. Denote by*

$$u_m(f) :=$$
$$\left| \int_{\alpha_1}^{\beta_1} \overline{f}^{(m)}(x)\psi(x)dx - (-1)^m \int_{\alpha}^{\beta} f(y)\omega^{(m)}(y)dy - \int_{\alpha_1}^{\beta_1} \left(Q_m^{l-1} f(x)\right)\psi(x)dx \right|, \tag{3.51}$$

and

$$\rho_m(f) := \left| \int_{\alpha_1}^{\beta_1} \overline{f}^{(m)}(x)\psi(x)dx - (-1)^m \int_{\alpha}^{\beta} f(y)\omega^{(m)}(y)dy \right|. \tag{3.52}$$

i) it holds

$$u_m(f) \leq E_1, \tag{3.53}$$

where E_1 *as in (*3.28*),*
and

$$\rho_m(f) \leq \left(\sum_{k=1}^{l-m-1} \frac{1}{k!} \sup_{x\in[\alpha_1,\beta_1]} \left\| \left[(x-y)^k \omega(y) \right]_y^{(k+m)} \right\|_{\infty} \right) \|f\|_{L_1(\alpha,\beta)} + E_1, \tag{3.54}$$

ii) if $\overline{f}^{(l)} \in L_\infty(\alpha,\beta)$, *then*

$$u_m(f) \leq \frac{\|\omega\|_{L_1(\mathbb{R})} \left\| \overline{f}^{(l)} \right\|_{L_\infty(\alpha,\beta)}}{(l-m)!} (\beta-\alpha)^{l-m}, \tag{3.55}$$

and if additionally assume $f \in L_\infty(\alpha,\beta)$, *then*

$$\rho_m(f) \leq \left(\sum_{k=1}^{l-m-1} \frac{1}{k!} \sup_{x\in[\alpha_1,\beta_1]} \left\| \left[(x-y)^k \omega(y) \right]_y^{(k+m)} \right\|_{L_1(\alpha,\beta)} \right) \|f\|_{L_\infty(\alpha,\beta)}$$
$$+ \frac{\|\omega\|_{L_1(\mathbb{R})} \left\| \overline{f}^{(l)} \right\|_{L_\infty(\alpha,\beta)} (\beta-\alpha)^{l-m}}{(l-m)!}, \tag{3.56}$$

iii) let $p,q>1 : \frac{1}{p}+\frac{1}{q}=1$, *assume further* $\overline{f}^{(l)} \in L_p(\alpha,\beta)$, *then*

$$u_m(f) \leq \int_{\alpha_1}^{\beta_1} E_3(x)\psi(x)dx, \tag{3.57}$$

where $E_3(x)$ *as in (*3.32*),*
and if additionally $f \in L_p(\alpha,\beta)$, *then*

$$\rho_m(f) \leq \int_{\alpha_1}^{\beta_1} G_3(x)\psi(x)dx, \tag{3.58}$$

where $G_3(x)$ *as in (*3.33*).*

We need

Background 3.30. Let f be a convex function from $(0,+\infty)$ into $\mathbb{R}$ which is strictly convex at 1 with $f(1)=0$. Let (X,A,λ) be a measure space, where λ is a finite or a σ-finite measure on (X,A). And let μ_1, μ_2 be two probability measures on (X,A) such that $\mu_1 \ll \lambda$, $\mu_2 \ll \lambda$ (absolutely continuous), e.g. $\lambda = \mu_1 + \mu_2$.

Denote by $p = \frac{d\mu_1}{d\lambda}$, $q = \frac{d\mu_2}{d\lambda}$ the (densities) Radon-Nikodym derivatives of μ_1, μ_2 with respect to λ. Here we suppose that

$$0 < \alpha \leq \frac{p}{q} \leq \beta, \text{ a.e. on } X \text{ and } \alpha \leq 1 \leq \beta.$$

The quantity

$$\Gamma_f(\mu_1,\mu_2) = \int_X q(x) f\left(\frac{p(x)}{q(x)}\right) d\lambda(x), \tag{3.59}$$

was introduced by I. Csiszar in 1967, see [8], and is called f-divergence of the probability measures μ_1 and μ_2. By Lemma 1.1 of [8], the integral (3.59) is well-defined and $\Gamma_f(\mu_1,\mu_2) \geq 0$ with equality only when $\mu_1 = \mu_2$. Furthermore $\Gamma_f(\mu_1,\mu_2)$ does not depend on the choice of λ. The concept of f-divergence was introduced first in [7] as a generalization of Kullback's "information for discrimination" or I-divergence (generalized entropy) [12], [11] and of Rényi's "information gain" (I-divergence of order δ) [14]. In fact the I-divergence of order 1 equals $\Gamma_{u\log_2 u}(\mu_1,\mu_2)$. The choice $f(x) = (u-1)^2$ produces again a known measure of difference of distributions that is called χ^2-divergence. Of course the total variation distance $|\mu_1 - \mu_2| = \int_X |p(x) - q(x)| \, d\lambda(x)$ is equal to $\Gamma_{|u-1|}(\mu_1,\mu_2)$.

Here by supposing $f(1) = 0$ we can consider $\Gamma_f(\mu_1,\mu_2)$, the f-divergence as a measure of the difference between the probability measures μ_1,μ_2. The f-divergence is in general asymmetric in μ_1 and μ_2. But since f is convex and strictly convex at 1 so is

$$f^*(u) = uf\left(\frac{1}{u}\right) \tag{3.60}$$

and as in [8] we obtain

$$\Gamma_f(\mu_2,\mu_1) = \Gamma_{f^*}(\mu_1,\mu_2). \tag{3.61}$$

In Information Theory and Statistics many other divergences are used which are special cases of the above general Csiszar f-divergence, e.g. Hellinger distance D_H, α-distance D_α, Bhattacharyya distance D_B, Harmonic distance D_{Ha}, Jeffrey's distance D_J, triangular discrimination D_Δ, for all these see, e.g. [4], [9]. The problem of finding and estimating the proper distance (or difference or discrimination) of two probability distributions is one of the major ones in Probability Theory.

Here we provide a general probabilistic representation formula for $\Gamma_f(\mu_1,\mu_2)$. Then we present tight estimates for the remainder involving a variety of norms of the engaged functions. Also are implied some direct general approximations for the Csiszar's f-divergence. We give some applications.

We make

Remark 3.31. Here $0 < a < \alpha \leq \frac{p(x)}{q(x)} \leq \beta < b < +\infty$, a e. on X and $\alpha \leq 1 \leq \beta$. Also assume that $f^{(l-1)}$ exists and is absolutely continuous on $[a,b]$, $l \in \mathbb{N}$. Furthermore f is convex from $(0,+\infty)$ into $\mathbb{R}$, strictly convex at 1 with $f(1)=0$. Let $\omega \in L_1(\mathbb{R})$, $\sup p\omega \subset [\alpha,\beta]$, $\int_{\mathbb{R}} \omega(x)\,dx = 1$.

Then $\forall\ x \in (\alpha,\beta)$ we get by Theorem 2.7, as in (2.38), that

$$f(x) = \int_\alpha^\beta f(y)\,\omega(y)\,dy + Q^{l-1}f(x) + R_{0,l}f(x).$$

Therefore

$$f\left(\frac{p(x)}{q(x)}\right) = \int_\alpha^\beta f(y)\,\omega(y)\,dy + Q^{l-1}f\left(\frac{p(x)}{q(x)}\right) + R_{0,l}f\left(\frac{p(x)}{q(x)}\right),$$

a.e. on X.

Hence

$$q(x)\,f\left(\frac{p(x)}{q(x)}\right) =$$

$$q(x)\int_\alpha^\beta f(y)\,\omega(y)\,dy + q(x)\,Q^{l-1}f\left(\frac{p(x)}{q(x)}\right) + q(x)\,R_{0,l}f\left(\frac{p(x)}{q(x)}\right),$$

a.e. on X.

Therefore we get the representation of f-divergence of μ_1 and μ_2,

$$\Gamma_f(\mu_1,\mu_2) = \int_X q(x)\,f\left(\frac{p(x)}{q(x)}\right)d\lambda(x)$$

$$= \int_\alpha^\beta f(y)\,\omega(y)\,dy + \int_X q(x)\,Q^{l-1}f\left(\frac{p(x)}{q(x)}\right)d\lambda(x)$$

$$+ \int_X q(x)\,R_{0,l}f\left(\frac{p(x)}{q(x)}\right)d\lambda(x). \tag{3.62}$$

Call

$$Q_\Gamma := \int_X q(x)\,Q^{l-1}f\left(\frac{p(x)}{q(x)}\right)d\lambda(x), \tag{3.63}$$

and

$$R_\Gamma := \int_X q(x)\,R_{0,l}f\left(\frac{p(x)}{q(x)}\right)d\lambda(x). \tag{3.64}$$

We estimate Q_Γ and R_Γ.

If $\|\omega\|_{L_\infty(\mathbb{R})} < \infty$, we get by (2.26) that

$$|Q_\Gamma| \leq \left(\sum_{k=1}^{l-1} \frac{(\beta-\alpha)^k}{k!}\left\|f^{(k)}\right\|_{L_1(\alpha,\beta)}\right)\|\omega\|_{L_\infty(\mathbb{R})}. \tag{3.65}$$

Notice if $l=1$, then always $Q_\Gamma = 0$.

Next if again $\|\omega\|_{L_\infty(\mathbb{R})} < \infty$, then (by (2.27))

$$|Q_\Gamma| \leq \left(\int_X q(x)\left(\sum_{k=1}^{l-1} \frac{\left(\beta - \frac{p(x)}{q(x)}\right)^{k+1} + \left(\frac{p(x)}{q(x)} - \alpha\right)^{k+1}}{(k+1)!}\left\|f^{(k)}\right\|_{L_\infty(\alpha,\beta)}\right)d\lambda(x)\right)$$

$$\cdot \|\omega\|_{L_\infty(\mathbb{R})} \cdot \tag{3.66}$$

Let now $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$ and again $\|\omega\|_{L_\infty(\mathbb{R})} < \infty$. Then (by (2.28))

$$|Q_\Gamma| \leq \left(\int_X q(x) \left(\sum_{k=1}^{l-1} \left(\frac{\left(\beta - \frac{p(x)}{q(x)}\right)^{(kq+1)} + \left(\frac{p(x)}{q(x)} - \alpha\right)^{(kq+1)}}{kq+1} \right)^{\frac{1}{q}} \cdot \right.\right.$$

$$\left.\left. \frac{\left\|f^{(k)}\right\|_{L_p(\alpha,\beta)}}{k!} \right) d\lambda(x) \right) \|\omega\|_{L_\infty(\mathbb{R})} \cdot \tag{3.67}$$

Next assume $\omega \in L_1(\mathbb{R})$, then (by (2.29))

$$|Q_\Gamma| \leq \left(\sum_{k=1}^{l-1} \frac{\left\|f^{(k)}\right\|_{L_\infty(\alpha,\beta)} (\beta-\alpha)^k}{k!} \right) \|\omega\|_{L_1(\mathbb{R})} \cdot \tag{3.68}$$

If $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$ and $\omega \in L_q(\alpha, \beta)$, then (by (2.30))

$$|Q_\Gamma| \leq \left(\sum_{k=1}^{l-1} \frac{(\beta-\alpha)^k}{k!} \left\|f^{(k)}\right\|_{L_p(\alpha,\beta)} \right) \|\omega\|_{L_q(\alpha,\beta)} \cdot \tag{3.69}$$

Assume $p, q, r > 1 : \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$ and $\omega \in L_q(\alpha, \beta)$, then (by (2.31))

$$|Q_\Gamma| \leq \left(\int_X q(x) \left(\sum_{k=1}^{l-1} \frac{\left\|f^{(k)}\right\|_{L_p(\alpha,\beta)}}{k!} \cdot \right.\right.$$

$$\left.\left. \left(\frac{\left(\beta - \frac{p(x)}{q(x)}\right)^{(kr+1)} + \left(\frac{p(x)}{q(x)} - \alpha\right)^{(kr+1)}}{kr+1} \right)^{\frac{1}{r}} \right) d\lambda(x) \right) \|\omega\|_{L_q(\alpha,\beta)} \cdot \tag{3.70}$$

We make

Remark 3.32. (continuation of Remark 3.31) Here $l > 1$, $\omega \in C^{(l-2)}(\mathbb{R})$, $\sup p\omega \subset [\alpha, \beta]$, $\int_{\mathbb{R}} \omega(x)\,dx = 1$, and $\omega^{(l-2)}$ is absolutely continuous on $[a, b]$. Then (by (2.32))

$$Q_\Gamma = \int_X q(x) \left(\sum_{k=1}^{l-1} \frac{(-1)^k}{k!} \int_\alpha^\beta \left(\left[\left(\frac{p(x)}{q(x)} - y \right)^k \omega(y) \right]_y^{(k)} \right) f(y)\,dy \right) d\lambda(x). \tag{3.71}$$

Hence by (2.34) we obtain

$$|Q_\Gamma| \leq \min \text{ of}$$

$$\begin{cases} \left(\int_X q(x)\left(\sum_{k=1}^{l-1}\frac{1}{k!}\left\|\left[\left(\frac{p(x)}{q(x)}-y\right)^k\omega(y)\right]_y^{(k)}\right\|_\infty\right)d\lambda(x)\right)\|f\|_{L_1(\alpha,\beta)}, \\ \left(\int_X q(x)\left(\sum_{k=1}^{l-1}\frac{1}{k!}\left\|\left[\left(\frac{p(x)}{q(x)}-y\right)^k\omega(y)\right]_y^{(k)}\right\|_{L_1(\alpha,\beta)}\right)d\lambda(x)\right)\|f\|_{L_\infty(\alpha,\beta)}, \\ \text{when } p,q>1:\frac{1}{p}+\frac{1}{q}=1, \text{ we have} \\ \left(\int_X q(x)\left(\sum_{k=1}^{l-1}\frac{1}{k!}\left\|\left[\left(\frac{p(x)}{q(x)}-y\right)^k\omega(y)\right]_y^{(k)}\right\|_{L_q(\alpha,\beta)}\right)d\lambda(x)\right)\|f\|_{L_p(\alpha,\beta)}. \end{cases} \tag{3.72}$$

We also make

Remark 3.33. (another continuation of Remark 3.31) Here we estimate the remainder R_Γ of (3.62). By (2.20), (3.64) we obtain

$$|R_\Gamma| \leq \frac{\|\omega\|_{L_1(a,b)}\left\|f^{(l)}\right\|_{L_1(\alpha,\beta)}(\beta-\alpha)^{l-1}}{(l-1)!}. \tag{3.73}$$

If $f^{(l)} \in L_\infty(\alpha,\beta)$, then (by (2.21)) we obtain

$$|R_\Gamma| \leq \frac{\|\omega\|_{L_1(a,b)}\left\|f^{(l)}\right\|_{L_\infty(\alpha,\beta)}}{l!}\cdot$$

$$\left(\int_X q(x)\left(\left(\beta-\frac{p(x)}{q(x)}\right)^l+\left(\frac{p(x)}{q(x)}-\alpha\right)^l\right)d\lambda(x)\right). \tag{3.74}$$

Let now $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Here $f^{(l)} \in L_p(\alpha,\beta)$, then (by (2.22)) we get

$$|R_\Gamma| \leq \frac{\|\omega\|_{L_1(a,b)}\left\|f^{(l)}\right\|_{L_p(\alpha,\beta)}}{(q(l-1)+1)^{\frac{1}{q}}(l-1)!}\cdot$$

$$\left(\int_X q(x)\left(\left(\beta-\frac{p(x)}{q(x)}\right)^{(q(l-1)+1)}+\left(\frac{p(x)}{q(x)}-\alpha\right)^{(q(l-1)+1)}\right)^{\frac{1}{q}}d\lambda(x)\right). \tag{3.75}$$

Finally we see that

$$\Gamma_f(\mu_1,\mu_2)-\int_\alpha^\beta f(y)\,\omega(y)\,dy = Q_\Gamma + R_\Gamma, \tag{3.76}$$

and

$$T := \left|\Gamma_f(\mu_1,\mu_2)-\int_\alpha^\beta f(y)\,\omega(y)\,dy\right| \leq |Q_\Gamma|+|R_\Gamma|. \tag{3.77}$$

Then one by the above estimates of $|Q_\Gamma|$ and $|R_\Gamma|$ can estimate T, in a number of cases.

4. Applications

Example 4.1. Let $V := \{x \in \mathbb{R} : |x - x_0| < \rho\}$, $x_0 \in \mathbb{R}$, and

$$\varphi(x) := \begin{cases} e^{-\left(1-\frac{(x-x_0)^2}{\rho^2}\right)^{-1}}, & \text{if } |x-x_0| < \rho, \\ 0, & \text{if } |x-x_0| \geq \rho. \end{cases} \tag{4.1}$$

Call $c := \int_{\mathbb{R}} \varphi(x)\,dx > 0$, then $\Phi(x) := \frac{1}{c}\varphi(x) \in C_0^{\infty}(\mathbb{R})$ (space of continuously infinitely many times differentiable functions of compact support) with $\sup p\Phi = \overline{V}$ and $\int_{-\infty}^{\infty} \Phi(x)\,dx = 1$ and $\max|\Phi| \leq cons\tan t \cdot \rho^{-1}$. We call Φ a cut-off function.

One for this article's results by choosing $\omega(x) = \Phi(x)$ or $\omega(x) = \frac{1}{2\rho}$, etc., can give lots of applications. Due to lack of space we avoid it.

Instead, selectively, we give some special cases inequalities. We start with Chebyshev-Grüss type inequalities.

Corollary 4.2. *(to Theorem 3.3) Let $f, g \in C^1([a,b])$, $[a,b] \subset \mathbb{R}$, $(\alpha,\beta) \subset (a,b)$. Let also $\omega \in L_{\infty}(\mathbb{R})$, $\sup p\omega \subset [\alpha,\beta]$, $\int_{\mathbb{R}} \omega(x)\,dx = 1$. Then*

$$\left| \int_{\alpha}^{\beta} \omega(x) f(x) g(x)\,dx - \left(\int_{\alpha}^{\beta} \omega(x) f(x)\,dx\right)\left(\int_{\alpha}^{\beta} \omega(x) g(x)\,dx\right)\right| \leq$$

$$\|\omega\|_{L_1(\mathbb{R})}\|\omega\|_{\infty,(\alpha,\beta)}\frac{(\beta-\alpha)^2}{2}(\|g\|_{\infty,(\alpha,\beta)}\|f'\|_{\infty,(\alpha,\beta)} + \|f\|_{\infty,(\alpha,\beta}\|g'\|_{\infty,(\alpha,\beta)}). \tag{4.2}$$

If $f = g$, then

$$\left| \int_{\alpha}^{\beta} \omega(x) f^2(x)\,dx - \left(\int_{\alpha}^{\beta} \omega(x) f(x)\,dx\right)^2\right| \leq$$

$$\|\omega\|_{L_1(\mathbb{R})}\|\omega\|_{\infty,(\alpha,\beta)}(\beta-\alpha)^2\|f\|_{\infty,(\alpha,\beta)}\|f'\|_{\infty,(\alpha,\beta)}. \tag{4.3}$$

Corollary 4.3. *(to Theorem 3.4) Let $f \in \left(W_1^1\right)^{loc}(a,b)$; $a, b \in \mathbb{R}$; $(\alpha,\beta) \subset (a,b)$, $\omega(x) := \frac{1}{\beta-\alpha}$ for $x \in [\alpha,\beta]$, and zero elsewhere. Then*

$$\left|\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} f^2(x)\,dx - \frac{1}{(\beta-\alpha)^2}\left(\int_{\alpha}^{\beta} f(x)\,dx\right)^2\right| \leq$$

$$\frac{\|f\|_{L_1(\alpha,\beta)}\left\|f_w^{(1)}\right\|_{L_1(\alpha,\beta)}}{(\beta-\alpha)}. \tag{4.4}$$

We continue with an Ostrowski type inequality.

Corollary 4.4. *(to Theorem 3.11) All as in Theorem 3.11. Case of $l = 1$. Then, for any $x \in (\alpha,\beta)$ (or for almost every $x \in (\alpha,\beta)$, respectively), we get*

$$\left|f(x) - \int_{\alpha}^{\beta} f(y)\,\omega(y)\,dy\right| \leq \|\omega\|_{L_1(\mathbb{R})}\left\|\overline{f}'\right\|_{L_1(\alpha,\beta)}. \tag{4.5}$$

Next comes a comparison of means inequality.

Corollary 4.5. *All here as in Corollary* 4.4 *and Remark* 3.19. *Then*

$$\left|\int_{\alpha_1}^{\beta_1} f(x)\psi(x)\,dx - \int_{\alpha}^{\beta} f(y)\omega(y)\,dy\right| \leq \|\omega\|_{L_1(\mathbb{R})}\left\|\overline{f}'\right\|_{L_1(\alpha,\beta)}. \tag{4.6}$$

Proof. By (4.5). □

We finish with an application of f-divergence.

Remark 4.6. All here as in Background 3.30 and Remark 3.31. Case of $l=1$. By (3.62) we get

$$\Gamma_f(\mu_1,\mu_2) = \int_{\alpha}^{\beta} f(y)\omega(y)\,dy + \int_X q(x) R_{0,1} f\left(\frac{p(x)}{q(x)}\right) d\lambda(x). \tag{4.7}$$

That is here

$$R_\Gamma = \int_X q(x) R_{0,1} f\left(\frac{p(x)}{q(x)}\right) d\lambda(x). \tag{4.8}$$

By (3.73) here we get that

$$|R_\Gamma| \leq \|\omega\|_{L_1(a,b)} \|f'\|_{L_1(\alpha,\beta)}. \tag{4.9}$$

If $f' \in L_\infty(\alpha,\beta)$, then here we get

$$|R_\Gamma| \leq \|\omega\|_{L_1(a,b)} \|f'\|_{L_\infty(\alpha,\beta)} (\beta-\alpha). \tag{4.10}$$

Let now $p,q>1 : \frac{1}{p}+\frac{1}{q}=1$ and assume $f' \in L_p(\alpha,\beta)$, then here we obtain

$$|R_\Gamma| \leq \|\omega\|_{L_1(a,b)} \|f'\|_{L_p(\alpha,\beta)} (\beta-\alpha)^{\frac{1}{q}}. \tag{4.11}$$

Notice also here that

$$K := \Gamma_f(\mu_1,\mu_2) - \int_{\alpha}^{\beta} f(y)\omega(y)\,dy = R_\Gamma, \tag{4.12}$$

($l=1$ case).

So the estimates (4.9), (4.10) and (4.11) are also estimates for K.

References

[1] Anastassiou, G.A., *Quantitative Approximations*, Chapman & Hall / CRC, Boca Raton, New York, 2000.

[2] Anastassiou, G.A., *Probabilistic Inequalities*, World Scientific, Singapore, New Jersey, 2010.

[3] Anastassiou, G.A., *Advanced Inequalities*, World Scientific, Singapore, New Jersey, 2011.

[4] Barnett, N.S., Cerone, P., Dragomir, S.S., Sofo, A., *Approximating Csiszar's f-divergence by the use of Taylor's formula with integral remainder*, (paper #10, pp. 16), Inequalities for Csiszar's f-Divergence in Information Theory, S.S. Dragomir (ed.), Victoria University, Melbourne, Australia, 2000. On line: http://rgmia.vu.edu.au

[5] Burenkov, V., *Sobolev spaces and domains*, B.G. Teubner, Stuttgart, Leipzig, 1998.

[6] Chebyshev, P.L., *Sur les expressions approximatives des integrales definies par les autres prises entre les mêmes limites*, Proc. Math. Soc. Charkov, **2**(1882), 93-98.

[7] Csiszar, I., *Eine Informationstheoretische Ungleichung und ihre Anwendung auf den Beweis der Ergodizität von Markoffschen Ketten*, Magyar Tud. Akad. Mat. Kutato Int. Közl., **8**(1963), 85-108.

[8] Csiszar, I., *Information-type measures of difference of probability distributions and indirect observations*, Studia Math. Hungarica, **2**(1967), 299-318.

[9] Dragomir, S.S., (ed.), *Inequalities for Csiszar f-Divergence in Information Theory*, Victoria University, Melbourne, Australia, 2000.
On-line: http://rgmia.vu.edu.au

[10] Grüss, G., *Über das Maximum des absoluten Betrages von* $\left[\left(\frac{1}{b-a}\right)\int_a^b f(x)\,g(x)\,dx-\left(\frac{1}{(b-a)^2}\int_a^b f(x)\,dx\int_a^b g(x)\,dx\right)\right]$, Math. Z., **39**(1935), 215-226.

[11] Kullback, S., *Information Theory and Statistics*, Wiley, New York, 1959.

[12] Kullback, S., Leibler, R., *On information and sufficiency*, Ann. Math. Statist., **22**(1951), 79-86.

[13] Ostrowski, A., *Uber die Absolutabweichung einer differentiabaren Funcktion von ihrem Integralmittelwert*, Comment. Math. Helv., **10**(1938), 226-227.

[14] Rényi, A., *On measures of entropy and information*, Proceedings of the 4th Berkeley Symposium on Mathematical Statistics and Probability, I, Berkeley, CA, 1960, 547-561.

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