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Bezier blending surfaces on astroid

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Abstract. In this article we construct Bezier surfaces on a domain bounded by an astroid using the univariate polynomial Bernstein operator. We study the monotonicity and we give conditions of convexity in some directions for the constructed surfaces. Also, we give conditions for obtaining hyperbolic, parabolic and elliptic surfaces.

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1. Introduction

The surfaces of blending type have been introduced by Coons in [5]. They have the property of matching some given curves. In some previous papers [1, 2, 3, 4, 9] there were constructed the blending surfaces with the support on the border of a rectangular, triangular or circular domain and having a fixed height in a point from the domain. In this paper we obtained the Bezier surfaces which stay on an astroid. We construct the surfaces using the univariate Bernstein operator. The obtained surfaces are defined on a domain bounded by an astroid, they stay on the border of the domain and have a fixed height in the center of the domain. Instead of the control points from the case of classical Bezier surfaces we use a curves network (one of the curves from network is reduced to a point). We study the monotonicity and the convexity using the first and the second directional derivatives respectively (like in [7, 8, 10]).

These surfaces can be used in civil engineering (as roofs for buildings) or in Computer Aided Geometric Design (CAGD). For roof surfaces the maximal stress acts in the parabolical points (see [3, 4, 9, 11]). It is preferable to avoid having the parabolic points among the points of other type (hyperbolic, elliptic). We give conditions for obtaining the surfaces of hyperbolic, parabolic or elliptic type.

2. Construction of the surfaces

Let $n \in \mathbb{N}$, $n \geq 2$ and $h_i, h \in \mathbb{R}$, $i = 1, ..., n-1$ be such that

$$0 = h_n < h_{n-1} < ... < h_1 < h_0 = h \tag{2.1}$$

and let $f : [0,1] \to \mathbb{R}$ be a function with the properties

$$\begin{array}{l} f(0) = h, \\ f(\frac{j}{n}) = h_j, \ j = 1, ..., n-1, \\ f(1) = 0. \end{array} \tag{2.2}$$

Let B_n the univariate Bernstein operator on the interval $[0,1]$,

$$(B_n f)(y) = \sum_{j=0}^{n} b_{jn}(y) f(\frac{j}{n}),$$

where the functions b_{jn} are given by formula

$$b_{jn}(y) = \binom{n}{j} y^j (1-y)^{n-j}, \ for \ j = 0, ..., n.$$

Taking into account (2.2), we obtain

$$(B_n f)(y) = b_{0n}(y) h + \sum_{j=1}^{n-1} b_{jn}(y) h_j. \tag{2.3}$$

The function in (2.3) has the properties

$$(B_n f)(0) = h, \ (B_n f)(1) = 0.$$

Let $D = \{(X,Y) \in \mathbb{R}^2 : \ X^{\frac{2}{3}} + Y^{\frac{2}{3}} \leq 1\}$ be a domain in the XOY plane (the domain bounded by the astroid $X^{\frac{2}{3}} + Y^{\frac{2}{3}} = 1$).

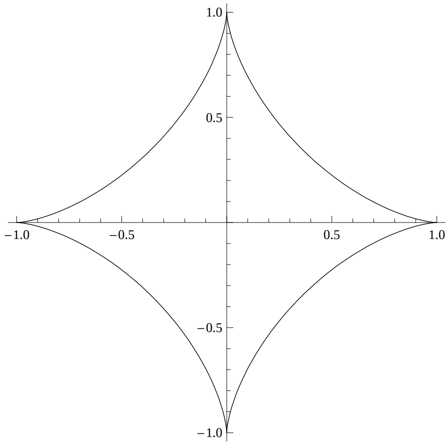

FIGURE 1. The astroid

If we make the substitution $y = \left(X^{\frac{2}{3}} + Y^{\frac{2}{3}}\right)^{\alpha}$, $\alpha > 0$ in (2.3), we obtain the surfaces

$$F(X,Y) := (B_n f)\left(\left(X^{\frac{2}{3}} + Y^{\frac{2}{3}}\right)^{\alpha}\right) = \tag{2.4}$$

$$= b_{0n}\left(\left(X^{\frac{2}{3}} + Y^{\frac{2}{3}}\right)^{\alpha}\right) h + \sum_{j=1}^{n-1} b_{jn}\left(\left(X^{\frac{2}{3}} + Y^{\frac{2}{3}}\right)^{\alpha}\right) h_j,\ (X,Y) \in D.$$

The surfaces (2.4) have the properties

$$F|_{\partial D} = 0,$$

$$F(0,0) = h.$$

It follows that the surfaces F match the astroid $X^{\frac{2}{3}} + Y^{\frac{2}{3}} = 1$, $Z = 0$ (the surfaces stay on the border of domain D) and the height of the surfaces in the point $(0,0)$ is h.

We can give a parametrical representation for these surfaces

$$\begin{cases} X = u\cos^3 v, \\ Y = u\sin^3 v, \\ Z = b_{0n}(u^{\frac{2\alpha}{3}})h + \sum\limits_{j=1}^{n-1} b_{jn}(u^{\frac{2\alpha}{3}})h_j \end{cases} \quad u \in [0,1], \quad v \in [0,2\pi].$$

Next sections, we study the monotonicity and the convexity using the directional derivative of the first and the second order respectively. The domain D is not convex but it is a star convex set with respect to the point $(0,0)$. We will use directions that pass by the point $(0,0)$. Also, some results about the type of the points of the surfaces F on the domain $D \setminus D_1$, where $D_1 = \{(x,0), x \in [-1,1]\} \cup \{(0,y), y \in [-1,1]\}$, are given.

3. Monotonicity of the surfaces

We denote

$$\Delta_1 h_j = h_{j+1} - h_j,\ j = 0, ..., n-1.$$

We recall that a bivariate function G is increasing (decreasing) in the direction $d = (d_1, d_2) \in \mathbb{R}^2$ if

$$G(X + \lambda d_1, Y + \lambda d_2) \geq (\leq) G(X,Y), \tag{3.1}$$

for every $(X,Y) \in A \subset \mathbb{R}^2$ and every $\lambda > 0$ such that $(X + \lambda d_1, Y + \lambda d_2) \in A$. The first order directional derivative in the direction $d = (d_1, d_2)$ of a C^1 function G is

$$D_d G = d_1 G_X + d_2 G_Y.$$

The conditions (3.1) are equivalent to

$$D_d G \geq 0 (\leq 0),\ \text{on } A.$$

Next theorem gives conditions for the monotonicity in some directions of the surfaces F.

Theorem 3.1. *If* $\left(d_1X^{-\frac{1}{3}}+d_2Y^{-\frac{1}{3}}\right)<0$ (>0) *on* $D\setminus D_1$*, then* F *is increasing (decreasing) in the direction* (d_1,d_2) *on* $D\setminus D_1$*, where* (d_1,d_2) *is a direction that pass by the point* $(0,0)$.

Proof. Let $(X,Y)\in D\setminus D_1$ and (d_1,d_2) a direction that pass by the point $(0,0)$. Using some results from [6], it follows that the first partial derivatives of the function F are given by

$$F_X(X,Y)=\frac{2n\alpha X^{-\frac{1}{3}}}{3}\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha-1}\sum_{j=0}^{n-1}b_{j,n-1}\left(\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha}\right)\Delta_1h_j,$$

$$F_Y(X,Y)=\frac{2n\alpha Y^{-\frac{1}{3}}}{3}\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha-1}\sum_{j=0}^{n-1}b_{j,n-1}\left(\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha}\right)\Delta_1h_j.$$

If we compute the first order directional derivative of the function F in the direction $d=(d_1,d_2)$, we obtain

$$D_dF(X,Y)=$$

$$=\frac{2n\alpha}{3}\left(d_1X^{-\frac{1}{3}}+d_2Y^{-\frac{1}{3}}\right)\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha-1}\sum_{j=0}^{n-1}b_{j,n-1}\left(\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha}\right)D_1h_j.$$

Taking into account (2.1), the condition $d_1X^{-\frac{1}{3}}+d_2Y^{-\frac{1}{3}}<0$ (>0) on $D\setminus D_1$ implies $D_dF>0$ (<0) on $D\setminus D_1$, and the theorem is proved. □

4. Convexity and type of the surfaces

We denote

$$\Delta_2h_{j+1}=h_{j+2}-2h_{j+1}+h_j,\ j=0,...,n-2.$$

We recall that a bivariate C^2 function G is convex (concave) in the the direction $d=(d_1,d_2)\in\mathbb{R}^2$ if and only if $D_d^2G\geq 0$ (≤ 0) on $A\subset\mathbb{R}^2$, where D_d^2G is the second order directional derivative in direction $d=(d_1,d_2)$ of the function G,

$$D_d^2G=d_1^2G_{XX}+2d_1d_2G_{XY}+d_2^2G_{YY}.$$

We give sufficient conditions for convexity in some directions of the surfaces F.

Theorem 4.1. *If* $\alpha\in(0,1]$ *and* $\Delta_2h_j\geq 0$, $j=0,...,n-2$*, then the function* F *is convex in the direction* (d_1,d_2) *on* $D\setminus D_1$*, where* (d_1,d_2) *is a direction that pass by the point* $(0,0)$.

Proof. Let $(X,Y)\in D\setminus D_1$ and (d_1,d_2) a direction that pass by the point $(0,0)$. Taking into account results from [6], the second order derivatives of the function F are

$$F_{XX}(X,Y)= \tag{4.1}$$

$$=\frac{4X^{-\frac{2}{3}}n(n-1)\alpha^2}{9}\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{2\alpha-2}\sum_{j=0}^{n-2}b_{j,n-2}\left(\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha}\right)\Delta_2h_j+$$

$$+\left(\frac{4X^{-\frac{2}{3}}\alpha(\alpha-1)}{9}\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha-2}-\frac{2X^{-\frac{4}{3}}\alpha}{9}\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha-1}\right)\times$$

$$\times n\sum_{j=0}^{n-1}b_{j,n-1}\left(\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha}\right)\Delta_1 h_j,$$

$$F_{XY}(X,Y)=\frac{4X^{-\frac{1}{3}}Y^{-\frac{1}{3}}\alpha^2}{9}\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{2\alpha-2}\times \tag{4.2}$$

$$\times n(n-1)\sum_{j=0}^{n-2}b_{j,n-2}\left(\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha}\right)\Delta_2 h_j+$$

$$+\frac{4X^{-\frac{1}{3}}Y^{-\frac{1}{3}}n\alpha(\alpha-1)}{9}\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha-2}\sum_{j=0}^{n-1}b_{j,n-1}\left(\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha}\right)\Delta_1 h_j,$$

$$F_{YY}(X,Y)= \tag{4.3}$$

$$=\frac{4X^{-\frac{2}{3}}n(n-1)\alpha^2}{9}\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{2\alpha-2}\sum_{j=0}^{n-2}b_{j,n-2}\left(\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha}\right)\Delta_2 h_j+$$

$$+\left(\frac{4Y^{-\frac{2}{3}}\alpha(\alpha-1)}{9}\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha-2}-\frac{2Y^{-\frac{4}{3}}\alpha}{9}\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha-1}\right)\times$$

$$\times n\sum_{j=0}^{n-1}b_{j,n-1}\left(\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha}\right)\Delta_1 h_j.$$

If we compute the second order directional derivative in the direction $d=(d_1,d_2)$ of the function F, we obtain

$$D_d^2F(X,Y)=\frac{4\alpha^2}{9}\left(d_1X^{-\frac{1}{3}}+d_2Y^{-\frac{1}{3}}\right)^2\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{2\alpha-2}\times$$

$$\times n(n-1)\sum_{j=0}^{n-2}b_{j,n-2}\left(\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha}\right)\Delta_2 h_j+$$

$$+\frac{4\alpha(\alpha-1)}{9}\left(d_1X^{-\frac{1}{3}}+d_2Y^{-\frac{1}{3}}\right)^2\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha-2}\times$$

$$\times n\sum_{j=0}^{n-1}b_{j,n-1}\left(\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha}\right)\Delta_1 h_j-$$

$$-\frac{2n\alpha}{9}\left(d_1^2X^{-\frac{4}{3}}+d_2^2Y^{-\frac{4}{3}}\right)\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha-1}\sum_{j=0}^{n-1}b_{j,n-1}\left(\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha}\right)\Delta_1 h_j.$$

From $\alpha\in(0,1]$, $\Delta_2 h_j\geq 0$, $j=0,...,n-2$ and the condition (2.1), it follows $D_d^2F\geq 0$ on $D\setminus D_1$. Thus, the conclusion of the theorem holds. □

We recall that a point of a surface $Z = G(X, Y)$, $(X, Y) \in A \subset R^2$ is parabolic point if $PG(X, Y) = 0$, where

$$PG(X,Y) = G_{XX}(X,Y)G_{YY}(X,Y) - (G_{XY}(X,Y))^2. \tag{4.4}$$

If we have $PG(X, Y) < 0$ (> 0) the point (X, Y) is called hyperbolic point (elliptic point). The surface G is called of parabolic (hyperbolic, elliptic) type if all the points of the surface are parabolic (hyperbolic, elliptic).

The following theorem gives conditions for obtaining the surfaces F of different types on $D \setminus D_1$.

Theorem 4.2. *We have:*

1. *If* $\alpha \in \left(0, \frac{3}{2}\right)$ *and* $\Delta_2 h_j \geq 0$, $j = 0, ..., n-2$, *then the surfaces* F *are of elliptic type on* $D \setminus D_1$.
2. *If* $\alpha = \frac{3}{2}$ *and* $\Delta_2 h_j \geq 0$, $j = 0, ..., n-2$ *and there exists* $j_0 \in \{0, ..., n-2\}$ *such that* $\Delta_2 h_{j_0} \neq 0$, *then the surfaces* F *are of elliptic type on* $D \setminus D_1$.
3. *If* $\alpha = \frac{3}{2}$ *and* $\Delta_2 h_j = 0$, $j = 0, ..., n-2$, *then the surfaces* F *are of parabolic type on* $D \setminus D_1$.
4. *If* $\alpha = \frac{3}{2}$ *and* $\Delta_2 h_j \leq 0$, $j = 0, ..., n-2$ *and there exists* $j_0 \in \{0, ..., n-2\}$ *such that* $\Delta_2 h_{j_0} \neq 0$, *then the surfaces* F *are of hyperbolic type on* $D \setminus D_1$.
5. *If* $\alpha \in \left(\frac{3}{2}, \infty\right)$ *and* $\Delta_2 h_j \leq 0$, $j = 0, ..., n-2$, *then the surfaces* F *are of hyperbolic type on* $D \setminus D_1$.

Proof. Let $(X, Y) \in D \setminus D_1$. From (4.4) and (4.1)-(4.3) we obtain

$$PF(X,Y) =$$

$$= -\frac{4\alpha^2(2\alpha-3)\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{2\alpha-2}}{81X^{\frac{4}{3}}Y^{\frac{4}{3}}}\left(n\sum_{j=0}^{n-1} b_{j,n-1}\left(\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha}\right)\Delta_1 h_j\right)^2 +$$

$$-\frac{8\alpha^3\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{3\alpha-2}}{81X^{\frac{4}{3}}Y^{\frac{4}{3}}}\left(n\sum_{j=0}^{n-1} b_{j,n-1}\left(\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha}\right)\Delta_1 h_j\right) \times$$

$$\times\left(n(n-1)\sum_{j=0}^{n-2} b_{j,n-2}\left(\left(X^{\frac{2}{3}}+Y^{\frac{2}{3}}\right)^{\alpha}\right)\Delta_2 h_j\right),$$

The conclusions of the theorem follow using the condition (2.1). □

Remark 4.3. The conditions from Theorem 4.1 and Theorem 4.2 depend only the parameters h_j (i.e. they depend only on the control network).

We have plotted the surface F for $n = 3$.

In Figure 2.a we take $h = h_0 = 3$, $h_1 = 1.5$, $h_2 = 0.5$, $h_3 = 0$ and $\alpha = 1$; we have $\Delta_2 h_j > 0$, $j = 0, 1$. The surface is of elliptic type.

In Figure 2.b we take $h = h_0 = 3$, $h_1 = 1.5$, $h_2 = 0.5$, $h_3 = 0$ and $\alpha = 1.5$; we have $\Delta_2 h_j > 0$, $j = 0, 1$. The surface is of elliptic type.

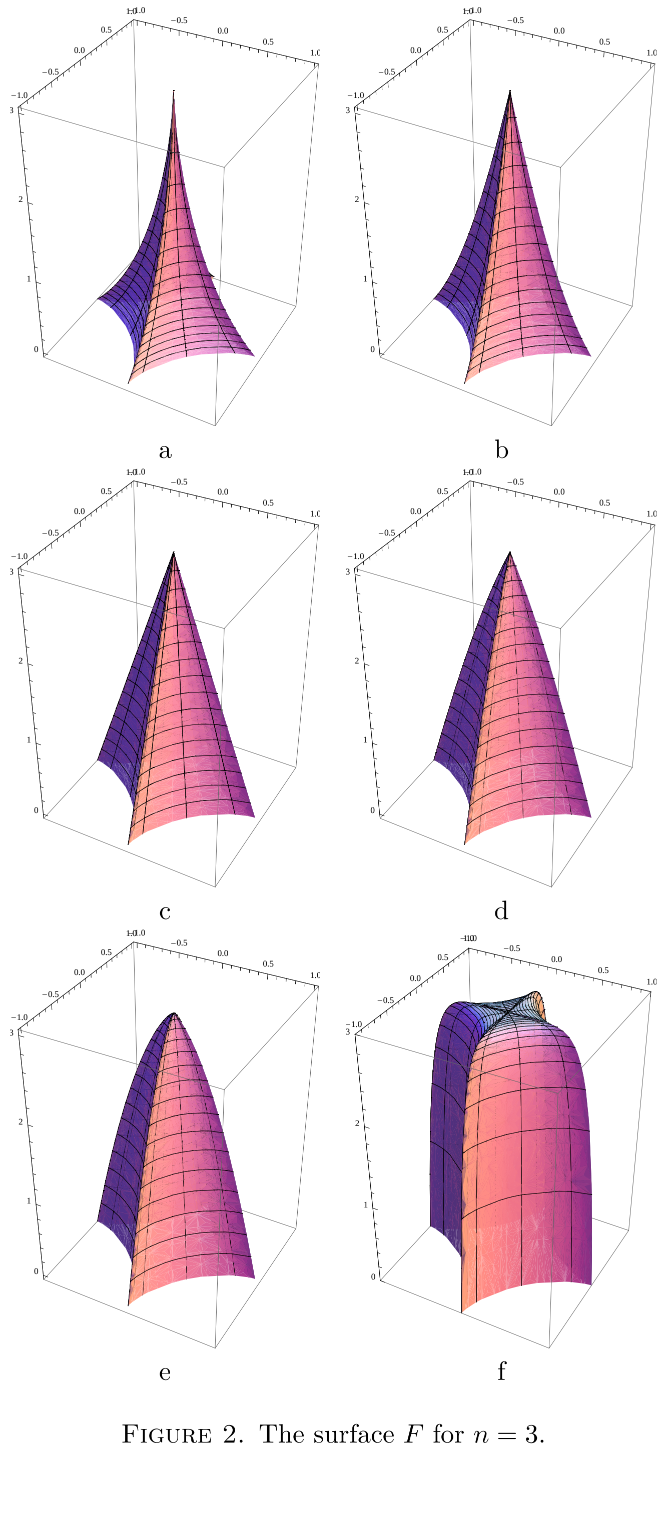

FIGURE 2. The surface F for $n = 3$.

In Figure 2.c we take $h = h_0 = 3$, $h_1 = 2$, $h_2 = 1$, $h_3 = 0$ and $\alpha = 1.5$; we have $\Delta_2 h_j = 0$, $j = 0, 1$. The surface is of parabolic type.

In Figure 2.d we take $h = h_0 = 3$, $h_1 = 2.3$, $h_2 = 1.2$, $h_3 = 0$ and $\alpha = 1.5$; we have $\Delta_2 h_j < 0$, $j = 0, 1$. The surface is of hyperbolic type.

In Figure 2.e we take $h = h_0 = 3$, $h_1 = 2.3$, $h_2 = 1.2$, $h_3 = 0$ and $\alpha = 2$; we have $\Delta_2 h_j < 0$, $j = 0, 1$. The surface is of hyperbolic type.

In Figure 2.f we take $h = h_0 = 3$, $h_1 = 2.3$, $h_2 = 1.2$, $h_3 = 0$ and $\alpha = 10$; we have $\Delta_2 h_j < 0$, $j = 0, 1$. The surface is of hyperbolic type.

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