Stud. Univ. Babeş-Bolyai Math. 56(2011), No. 3, 95–104

Four-dimensional matrix transformation and rate of A-statistical convergence of Bögel-type continuous functions

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Abstract. The purpose of this paper is to investigate the effects of four-dimensional summability matrix methods on the A-statistical approximation of sequences of positive linear operators defined on the space of all real valued Bögel-type continuous functions on a compact subset of the real line. Furthermore, we study the rates of A-statistical convergence in our approximation.

Mathematics Subject Classification (2010): 41A25, 41A36.

Keywords: The Korovkin theorem, B-continuous functions, rates of A-statistical convergence for double sequences, regularity for double sequences.

1. Introduction

In order to improve the classical Korovkin theory, the space of Bögel-type continuous (or, simply, B-continuous) functions instead of the classical one has been used in [2, 3, 4, 5]. Recall that the concept of B-continuity was first introduced in 1934 by Bögel [6] (see also [7, 8]). On the other hand, this Korovkin theory has also been generalized via the concept of statistical convergence (see, for instance, [11, 12]). It is well-known that every convergent sequence (in the usual sense) is statistically convergent but its converse is not always true. Also, statistical convergent sequences do not need to be bounded. With these properties, the usage of the statistical convergence in the approximation theory leads us to more powerful results than the classical aspects.

We now recall some basic definitions and notations used in the paper.

A double sequence

$$x = \{x_{m,n}\}, \quad m, n \in \mathbb{N},$$

is convergent in Pringsheim's sense if, for every $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $|x_{m,n} - L| < \varepsilon$ whenever $m, n > N$. Then, L is called the Pringsheim limit of x and is denoted by $P - \lim x = L$ (see [19]). In this case, we say that $x = \{x_{m,n}\}$ is "P-convergent to L". Also, if there exists a positive number M such that $|x_{m,n}| \leq M$ for all $(m, n) \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$, then $x = \{x_{m,n}\}$ is said to be bounded. Note that in contrast to the case for single sequences, a convergent double sequence not to be bounded.

Now let

$$A = [a_{j,k,m,n}], \quad j, k, m, n \in \mathbb{N},$$

be a four-dimensional summability matrix. For a given double sequence $x = \{x_{m,n}\}$, the A-transform of x, denoted by $Ax := \{(Ax)_{j,k}\}$, is given by

$$(Ax)_{j,k} = \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} x_{m,n}, \quad j, k \in \mathbb{N},$$

provided the double series converges in Pringsheim's sense for every $(j, k) \in \mathbb{N}^2$. In summability theory, a two-dimensional matrix transformation is said to be regular if it maps every convergent sequence into a convergent sequence with the same limit. The well-known characterization for two-dimensional matrix transformations is known as Silverman-Toeplitz conditions (see, for instance, [16]). In 1926, Robison [20] presented a four-dimensional analog of the regularity by considering an additional assumption of boundedness. This assumption was made because a double P-convergent sequence is not necessarily bounded. The definition and the characterization of regularity for four-dimensional matrices is known as Robison-Hamilton conditions, or briefly, RH-regularity (see, [15, 20]).

Recall that a four dimensional matrix $A = [a_{j,k,m,n}]$ is said to be RH-regular, if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit. The Robison-Hamilton conditions state that a four dimensional matrix $A = [a_{j,k,m,n}]$ is RH-regular if and only if

(i) $P - \lim_{j,k} a_{j,k,m,n} = 0$ for each $(m, n) \in \mathbb{N}^2$,

(ii) $P - \lim_{j,k} \sum_{(m,n)\in\mathbb{N}^2} a_{j,k,m,n} = 1$,

(iii) $P - \lim_{j,k} \sum_{m\in\mathbb{N}} |a_{j,k,m,n}| = 0$ for each $n \in \mathbb{N}$,

(iv) $P - \lim_{j,k} \sum_{n\in\mathbb{N}} |a_{j,k,m,n}| = 0$ for each $m \in \mathbb{N}$,

(v) $\sum_{(m,n)\in\mathbb{N}^2} |a_{j,k,m,n}|$ is $P-$convergent for each $(j, k) \in \mathbb{N}^2$,

(vi) there exist finite positive integers A and B such that

$$\sum_{m,n>B} |a_{j,k,m,n}| < A$$

holds for every $(j, k) \in \mathbb{N}^2$.

Now let $A = [a_{j,k,m,n}]$ be a non-negative RH-regular summability matrix, and let $K \subset \mathbb{N}^2$. Then, a real double sequence $x = \{x_{m,n}\}$ is said to be

A-statistically convergent to a number L if, for every $\varepsilon > 0$,

$$P - \lim_{j,k} \sum_{(m,n)\in K(\varepsilon)} a_{j,k,m,n} = 0,$$

where

$$K(\varepsilon) := \{(m,n) \in \mathbb{N}^2 : |x_{m,n} - L| \geq \varepsilon\}.$$

In this case, we write $st_A^{(2)} - \lim x_{m,n} = L$. Observe that, a P-convergent double sequence is A-statistically convergent to the same value but the converse does not hold. For example, consider the double sequence $x = \{x_{m,n}\}$ given by

$$x_{m,n} = \begin{cases} mn, & \text{if } m \text{ and } n \text{ are squares,} \\ 1, & \text{otherwise.} \end{cases}$$

We should note that if we take $A = C(1,1)$, which is the double Cesáro matrix, then $C(1,1)$-statistical convergence coincides with the notion of statistical convergence for a double sequence, which was introduced in [17, 18]. Finally, if we replace the matrix A by the identity matrix for four-dimensional matrices, then A-statistical convergence reduces to the Pringsheim convergence.

In most investigations the approximated functions are assumed to be continuous. However, the considered approximation processes are often meaningful for a bigger class of functions, namely for so-called $B-$continuous functions introduced by Bögel [6, 7, 8].

The definition of $B-$continuous was introduced by Bögel as follows:

Let X and Y be compact subsets of the real numbers, and let $D = X \times Y$. Then, a function $f : D \to \mathbb{R}$ is called $B-$continuous at a point $(x,y) \in D$ if, for every $\varepsilon > 0$, there exists a positive number $\delta = \delta(\varepsilon)$ such that

$$|\Delta_{x,y}[f(u,v)]| < \varepsilon,$$

for any $(u,v) \in D$ with $|u - x| < \delta$ and $|v - y| < \delta$, where the symbol $\Delta_{x,y}[f(u,v)]$ denotes the mixed difference of f defined by

$$\Delta_{x,y}[f(u,v)] = f(u,v) - f(u,y) - f(x,v) + f(x,y).$$

By $C_b(D)$ we denote the space of all B-continuous functions on D. Recall that $C(D)$ and $B(D)$ denote the space of all continuous (in the usual sense) functions and the space of all bounded functions on D, respectively. Then, notice that $C(D) \subset C_b(D)$. Moreover, one can find an unbounded $B-$continuous function, which follows from the fact that, for any function of the type $f(u,v) = g(u) + h(v)$, we have $\Delta_{x,y}[f(u,v)] = 0$ for all $(x,y), (u,v) \in D$.

The usual supremum norm on the spaces $B(D)$ is given by

$$\|f\| := \sup_{(x,y)\in D} |f(x,y)| \quad \text{for } f \in B(D).$$

Throughout the paper, for fixed $(x,y) \in D$ and $f \in C_b(D)$, we use the function $F_{x,y}$ defined as follows:

$$F_{x,y}(u,v) = f(u,y) + f(x,v) - f(u,v) \qquad \text{for } (u,v) \in D. \tag{1.1}$$

Since

$$\Delta_{x,y}\left[F_{x,y}(u,v)\right] = -\Delta_{x,y}\left[f(u,v)\right]$$

holds for all (x,y), $(u,v) \in D$, the $B-$continuity of f implies the $B-$continuity of $F_{x,y}$ for every fixed $(x,y) \in D$. We also use the following test functions

$$e_0(u,v) = 1,\ e_1(u,v) = u,\ e_2(u,v) = v \text{ and } e_3(u,v) = u^2 + v^2.$$

With this terminology the authors [14] proved the following theorem, which corresponds to the A-statistical formulation of the problem above studied by Badea et. al. [3].

Theorem 1.1. [14] *Let $\{L_{m,n}\}$ be a double sequence of positive linear operators acting from $C_b(D)$ into $B(D)$, and let $A = [a_{j,k,m,n}]$ be a non-negative $RH-$regular summability matrix method. Assume that the following conditions hold:*

$$\delta_A^{(2)}\left\{(m,n) \in \mathbb{N}^2 : L_{m,n}(e_0; x,y) = e_0(x,y) \quad \text{for all } (x,y) \in D\right\} = 1$$

and

$$st_A^{(2)} - \lim_{m,n} \|L_{m,n}(e_i) - e_i\| = 0 \quad \text{for } i = 1,2,3.$$

Then, for all $f \in C_b(D)$, we have

$$st_A^{(2)} - \lim_{m,n} \|L_{m,n}(F_{x,y}) - f\| = 0,$$

where $F_{x,y}$ is given by (1.1)*.*

The aim of the present paper is to compute the rates of A-statistical approximation in Theorem 1.1 with the help of mixed modulus of smoothness.

2. Rate of A-statistical convergence

Various ways of defining rates of convergence in the A-statistical sense for two-dimensional summability matrix were introduced in [10]. In a similar way, for four-dimensional summability matrix, defining rates of convergence in the A-statistical sense introduced in [13]. In this section, we compute the corresponding rates of A-statistical convergence in Theorem 1.1 by means of four different ways.

Definition 2.1. [13] *Let $A = [a_{j,k,m,n}]$ be a non-negative RH-regular summability matrix and let $\{\alpha_{m,n}\}$ be a positive non-increasing double sequence. A double sequence $x = \{x_{m,n}\}$ is A-statistical convergent to a number L with the rate of $o(\alpha_{m,n})$, if for every $\varepsilon > 0$,*

$$P - \lim_{j,k\to\infty} \frac{1}{\alpha_{j,k}} \sum_{(m,n)\in K(\varepsilon)} a_{j,k,m,n} = 0,$$

where

$$K(\varepsilon) := \left\{(m,n) \in \mathbb{N}^2 : |x_{m,n} - L| \geq \varepsilon\right\}.$$

In this case, it is denoted by

$$x_{m,n} - L = st_A^{(2)} - o(\alpha_{m,n}) \quad as \quad m,n \to \infty.$$

Definition 2.2. [13] *Let $A = [a_{j,k,m,n}]$ and $\{\alpha_{m,n}\}$ be the same as in Definition 2.1. Then, a double sequence $x = \{x_{m,n}\}$ is A-statistical bounded with the rate of $O(\alpha_{m,n})$ if for every $\varepsilon > 0$,*

$$\sup_{j,k} \frac{1}{\alpha_{j,k}} \sum_{(m,n)\in L(\varepsilon)} a_{j,k,m,n} < \infty,$$

where

$$L(\varepsilon) := \left\{(m,n) \in \mathbb{N}^2 : |x_{m,n}| \geq \varepsilon \right\}.$$

In this case, it is denoted by

$$x_{m,n} = st_A^{(2)} - O(\alpha_{m,n}) \quad as \quad m,n \to \infty.$$

Definition 2.3. [13] *Let $A = [a_{j,k,m,n}]$ and $\{\alpha_{m,n}\}$ be the same as in Definition 2.1. Then, a double sequence $x = \{x_{m,n}\}$ is A-statistical convergent to a number L with the rate of $o_{m,n}(\alpha_{m,n})$ if for every $\varepsilon > 0$,*

$$P - \lim_{j,k\to\infty} \sum_{(m,n)\in M(\varepsilon)} a_{j,k,m,n} = 0,$$

where

$$M(\varepsilon) := \left\{(m,n) \in \mathbb{N}^2 : |x_{m,n} - L| \geq \varepsilon \ \alpha_{m,n}\right\}.$$

In this case, it is denoted by

$$x_{m,n} - L = st_A^{(2)} - o_{m,n}(\alpha_{m,n}) \quad as \quad m,n \to \infty.$$

Definition 2.4. [13] *Let $A = [a_{j,k,m,n}]$ and $\{\alpha_{m,n}\}$ be the same as in Definition 2.1. Then, a double sequence $x = \{x_{m,n}\}$ is A-statistical bounded with the rate of $O_{m,n}(\alpha_{m,n})$ if for every $\varepsilon > 0$,*

$$P - \lim_{j,k} \sum_{(m,n)\in N(\varepsilon)} a_{j,k,m,n} = 0,$$

where

$$N(\varepsilon) := \left\{(m,n) \in \mathbb{N}^2 : |x_{m,n}| \geq \varepsilon \ \alpha_{m,n}\right\}.$$

In this case, it is denoted by

$$x_{m,n} = st_A^{(2)} - O_{m,n}(\alpha_{m,n}) \quad as \quad m,n \to \infty.$$

We see from the above statements that, in Definitions 2.1 and 2.2 the rate sequence $\{\alpha_{m,n}\}$ directly effects the entries of the matrix $A = [a_{j,k,m,n}]$ although, according to Definitions 2.3 and 2.4, the rate is more controlled by the terms of the sequence $x = \{x_{m,n}\}$.

Using these definitions we have the following auxiliary result [13].

Lemma 2.5. [13] *Let $\{x_{m,n}\}$ and $\{y_{m,n}\}$ be double sequences. Assume that let $A = [a_{j,k,m,n}]$ be a non-negative RH-regular summability matrix and let $\{\alpha_{m,n}\}$ and $\{\beta_{m,n}\}$ be positive non-increasing sequences. If $x_{m,n} - L_1 = st_A^{(2)} - o(\alpha_{m,n})$ and $y_{m,n} - L_2 = st_A^{(2)} - o(\beta_{m,n})$, then we have* *(i)* $(x_{m,n} - L_1) \mp (y_{m,n} - L_2) = st_A^{(2)} - o(\gamma_{m,n})$ *as* $m, n \to \infty$ *, where* $\gamma_{m,n} := \max\{\alpha_{m,n}, \beta_{m,n}\}$ *for each* $(m, n) \in \mathbb{N}^2$,

(ii) $\lambda(x_{m,n} - L_1) = st_A^{(2)} - o(\alpha_{m,n})$ *as* $m, n \to \infty$ *for any real number* λ.

Furthermore, similar conclusions hold with the symbol "o" replaced by "O".

The above result can easily be modified to obtain the following result similarly.

Lemma 2.6. [13] *Let* $\{x_{m,n}\}$ *and* $\{y_{m,n}\}$ *be double sequences. Assume that* $A = [a_{j,k,m,n}]$ *is a non-negative RH-regular summability matrix and let* $\{\alpha_{m,n}\}$ *and* $\{\beta_{m,n}\}$ *be positive non-increasing sequences. If* $x_{m,n} - L_1 = st_A^{(2)} - o_{m,n}(\alpha_{m,n})$ *and* $y_{m,n} - L_2 = st_A^{(2)} - o_{m,n}(\beta_{m,n})$*, then we have*

(i) $(x_{m,n} - L_1) \mp (y_{m,n} - L_2) = st_A^{(2)} - o_{m,n}(\gamma_{m,n})$ *as* $m, n \to \infty$ *, where* $\gamma_{m,n} := \max\{\alpha_{m,n}, \beta_{m,n}\}$ *for each* $(m, n) \in \mathbb{N}^2$,

(ii) $\lambda(x_{m,n} - L_1) = st_A^{(2)} - o_{m,n}(\alpha_{m,n})$ *as* $m, n \to \infty$ *for any real number* λ.

Furthermore, similar conclusions hold with the symbol "$o_{m,n}$*" replaced by "*$O_{m,n}$*".*

Now we recall the concept of mixed modulus of smoothness. For $f \in C_b(D)$, the *mixed modulus of smoothness* of f, denoted by $\omega_{mixed}(f; \delta_1, \delta_2)$, is defined to be

$$\omega_{mixed}(f; \delta_1, \delta_2) = \sup\{|\Delta_{x,y}[f(u, v)]| : |u - x| \le \delta_1,\ |v - y| \le \delta_2\}$$

for $\delta_1, \delta_2 > 0$. In order to obtain our result, we will make use of the elementary inequality

$$\omega_{mixed}(f; \lambda_1\delta_1, \lambda_2\delta_2) \le (1 + \lambda_1)(1 + \lambda_2)\, \omega_{mixed}(f; \delta_1, \delta_2)$$

for $\lambda_1, \lambda_2 > 0$. The modulus ω_{mixed} has been used by several authors in the framework of "Boolean sum type" approximation (see, for example, [9]). Elementary properties of ω_{mixed} can be found in [21] (see also [1]) and in particular for the case of B-continuous functions in [2].

Then we have the following result.

Theorem 2.7. *Let* $\{L_{m,n}\}$ *be a sequence of positive linear operators acting from* $C_b(D)$ *into* $B(D)$ *and let* $A = [a_{j,k,m,n}]$ *be a non-negative RH−regular summability matrix. Let* $\{\alpha_{m,n}\}$ *and* $\{\beta_{m,n}\}$ *be a positive non-increasing double sequence. Assume that the following conditions hold:*

$$P - \lim_{j,k \to \infty} \frac{1}{\alpha_{j,k}} \sum_{(m,n) \in K} a_{j,k,m,n} = 1, \tag{2.1}$$

where $K = \left\{(m, n) \in \mathbb{N}^2 : L_{m,n}(e_0; x, y) = 1 \text{ for all } (x, y) \in D\right\}$*; and*

$$\omega_{mixed}(f; \gamma_{m,n}, \delta_{m,n}) = st_A^{(2)} - o(\beta_{m,n}) \quad as \quad m, n \to \infty, \tag{2.2}$$

where $\gamma_{m,n} := \sqrt{\|L_{m,n}(\varphi)\|}$ *and* $\delta_{m,n} := \sqrt{\|L_{m,n}(\Psi)\|}$ *with* $\varphi(u, v) = (u - x)^2$, $\Psi(u, v) = (v - y)^2$*. Then we have, for all* $f \in C_b(D)$,

$$\|L_{m,n}(F_{x,y}) - f\| = st_A^{(2)} - o(c_{m,n}) \quad as \quad m, n \to \infty,$$

where $F_{x,y}$ is given by (1.1) and $c_{m,n} := \max\{\alpha_{m,n}, \beta_{m,n}\}$ for each $(m,n) \in \mathbb{N}^2$. Furthermore, similar results hold when the symbol "o" is replaced by "O".

Proof. Let $(x,y) \in D$ and $f \in C_b(D)$ be fixed. It follows from (2.1) that

$$P - \lim_{j,k\to\infty} \frac{1}{\alpha_{j,k}} \sum_{(m,n)\in\mathbb{N}^2\setminus K} a_{j,k,m,n} = 0. \tag{2.3}$$

Also, since

$$\Delta_{x,y}\left[F_{x,y}(u,v)\right] = -\Delta_{x,y}\left[f(u,v)\right],$$

we observe that

$$L_{m,n}\left(F_{x,y}; x, y\right) - f(x,y) = L_{m,n}\left(\Delta_{x,y}\left[F_{x,y}(u,v)\right]; x, y\right)$$

holds for all $(m,n) \in K$. Then, using the properties of ω_{mixed} we obtain

$$\begin{aligned} \left|\Delta_{x,y}\left[F_{x,y}(u,v)\right]\right| &\le \omega_{mixed}\left(f; |u-x|, |v-y|\right) \\ &\le \left(1 + \frac{1}{\delta_1}|u-x|\right)\left(1 + \frac{1}{\delta_2}|v-y|\right) \\ &\quad \times \omega_{mixed}\left(f; \delta_1, \delta_2\right). \end{aligned} \tag{2.4}$$

Hence, using the monotonicity and the linearity of the operators $L_{m,n}$, for all $(m,n) \in K$, it follows from (2.4) that

$$\begin{aligned} & \left|L_{m,n}(F_{x,y}; x, y) - f(x,y)\right| \\ = \; & \left|L_{m,n}\left(\Delta_{x,y}\left[F_{x,y}(u,v)\right]; x, y\right)\right| \\ \le \; & L_{m,n}\left(\left|\Delta_{x,y}\left[F_{x,y}(u,v)\right]\right|; x, y\right) \\ \le \; & L_{m,n}\left(\left(1 + \frac{1}{\delta_1}|u-x|\right)\left(1 + \frac{1}{\delta_2}|v-y|\right); x, y\right)\omega_{mixed}\left(f; \delta_1, \delta_2\right) \\ = \; & \left\{1 + \frac{1}{\delta_1} L_{m,n}\left(|u-x|; x, y\right) + \frac{1}{\delta_2} L_{m,n}\left(|v-y|; x, y\right)\right. \\ & \left. \frac{1}{\delta_1\delta_2} L_{m,n}\left(|u-x| . |v-y|; x, y\right)\right\}\omega_{mixed}\left(f; \delta_1, \delta_2\right). \end{aligned}$$

Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} & \left|L_{m,n}(F_{x,y}; x, y) - f(x,y)\right| \\ & \le \left\{1 + \frac{1}{\delta_1}\sqrt{L_{m,n}\left(\varphi; x, y\right)} + \frac{1}{\delta_2}\sqrt{L_{m,n}\left(\Psi; x, y\right)}\right. \\ & \left. \frac{1}{\delta_1\delta_2}\sqrt{L_{m,n}\left(\varphi; x, y\right)}\sqrt{L_{m,n}\left(\Psi; x, y\right)}\right\}\omega_{mixed}\left(f; \delta_1, \delta_2\right) \end{aligned} \tag{2.5}$$

for all $(m,n) \in K$. Taking supremum over $(x,y) \in D$ on the both-sides of inequality (2.5) we obtain, for all $(m,n) \in K$, that

$$\left\|L_{m,n}(F_{x,y}) - f\right\| \le 4\omega_{mixed}\left(f; \gamma_{m,n}, \delta_{m,n}\right) \tag{2.6}$$

where $\delta_1 := \gamma_{m,n} := \sqrt{\|L_{m,n}(\varphi)\|}$ and $\delta_2 := \delta_{m,n} := \sqrt{\|L_{m,n}(\Psi)\|}$. Now, given $\varepsilon > 0$, define the following sets:

$$\begin{aligned} U \; &: = \left\{(m,n) \in \mathbb{N}^2 : \left\|L_{m,n}(F_{x,y}) - f\right\| \ge \varepsilon\right\}, \\ U_1 \; &: = \left\{(m,n) \in \mathbb{N}^2 : \omega_{mixed}\left(f; \gamma_{m,n}, \delta_{m,n}\right) \ge \frac{\varepsilon}{4}\right\}. \end{aligned}$$

Hence, it follows from (2.6) that

$$U \cap K \subseteq U_1 \cap K,$$

which gives, for all $(j,k) \in \mathbb{N}^2$,

$$\begin{aligned} \frac{1}{c_{j,k}} \sum_{(m,n) \in U \cap K} a_{j,k,m,n} &\leq \frac{1}{c_{j,k}} \sum_{(m,n) \in U_1 \cap K} a_{j,k,m,n} \\ &\leq \frac{1}{c_{j,k}} \sum_{(m,n) \in U_1} a_{j,k,m,n} \\ &\leq \frac{1}{\beta_{j,k}} \sum_{(m,n) \in U_1} a_{j,k,m,n}. \end{aligned} \tag{2.7}$$

where $c_{m,n} = \max\{\alpha_{m,n}, \beta_{m,n}\}$. Letting $j,k \to \infty$ (in any manner) in (2.7) and from (2.2), we conclude that

$$P - \lim_{j,k \to \infty} \frac{1}{c_{j,k}} \sum_{(m,n) \in U \cap K} a_{j,k,m,n} = 0. \tag{2.8}$$

Furthermore, we use the inequality

$$\begin{aligned} \sum_{(m,n) \in U} a_{j,k,m,n} &= \sum_{(m,n) \in U \cap K} a_{j,k,m,n} + \sum_{(m,n) \in U \cap (\mathbb{N}^2 \setminus K)} a_{j,k,m,n} \\ &\leq \sum_{(m,n) \in U \cap K} a_{j,k,m,n} + \sum_{(m,n) \in \mathbb{N}^2 \setminus K} a_{j,k,m,n} \end{aligned}$$

which gives,

$$\frac{1}{c_{j,k}} \sum_{(m,n) \in U} a_{j,k,m,n} \leq \frac{1}{c_{j,k}} \sum_{(m,n) \in U \cap K} a_{j,k,m,n} + \frac{1}{\alpha_{j,k}} \sum_{(m,n) \in \mathbb{N}^2 \setminus K} a_{j,k,m,n}. \tag{2.9}$$

Letting $j,k \to \infty$ (in any manner) in (2.9) and from (2.8) and (2.3), we conclude that

$$P - \lim_{j,k \to \infty} \frac{1}{c_{j,k}} \sum_{(m,n) \in U} a_{j,k,m,n} = 0.$$

The proof is completed. □

The following similar result holds.

Theorem 2.8. *Let $\{L_{m,n}\}$ be a sequence of positive linear operators acting from $C_b(D)$ into $B(D)$ and let $A = [a_{j,k,m,n}]$ be a non-negative $RH-$regular summability matrix. Let $\{\alpha_{m,n}\}$ and $\{\beta_{m,n}\}$ be a positive non-increasing double sequence. Assume that the following conditions holds:*

$$P - \lim_{j,k \to \infty} \sum_{(m,n) \in K} a_{j,k,m,n} = 1, \tag{2.10}$$

where $K = \left\{(m,n) \in \mathbb{N}^2 : L_{m,n}(e_0; x, y) = 1 \text{ for all } (x,y) \in \mathbb{R}^2\right\}$; and

$$\omega_{mixed}(f; \gamma_{m,n}, \delta_{m,n}) = st_A^{(2)} - o_{m,n}(\beta_{m,n}) \quad as \quad m,n \to \infty, \tag{2.11}$$

where $\gamma_{m,n} := \sqrt{\|L_{m,n}(\varphi)\|}$ *and* $\delta_{m,n} := \sqrt{\|L_{m,n}(\Psi)\|}$ *with* $\varphi(u,v) = (u-x)^2$, $\Psi(u,v) = (v-y)^2$. *Then we have, for all* $f \in C_b(D)$,

$$\|L_{m,n}(F_{x,y}) - f\| = st_A^{(2)} - o_{m,n}(\beta_{m,n}) \quad as \quad m,n \to \infty,$$

where $F_{x,y}$ *is given by* (1.1)*. Similar results hold when little "*$o_{m,n}$*" is replaced by capital "*$O_{m,n}$*".*

3. Concluding remarks

1) Specializing the sequences $\{\alpha_{m,n}\}$ and $\{\beta_{m,n}\}$ in Theorem 2.7 or Theorem 2.8 we can easily get Theorem 1.1. Thus, Theorem 2.7 gives us the rates of $A-$statistical convergence of the operators $L_{m,n}$ from $C_b(D)$ into $B(D)$.

2) Replacing the matrix A by a double identity matrix and taking $\alpha_{m,n} = \beta_{m,n} = 1$ for all $m,n \in \mathbb{N}$, we get the ordinary rate of convergence of the operators $L_{m,n}$.

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