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Solution of a nonlinear system of second kind Lagrange's equations by fixed-point method

Ljubomir Georgiev and Konstantin Kostov

Abstract. The effect of forces acting upon a ferromagnetic rotational ellipsoid located in a homogeneous rotating magnetic field is considered. Lagrange's equations of the second kind connecting the motion parameters of a particle with torques acting upon it are composed. A non-homogeneous nonlinear autonomous system of second-order differential equations is obtained. That system is not solvable by quadrature. A solution by fixed-point method is proposed in this paper.

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1. Introduction

The principle of rotating magnetic field is applied in designing machines that intensify some technological processes like milling, emulsifying, mixing, etc. Ferromagnetic working particles are placed in the so-called active volume of the machine where they are driven by the field and exert a force-applying effect upon the treated material. It is characteristic for their motion that due to frequent collisions these particles are always in transition mode, i. e. the angle between the field vector and the longitudinal axis of the working particle changes. It can be assumed that after each collision there emerges a motion of new initial conditions. Calculating precisely the technological effect obtained requires good knowledge of the law of motion at arbitrary initial conditions. Our goal is to establish the existence of unique solution of the initial value problem for the corresponding system of two nonlinear second-order differential equations. We take advantage of the fixed point method to do this. At the end of this paper we present a sequence of successive analytical approximations of the solution, which belongs to a suitable subset of the space $C([0,\infty))$.

2. Physical model

A ferromagnetic rotational ellipsoid, formed by the rotation of an ellipse of axes $2a$ and $2b$ around its long axis of length $2a$ and located in a homogeneous rotating magnetic field of flux-density modulus $\vec{B}_0$, is considered. In this case the ellipsoid is homogeneously magnetized, which makes possible the analytical determination of its electromagnetic torque.

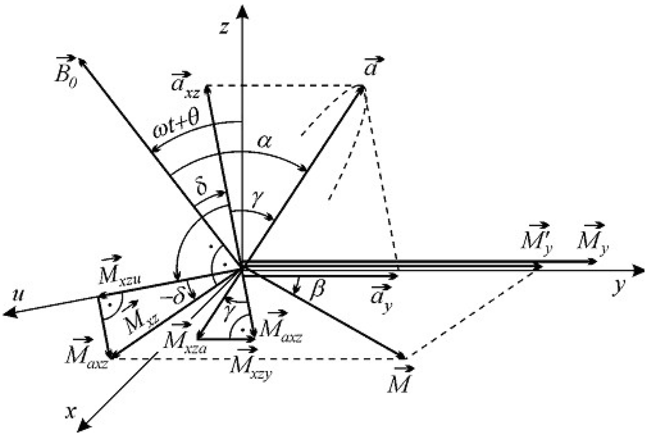

Fig. 1

Fig. 1 shows a layout of the particle, magnetic flux density and respective torque $\vec{M}$. The denotations ([3]) are as follows: $\vec{a}$ is a vector applied to the center of the ellipsoid and directed along its axis. It shows the spatial position of the particle considered ($\left|\vec{a}\right|$ is equal to the long (rotational) half-axis a of the ellipsoid). α is the smaller angle between vectors $\vec{B}_0$ and $\vec{a}$, ω is the angular velocity of the rotating magnetic field, $\omega t + \theta$ is the angle between the axis z and field vector $\vec{B}_0$, γ is the angle formed between the plane xOz and vector $\vec{a}$, $\vec{a}_{xz}$ is a vector component of $\vec{a}$ (its projection onto the plane xOz), δ is the angle between $\vec{B}_0$ and $\vec{a}_{xz}$. Denoted $\alpha, \gamma, \omega t + \theta$ and δ are oriented angles between vectors or between vectors and axes.

The synchronous reactive torque is determined in [3]:

$$M = -K{B_0}^2 \sin 2\alpha = -M_0 \sin 2\alpha \tag{2.1}$$

Vector $\vec{M}$ is perpendicular to the plane defined by $2\vec{a}$ and $\vec{B}_0$ and it is of the same direction as that of $\vec{a'} \times \vec{B}_0$. Here, $\vec{a'}$ is the vector along the ellipsoid's long axis, which makes with $\vec{B}_0$ an angle smaller than $\dfrac{\pi}{2}$.

Angles γ and δ are selected as generalized coordinates, defining uniquely the spatial position of the ellipsoid. An additional axis u lying in the plane

xOz and being at a positive angle $\frac{\pi}{2}$ with respect to $\vec{a}_{xz}$ is introduced. The synchronous torque $\vec{M}$ is decomposed along the axes y, u and $\vec{a}_{xz}$, ([5]): $\vec{M}=\vec{M}_y + \vec{M}_u + \vec{M}_a$, where $M_y = -M_0 \sin 2\delta, M_u = M_{xzu} = -M_0 \cos^2 \delta \sin 2\gamma$, and $M_a = M_{xza} = -M_0 \sin 2\delta \sin \gamma$ are scalar components of $\vec{M}$ along the respective axes.

The kinetic energy of the ellipsoid has the form:

$$T = \frac{1}{2} J \left(\omega + \dot{\delta}\right)^2 + \frac{1}{2} J \dot{\gamma}^2 + \frac{1}{2} J_a \dot{\varphi}^2, \tag{2.2}$$

where J_a is the inertia torque of the rotational ellipsoid with respect to the axis $2a$, J is the inertia torque of the rotational ellipsoid with respect to the axis $2b$ that goes through its center of gravity and is perpendicular to $\vec{a}$, $\dot{\varphi}$ is the angular velocity of the ellipsoid in its rotation around the axis $2a$. Therefore, φ is the third generalized coordinate. Let us read φ from the line, which is located further away from the plane xOz and in which the plane formed by $\vec{a}$ and $\vec{a}_{xz}$ crosses the ellipsoid at the initial time point $t = 0$. We assume the positive direction should be determined by the right-hand screw rule with axis $\vec{a}$.

A torque defined by currents acts on the ellipsoid as well. Its average value ([5]) is

$$\vec{M}_e = -\vec{j}\, M_\gamma \cos^3 \gamma \dot{\delta}, \tag{2.3}$$

where $M_\gamma = \dfrac{\pi \mu_r^2 \sigma B_0^2 d^4 l}{256(1 + \mu_r N_l)^2 (1 + k^4 d^4/256)}$ is the current torque for $\gamma = 0$ at $\dot{\delta} = 1\ rad/s$. For a cylinder of determined size and given magnetic permeability $M_\gamma = const.$

The torque M_e acts along the axis y and exhibits itself only when there is a difference between the angular velocities of the field and particle along the axis y. The negative sign indicates that this torque opposes the change in the angle δ. Besides the driving torques considered so far, there is also a hysteresis torque that will be neglected for we consider a particle made of soft-magnetic material and because of considerations related to its shape ([4]). There exist resisting torques as well, resulting from the friction forces. Due to the small size of the ellipsoid the linear velocities are of low values, and we can consequently assume that frictional torques are proportional to the first power of the respective angular velocity. Correspondingly, the proportionality factors for motions along δ and along γ are equal to each other for in both cases the rotation is realized around the axis $2b$ of the ellipsoid. Having in mind that torques M_u, M_y, and M_a act along the direction of generalized coordinates γ, δ, and φ, respectively, we compose the following system of

Lagrange's differential equations of the second order ([6]):

$$\left|\begin{array}{l} \dfrac{d}{dt}\dfrac{\partial T}{\partial \dot{\gamma}} - \dfrac{\partial T}{\partial \gamma} = M_u - k_1\dot{\gamma} \\[2ex] \dfrac{d}{dt}\dfrac{\partial T}{\partial \dot{\delta}} - \dfrac{\partial T}{\partial \delta} = M_y - M_\gamma \dot{\delta}\cos^3\gamma - k_1(\omega + \dot{\delta}) \\[2ex] \dfrac{d}{dt}\dfrac{\partial T}{\partial \dot{\varphi}} - \dfrac{\partial T}{\partial \varphi} = M_a - k_2\dot{\varphi}. \end{array}\right.$$

Here, $k_1\dot{\gamma}, k_1(\omega + \dot{\delta})$ and $k_2\dot{\varphi}$ are frictional torques in tracing out the respective angles, and k_1 and k_2 are proportionality factors. The negative signs before the frictional torques indicate that they are inversely proportional to respective angular velocities. We replace with the torques derived above and obtain

$$\left|\begin{array}{l} J\ddot{\gamma} = -M_0\cos^2\delta\sin 2\gamma - k_1\dot{\gamma} \\[2ex] J\ddot{\delta} = -M_0\sin 2\delta - M_\gamma\dot{\delta}\cos^3\gamma - k_1\dot{\delta} - k_1\omega \\[2ex] J_a\ddot{\varphi} = -M_0\sin 2\delta\sin\gamma - k_2\dot{\varphi}. \end{array}\right. \tag{2.4}$$

The Lagrange's equations (2.4) describe the motion of a rotational ellipsoid placed in general position in a homogeneous magnetic field, rotating with constant angular velocity, for every time instant. This is a non-homogeneous nonlinear autonomous system of differential equations of second order. The system is unsolvable by quadrature. We notice that the first two equations are independent of the third one. In addition, the latter does not contribute to solving the formulated problem as rotation around the axis $\vec{a}$ does not exert any technological effect.

3. Mathematical model

Let us consider the system composed by the first two equations, assuming that the current torque is much smaller than the synchronous one, which means we can neglect it. We obtain:

$$\left|\begin{array}{l} J\ddot{\gamma} = -M_0\cos^2\delta\sin 2\gamma - k_1\dot{\gamma} \\[2ex] J\ddot{\delta} = -M_0\sin 2\delta - k_1\dot{\delta} - k_1\omega. \end{array}\right. \tag{3.1}$$

The system (3.1) is unsolvable by quadrature, too. We seek a solution by means of contraction mapping principle ([1], [2], [7]) for it.

We denote by $M = \dfrac{M_0}{J} > 0, k = \dfrac{k_1}{J} > 0$ and obtain the system

$$\begin{cases} \ddot{\gamma} = -M\cos^2\delta\sin 2\gamma - k\dot{\gamma} \\ \ddot{\delta} = -M\sin 2\delta - k\dot{\delta} - k\omega. \end{cases} \tag{3.2}$$

where $\gamma = \gamma(t), \delta = \delta(t), t \in [0, \infty)$, with the corresponding initial conditions at $t = 0$.

Consider the second equation of the system (3.2).

If δ is a solution of

$$\ddot{\delta} = -M \sin 2\delta - k\dot{\delta} - k\omega \tag{3.3}$$

then $(\ddot{\delta}(s) + k\dot{\delta}(s))e^{ks} = -(M \sin 2\delta(s) + k\omega)e^{ks}$. After integrating along s from 0 to τ, we obtain: $\dot{\delta}(\tau) = (\omega + \dot{\delta}(0))e^{-k\tau} - \omega - Me^{-k\tau} \int_0^\tau e^{ks} \sin 2\delta(s)ds$ and integrating once again along τ from 0 to t, we obtain:

$$\delta(t) = \delta(0) + \frac{\omega + \dot{\delta}(0)}{k}(1 - e^{-kt}) - \omega t - M \int_0^t \int_0^\tau e^{-k(\tau - s)} \sin 2\delta(s) ds d\tau =$$
$$= \delta(0) + \frac{\omega + \dot{\delta}(0)}{k}(1 - e^{-kt}) - \omega t - M \int_0^t \left(\int_s^t e^{-k(\tau - s)} d\tau \right) \sin 2\delta(s)ds =$$
$$= \delta(0) + \frac{\omega + \dot{\delta}(0)}{k}(1 - e^{-kt}) - \omega t - \frac{M}{k} \int_0^t \left(1 - e^{-k(t-s)}\right) \sin 2\delta(s)ds,$$

which means that

$$\delta(t) = G(\delta)(t), \ \forall t \geq 0, \tag{3.4}$$

where the operator G is defined on a suitable subset $\mathbf{B}$ of the space of the functions continuous in $[0, \infty)$:

$$G(f)(t) = \delta(0) + \frac{\omega + \dot{\delta}(0)}{k}(1 - e^{-kt}) - \omega t - \frac{M}{k} \int_0^t (1 - e^{-k(t-s)}) \sin 2f(s)ds, t \geq 0. \tag{3.5}$$

If δ is a continuous solution of (3.4) then

$$\dot{\delta} = (\omega + \dot{\delta}(0))e^{-kt} - \omega - \frac{M}{k} \sin 2\delta(t) + \frac{M}{k} \frac{d}{dt} \left(e^{-kt} \int_0^t e^{ks} \sin 2\delta(s)ds \right) =$$
$$= (\omega + \dot{\delta}(0))e^{-kt} - \omega - M \int_0^t e^{-k(t-s)} \sin 2\delta(s)ds;$$
$$\ddot{\delta} = -k(\omega + \dot{\delta}(0))e^{-kt} + Mk \int_0^t e^{-k(t-s)} \sin 2\delta(s)ds - M \sin 2\delta(t) =$$
$$= -M \sin 2\delta(t) - k\dot{\delta} - k\omega.$$

In other words, δ is a twice-differentiable function satisfying (3.3).

By means of analogous transformations on the first of equations from (3.2) we reduce the system (3.2) to the following one:

$$\begin{cases} \gamma(t) = F_\delta(\gamma)(t), \ \forall t \geq 0 \\ \delta(t) = G(\delta)(t), \quad \forall t \geq 0 \end{cases}, \tag{3.6}$$

where G is defined as (3.5) and for any fixed function $f \in \mathbf{B}$ the operator F is defined on the same set $\mathbf{B}$ as follows: $F(g) = F_f(g)$, and for any $t \geq 0$:

$$F_f(g)(t) = \gamma(0) + \frac{\dot{\gamma}(0)}{k}(1 - e^{-kt}) - \frac{M}{k} \int_0^t (1 - e^{-k(t-s)}) \cos^2 f(s) \sin 2g(s)ds. \tag{3.7}$$

Remark 3.1. *One can try to use various kinds of schemes to find numerical approximations of the solution of the system* (3.1)*. For example, one can seek the approximations with some methods such as Runge-Kutta methods (or Euler's method, or Newton's method) for the corresponding system:*

$$\begin{cases} \dot{x}(t) = F(x(t)), \quad 0 < t < T_0 \\ x(0) = x_0, \end{cases}$$

where $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t))^T$*,*
$F(x) = (F_1(x), F_2(x), F_3(x), F_4(x))^T$*;* $x_1 = \delta, x_2 = \gamma, x_3 = \dot{\delta}, x_4 = \dot{\gamma}$*;*
$F_1 = x_3, F_2 = x_4, F_3 = -M sin 2x_1 - kx_3 - k\omega, F_4 = -M cos^2 x_1 sin 2x_2 - kx_4$.

But does the last nonlinear system have a solution and whether, if the system has a solution, it is only one?

We look for global solution of the system (3.6) (resp. of the system (3.1)).

In what follows we give a proof (by means of fixed point method) that there exists a unique solution.

Define the set $\mathbf{B} : \mathbf{B} = \left\{h \in C([0,\infty)) : |h(t)| \le Ce^{\lambda t},\ \forall t \ge 0\right\}$ with constants $\lambda > m$, $m = \max\left\{\frac{3M}{k},\ \omega + \frac{M}{k}\right\} = \max\left\{\frac{3M_0}{k_1},\ \omega + \frac{M_0}{k_1}\right\}$,

and

$$C = |\delta(0)| + \frac{\left|\omega + \dot{\delta}(0)\right|}{k} + |\gamma(0)| + \frac{|\dot{\gamma}(0)|}{k} + \frac{1}{2}.$$

Norm in $\mathbf{B}$ is introduced as follows:

$$\|f\|_B = \sup\left\{e^{-\lambda t}|f(t)| :\ t \ge 0\right\},\ f \in \mathbf{B},$$

and with the corresponding metrics: $d(f,\overline{f}) = \left\|f - \overline{f}\right\|_B$ $\left(f, \overline{f} \in \mathbf{B}\right)$ the set $\mathbf{B}$ becomes a complete metric space.

Define the product space $E = \mathbf{B} \times \mathbf{B}$ with a norm:

$$\|(g,f)\| = \|g\|_B + \|f\|_B\,.$$

With the corresponding metrics $d\left((g,f),(\overline{g},\overline{f})\right) = \|g - \overline{g}\|_B + \left\|f - \overline{f}\right\|_B$, E becomes a Banach space.

Define on E the operator $T : T((g,f)) = (F_f(g),\ G(f))\,,\ (g,f) \in E.$

It has the following properties: $T((g,f)) \in E,\ \ \forall (g,f) \in E.$

Indeed, $G(f), F_f(g)$ are continuous functions in $[0,\infty)$;

$$e^{-\lambda t}|G(f)(t)| \le e^{-\lambda t}|a(t)| + \frac{M}{k}te^{-\lambda t} \le |\delta(0)| + \frac{\left|\omega + \dot{\delta}(0)\right|}{k} + \left(\omega + \frac{M}{k}\right)\cdot\frac{1}{\lambda e},$$

consequently $e^{-\lambda t}|G(f)(t)| \le |\delta(0)| + \dfrac{\left|\omega + \dot{\delta}(0)\right|}{k} + \dfrac{1}{e} < C$

$$\left(a(t) = \delta(0) + \frac{\omega + \dot{\delta}(0)}{k}(1 - e^{-kt}) - \omega t,\ t \ge 0\right);$$

$$e^{-\lambda t}|F_f(g)(t)| \le |\gamma(0)| + \frac{|\dot{\gamma}(0)|}{k} + \frac{M}{k}\cdot\frac{1}{\lambda e} \le |\gamma(0)| + \frac{|\dot{\gamma}(0)|}{k} + \frac{1}{e} < C.$$

Moreover,

$$d\left(T((g,f)),T((\overline{g},\overline{f}))\right)=\left\|F_f(g)-F_{\overline{f}}(\overline{g})\right\|_B+\left\|G(f)-G(\overline{f})\right\|_B=$$
$$=\frac{M}{k}\sup_{t\geq 0}\left\{e^{-\lambda t}\int_0^t(1-e^{-k(t-s)})[\cos^2 f(s)\sin 2g(s)-\cos^2\overline{f}(s)\sin 2\overline{g}(s)]ds\right\}+$$
$$+\frac{M}{k}\sup_{t\geq 0}\left\{e^{-\lambda t}\int_0^t(1-e^{-k(t-s)})[\sin(2f(s))-\sin(2\overline{f}(s))]ds\right\}\leq$$
$$\leq\frac{M}{k}\sup_{t\geq 0}\left\{e^{-\lambda t}\int_0^t\left|2\sin(g(s)-\overline{g}(s))\cos^2(f(s))\cos(g(s)+\overline{g}(s))\right|ds\right\}+$$
$$+\frac{M}{k}\sup_{t\geq 0}\left\{e^{-\lambda t}\int_0^t\left|\sin(f(s)-\overline{f}(s))\sin(2\overline{g}(s))\sin(f(s)+\overline{f}(s))\right|ds\right\}+$$
$$+\frac{M}{k}\sup_{t\geq 0}\left\{e^{-\lambda t}\int_0^t(1-e^{-k(t-s)})\left|2\sin(f(s)-\overline{f}(s))\cos(f(s)+\overline{f}(s))\right|ds\right\}\leq$$
$$\leq\frac{2M}{k}\sup_{t\geq 0}\left\{e^{-\lambda t}\int_0^t|g(s)-\overline{g}(s)|\,ds\right\}+\frac{3M}{k}\sup_{t\geq 0}\left\{e^{-\lambda t}\int_0^t\left|f(s)-\overline{f}(s)\right|ds\right\},$$
from where we obtain:
$$d\left(T((g,f)),T((\overline{g},\overline{f}))\right)\leq$$
$$\leq\frac{M}{k}\left(2\left\|g-\overline{g}\right\|_B+3\left\|f-\overline{f}\right\|_B\right)\sup_{t\geq 0}\left\{e^{-\lambda t}\int_0^t e^{\lambda s}ds\right\}\leq$$
$$\leq\frac{M}{k\lambda}\left(2\left\|g-\overline{g}\right\|_B+3\left\|f-\overline{f}\right\|_B\right).$$

Therefore $d\left(T((g,f)),T((\overline{g},\overline{f}))\right)\leq\beta d\left((g,f),(\overline{g},\overline{f})\right)$,

i. e. T is a contraction operator on E with Lipschitz constant $\beta=\dfrac{3M}{k\lambda}<1$.

In view of contraction mapping principle T has a unique fixed point on E, which allows us making the following conclusion:

4. Conclusion

The system (3.1) has a unique solution (γ,δ) the coordinate functions of which belong to the set $\mathbf{B}$. The solution can be obtained as the limit (in $\mathbf{B}\times\mathbf{B}$) of the sequence of successive approximations $\{(g_n,f_n)\}_{n=0}^{\infty}$:

$$g_0(t)=\gamma(0)+\frac{\dot{\gamma}(0)}{k}\left(1-e^{-kt}\right);$$
$$f_0(t)=\delta(0)+\frac{\omega+\dot{\delta}(0)}{k}\left(1-e^{-kt}\right),\ \forall t\geq 0\quad((g_0,f_0)\in E);$$
$$f_n(t)=f_0(t)-\omega t-\frac{M}{k}\int_0^t\left(1-e^{-k(t-s)}\right)\sin[2f_{n-1}(s)]ds, n{=}1{,}2{,}...$$
$$g_n(t)=g_0(t)-\frac{M}{k}\int_0^t\left(1-e^{-k(t-s)}\right)\cos^2[f_{n-1}(s)]\sin[2g_{n-1}(s)]ds, n{=}1{,}2{,}...$$

As we have already shown, the limit of $\{(g_n,f_n)\}_{n=0}^{\infty}$ in E is the unique fixed point of the operator T. In particular, the limit of $\{f_n\}_{n=0}^{\infty}$ in $\mathbf{B}$ is the function δ and therefore, the limit of $\{g_n\}_{n=0}^{\infty}$ in $\mathbf{B}$ is the function γ, that is the unique fixed point of T is the ordered pair (γ, δ), which is the unique solution (in E) of the system $(5'')$, and respectively – of the system (3.1).

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Ljubomir Georgiev
UMG "St. I. Rilski",
Department of Mathematics
1700 Sofia,
Bulgaria
e-mail: `lubo_62@mgu.bg`

Konstantin Kostov
UMG "St. I. Rilski",
Department of Electrotechiques
1700 Sofia,
Bulgaria
e-mail: `costovs@yahoo.com`