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Subclasses of analytic functions associated with Fox-Wright's generalized hypergeometric functions based on Hilbert space operator

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Abstract. In this paper, we define a generalized class of starlike functions which are based upon some convolution operators on Hilbert space involving the Fox-Wright generalization of the classical hypergeometric pF_q function (with p numerator and q denominator parameters). The various results presented in this paper include (for example) normed coefficient inequalities and estimates, distortion theorems, and the radii of convexity and starlikeness for each of the analytic function classes which are investigated here. Also we obtain modified Hadamard product and integral means results.

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1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic and univalent in the open disc $U = \{z : z \in \mathcal{C};\ |z| < 1\}$. For functions $f \in A$ given by (1.1) and $g \in A$ given by $g(z) = z + \sum\limits_{k=2}^{\infty} b_k z^k$, we define the Hadamard product (or Convolution) of f and g by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k, \quad z \in U. \tag{1.2}$$

For positive real parameters $\alpha_1, A_1 \ldots, \alpha_l, A_l$ and $\beta_1, B_1 \ldots, \beta_m, B_m$, $(l, m \in N = 1, 2, 3, ...)$ such that

$$1 + \sum_{k=1}^{m} B_k - \sum_{k=1}^{l} A_k \geq 0, \quad z \in U, \tag{1.3}$$

the Wright generalized hypergeometric function [15]

$$\begin{aligned} &{}_l\Psi_m[(\alpha_1, A_1), \ldots, (\alpha_l, A_l); (\beta_1, B_1), \ldots, (\beta_m, B_m); z] \\ &=_l \Psi_m[(\alpha_t, A_t)_{1,l}(\beta_t, B_t)_{1,m}; z] \end{aligned}$$

is defined by

$${}_l\Psi_m[(\alpha_t, A_t)_{1,l}(\beta_t, B_t)_{1,m}; z] = \sum_{k=0}^{\infty}\{\prod_{t=0}^{l}\Gamma(\alpha_t + kA_t\}\{\prod_{t=0}^{m}\Gamma(\beta_t + kB_t\}^{-1}\frac{z^k}{k!},$$

$z \in U.$

If $A_t = 1(t = 1, 2, ..., l)$ and $B_t = 1(t = 1, 2, ..., m)$ we have the relationship:

$$\begin{aligned} \Omega_l\Psi_m[(\alpha_t, 1)_{1,l}(\beta_t, 1)_{1,m}; z] &\equiv {}_lF_m(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m; z) \\ &= \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \ldots (\alpha_l)_k}{(\beta_1)_k \ldots (\beta_m)_k} \frac{z^k}{k!} \end{aligned} \tag{1.4}$$

$(l \leq m + 1;\ l, m \in N_0 = N \cup \{0\}; z \in U)$ is the generalized hypergeometric function (see for details [6]), where N denotes the set of all positive integers and $(\alpha)_n$ is the Pochhammer symbol and

$$\Omega = \left(\prod_{t=0}^{l}\Gamma(\alpha_t)\right)^{-1}\left(\prod_{t=0}^{m}\Gamma(\beta_t)\right). \tag{1.5}$$

By using the generalized hypergeometric function, Dziok and Srivastava [6] introduced a linear operator which was subsequently extended by Dziok and Raina [5] by using the Fox-Wright generalized hypergeometric function.

Let $\mathcal{W}[(\alpha_t, A_t)_{1,l}; (\beta_t, B_t)_{1,m}] : A \to A$ be a linear operator defined by

$$\mathcal{W}[(\alpha_t, A_t)_{1,p}; (\beta_t, B_t)_{1,q}]f(z) \quad := \quad z\ {}_l\phi_m[(\alpha_t, A_t)_{1,l}; (\beta_t, B_t)_{1,m}; z] * f(z)$$

We observe that, for $f(z)$ of the form(1.1), we have

$$\mathcal{W}[(\alpha_t, A_t)_{1,l}; (\beta_t, B_t)_{1,m}]f(z) = z + \sum_{k=2}^{\infty}\sigma_k(\alpha_1)\ a_k z^k \tag{1.6}$$

where $\sigma_k(\alpha_1)$ is defined by

$$\sigma_k(\alpha_1) = \frac{\Omega\Gamma(\alpha_1 + A_1(k-1)) \ldots \Gamma(\alpha_l + A_l(k-1))}{(k-1)!\Gamma(\beta_1 + B_1(k-1)) \ldots \Gamma(\beta_m + B_m(k-1))}. \tag{1.7}$$

For convenience, we adopt the contracted notation $\mathcal{W}[\alpha_1]f(z)$ to represent the following:

$$\mathcal{W}[\alpha_1]f(z) = \mathcal{W}[(\alpha_1, A_1), \ldots, (\alpha_l, A_l); (\beta_1, B_1), \ldots, (\beta_m, B_m)]f(z) \tag{1.8}$$

throughout the sequel. The linear operator $\mathcal{W}[\alpha_1]f(z)$ contains the Dziok-Srivastava operator (see [6]), and as its various special cases contain such linear operators as the Hohlov operator, Carlson-Shaffer operator [3], Ruscheweyh derivative operator [14], generalized Bernardi-Libera-Livingston operator and fractional derivative operator [9]. Details and references about these operators can be found in [5] and [6].

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on H. For a complex-valued function $f$analytic in a domain $\mathbb{E}$ of the complex z-plane containing the spectrum $\sigma(\mathbb{P})$ of the bounded linear operator$\mathbb{P}$, let $f(\mathbb{P})$ denote the operator on H defined by [[2], p. 568]

$$f(\mathbb{P}) = \frac{1}{2\pi i}\int_{\mathcal{C}} (z\mathbb{I}-\mathbb{P})^{-1} f(z)dz,. \tag{1.9}$$

where $\mathbb{I}$ is the identity operator on $\mathcal{H}$ and $\mathcal{C}$ is a positively-oriented simple rectifiable closed contour containing the spectrum $\sigma(\mathbb{P})$ in the interior domain. The operator $f(\mathbb{P})$ can also be defined by the following series:

$$f(\mathbb{P}) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\mathbb{P}^n$$

which converges in the normed topology (cf. [4]).

We introduced a new subclass of analytic functions with negative coefficients and discuss some interesting properties of this generalized function class.

For $0 \le \alpha < 1$ and $0 < \beta \le 1$, we let $\mathcal{W}(\alpha,\beta)$ be the subclass of A consisting of functions of the form (1.1) and satisfying the inequality

$$\left\| \frac{\mathcal{J}_\lambda(\mathbb{P}) - 1}{\mathcal{J}(\mathbb{P}) - (2\alpha-1)} \right\| < \beta \tag{1.10}$$

where

$$\mathcal{J}_\lambda(\mathbb{P}) = (1-\lambda)\frac{\mathcal{W}[\alpha_1]f(\mathbb{P})}{\mathbb{P}} + \lambda(\mathcal{W}[\alpha_1]f(\mathbb{P}))', \tag{1.11}$$

$0 < \gamma \le 1$, $\mathcal{W}[\alpha_1]f(z)$ is given by (1.8).

We further let $\mathcal{W}(\alpha,\beta) = \mathcal{WT}(\alpha,\beta) \cap T$, where

$$T := \left\{ f \in A : f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \ge 0; z \in U \right\} \tag{1.12}$$

is a subclass of A introduced and studied by Silverman [10].

In the following section we obtain coefficient estimates and extreme points for the class $\mathcal{WT}(\lambda,\alpha,\beta,)$.

2. Coefficient bounds

Theorem 2.1. *Let the function f be defined by (1.12). Then $f \in \mathcal{WT}(\lambda,\alpha,\beta,)$ if and only if*

$$\sum_{k=2}^{\infty}(1+\lambda(k-1))[1+\beta]\sigma_k(\alpha_1)a_k \le 2\beta(1-\alpha). \tag{2.1}$$

The result is sharp for the function

$$f(z) = z - \frac{2\beta(1-\alpha)}{(1+\lambda(k-1))[1+\beta]\Omega\sigma_k(\alpha_1)} z^k, \quad k \geq 2. \tag{2.2}$$

Proof. Suppose f satisfies (2.1). Then for $\|z\| = \mathbb{P} = r\mathbb{I}$,

$$\begin{aligned}
&\|\mathcal{J}_\lambda(\mathbb{P}) - 1\| - \beta\,\|\mathcal{J}_\lambda(\mathbb{P}) + 1 - 2\alpha\| \\
&= \left\| -\sum_{k=2}^{\infty} [1+\lambda(k-1)]\sigma_k(\alpha_1) a_k \mathbb{P}^{k-1} \right\| \\
&-\beta \left\| 2(1-\alpha) - \sum_{k=2}^{\infty} [1+\lambda(k-1)]\sigma_k(\alpha_1) a_k \mathbb{P}^{k-1} \right\| \\
&\leq \sum_{k=2}^{\infty} [1+\lambda(k-1)] a_k\ \sigma_k(\alpha_1) r^{k-1} - 2\beta(1-\alpha) \\
&+ \sum_{k=2}^{\infty} [1+\lambda(k-1)]\beta\sigma_k(\alpha_1) a_k r^{k-1} \\
&= \sum_{k=2}^{\infty} [1+\lambda(k-1)][1+\beta]\sigma_k(\alpha_1) a_k - 2\beta(1-\alpha) \leq 0, \qquad \text{by (2.1)}.
\end{aligned}$$

Hence, by maximum modulus theorem and (1.10), $f \in \mathcal{WT}(\alpha, \beta)$.

To prove the converse, assume that

$$\left\| \frac{\mathcal{J}_\lambda(\mathbb{P}) - 1}{\mathcal{J}_\lambda(\mathbb{P}) + 1 - 2\alpha} \right\| = \left\| \frac{-\sum\limits_{k=2}^{\infty} [1+\lambda(k-1)]\sigma_k(\alpha_1) a_k \mathbb{P}^{k-1}}{2(1-\alpha) - \sum\limits_{k=2}^{\infty} [1+\lambda(k-1)]\sigma_k(\alpha_1) a_k \mathbb{P}^{k-1}} \right\|$$

$$\leq \beta, \quad z \in U.$$

Putting $\mathbb{P} = r\mathbb{I}(0 < r < 1$, and upon letting $r \to 1-$, yields the assertion (2.1) of Theorem 2.1. □

Corollary 2.2. *If $f(z)$ of the form (1.12) is in $\mathcal{WT}(\lambda, \alpha, \beta)$, then*

$$a_k \leq \frac{2\beta(1-\alpha)}{(1+k\lambda-\lambda)[1+\beta]\sigma_k(\alpha_1)}, \quad k \geq 2, \tag{2.3}$$

with equality only for functions of the form (2.2).

Theorem 2.3. (Extreme Points) *Let*

$$f_1(z) = z \quad \text{and}$$

$$f_k(z) = z - \frac{2\beta(1-\alpha)}{[1+\lambda(k-1)][1+\beta]\sigma_k(\alpha_1)} z^k, \;\; k \geq 2, \tag{2.4}$$

for $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $\lambda \geq 0$. Then $f(z)$ is in the class $\mathcal{WT}(\alpha, \beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z), \tag{2.5}$$

where $\mu_k \geq 0$ *and* $\sum\limits_{k=1}^{\infty} \mu_k = 1.$

Proof. Suppose $f(z)$ can be written as in (2.5). Then

$$f(z) \quad = \quad z - \sum_{k=2}^{\infty} \mu_k \frac{2\beta(1-\alpha)}{[1+\lambda(k-1)][1+\beta]\sigma_k(\alpha_1)} z^k.$$

Now,

$$\sum_{k=2}^{\infty} \frac{[1+\lambda(k-1)][1+\beta]\sigma_k(\alpha_1)}{2\beta(1-\alpha)} \mu_k \frac{2\beta(1-\alpha)}{[1+\lambda(k-1)][1+\beta]\sigma_k(\alpha_1)}$$

$$= \sum_{k=2}^{\infty} \mu_k = 1 - \mu_1 \leq 1.$$

Thus $f \in \mathcal{WT}(\alpha, \beta)$. Conversely, let us have $f \in \mathcal{WT}(\alpha, \beta)$. Then by using (2.3), we set

$$\mu_k = \frac{[1+\lambda(k-1)][1+\beta]\sigma_k(\alpha_1)}{2\beta(1-\alpha)} a_k, \quad k \geq 2$$

and

$$\mu_1 = 1 - \sum_{k=2}^{\infty} \mu_k.$$

Then we have

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z)$$

and hence this completes the proof of Theorem 2.3. □

3. Distortion bounds

In this section we obtain distortion bounds for the class $\mathcal{WT}(\alpha, \beta)$.

Theorem 3.1. *If* $f \in \mathcal{WT}(\alpha, \beta)$, *then*

$$r - \frac{2\beta(1-\alpha)}{(1+\lambda)[1+\beta]\sigma_2(\alpha_1)} r^2 \leq \|f(\mathbb{P})\| \ \leq r + \frac{2\beta(1-\alpha)}{(1+\lambda)[1+\beta]\sigma_2(\alpha_1)} r^2 \quad (3.1)$$

$$1 - \frac{4\beta(1-\alpha)}{(1+\lambda)[1+\beta]\sigma_2(\alpha_1)} r \leq \|f'(\mathbb{P})\| \ \leq 1 + \frac{4\beta(1-\alpha)}{(1+\lambda)[1+\beta]\sigma_2(\alpha_1)} r, \quad (3.2)$$

$(\mathbb{P} = r(0 < r < 1))$. *The bounds in (*3.1*) and (*3.2*) are sharp, since the equalities are attained by the function*

$$f(z) = z - \frac{2\beta(1-\alpha)}{(1+\lambda)[1+\beta]\sigma_2(\alpha_1)} z^2 \quad z = \pm r. \quad (3.3)$$

Proof. In the view of Theorem 2.1, we have

$$\sum_{k=2}^{\infty} a_k \leq \frac{2\beta(1-\alpha)}{(1+\lambda)[1+\beta]\sigma_2(\alpha_1)} \tag{3.4}$$

Using (1.12) and (3.4), we obtain

$$\begin{aligned} \|\mathbb{P}\| - \|\mathbb{P}\|^2 \sum_{k=2}^{\infty} a_k \quad &\leq \quad \|f(\mathbb{P})\| \ \leq \ \|\mathbb{P}\| + \|\mathbb{P}\|^2 \sum_{k=2}^{\infty} a_k \\ r - r^2 \frac{2\beta(1-\alpha)}{(1+\lambda)[1+\beta]\sigma_2(\alpha_1)} \quad &\leq \quad \|f(\mathbb{P})\| \ \leq \ r + r^2 \frac{2\beta(1-\alpha)}{(1+\lambda)[1+\beta]\sigma_2(\alpha_1)} \end{aligned} \tag{3.5}$$

Hence (3.1) follows from (3.5).

Further, since

$$\sum_{k=2}^{\infty} k a_k \leq \frac{4\beta(1-\alpha)}{(1+\lambda)[1+\beta]\sigma_2(\alpha_1)}.$$

Hence (3.2) follows from

$$1 - r\sum_{k=2}^{\infty} k a_k \leq \|f'(\mathbb{P})\| \ \leq \ 1 + r\sum_{k=2}^{\infty} k a_k. \qquad \square$$

4. Radius of starlikeness and convexity

The radii of close-to-convexity, starlikeness and convexity for the class $\mathcal{WT}(\alpha, \beta)$ are given in this section.

Theorem 4.1. *Let the function $f(z)$ defined by (1.12) belongs to the class $\mathcal{WT}(\alpha, \beta)$. Then $f(z)$ is close-to-convex of order δ $(0 \leq \delta < 1)$ in the disc $\|\mathbb{P}\| < r_1$, where*

$$r_1 := \left[\frac{(1-\delta)[1+\lambda(k-1)][1+\beta]\ \sigma_k(\alpha_1)}{2k\beta(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \geq 2). \tag{4.1}$$

The result is sharp, with extremal function $f(z)$ given by (2.4).

Proof. Given $f \in T$ and f is close-to-convex of order δ, we have

$$\|f'(\mathbb{P}) - 1\| < 1 - \delta. \tag{4.2}$$

For the left hand side of (4.2) we have

$$\|f'(\mathbb{P}) - 1\| \leq \sum_{k=2}^{\infty} k a_k \|\mathbb{P}\|^{k-1}.$$

The last expression is less than $1-\delta$ if

$$\sum_{k=2}^{\infty} \frac{k}{1-\delta} a_k \|\mathbb{P}\|^{k-1} < 1.$$

Using the fact, that $f \in \mathcal{WT}(\alpha, \beta)$ if and only if

$$\sum_{k=2}^{\infty} \frac{[1+\lambda(k-1)][1+\beta]a_k\ \sigma_k(\alpha_1)}{2\beta(1-\alpha)} \leq 1.$$

We can say (4.2) is true if

$$\frac{k}{1-\delta}\|\mathbb{P}\|^{k-1} \leq \frac{[1+\lambda(k-1)][1+\beta]\sigma_k(\alpha_1)}{2\beta(1-\alpha)}.$$

Or, equivalently,

$$\|\mathbb{P}\|^{k-1} = r^{k-1} < \left[\frac{(1-\delta)[1+\lambda(k-1)][1+\beta]\ \sigma_k(\alpha_1)}{2k\beta(1-\alpha)}\right]$$

which completes the proof. □

Theorem 4.2. *Let $f \in \mathcal{WT}(\alpha, \beta)$. Then*

1. *f is starlike of order $\delta (0 \leq \delta < 1)$ in the disc $|z| < r_2$; that is, $Re\left\{\frac{zf'(z)}{f(z)}\right\} > \delta, \quad (\|\mathbb{P}\| < r_2\ ;\ 0 \leq \delta < 1)$, where*

$$r_2 = \inf_{k\geq 2}\left\{\frac{(1-\delta)[1+\lambda(k-1)][1+\beta]\ \sigma_k(\alpha_1)}{2\beta(1-\alpha)(k-\delta)}\right\}^{\frac{1}{k-1}}.$$

2. *f is convex of order δ $(0 \leq \delta < 1)$ in the disc $|z| < r_3$, that is $Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \delta$, $(\|\mathbb{P}\| < r_3; 0 \leq \delta < 1)$, where*

$$r_3 = \inf_{k\geq 2}\left\{\frac{(1-\delta)[1+\lambda(k-1)][1+\beta]\ \sigma_k(\alpha_1)}{2\beta(1-\alpha)k(k-\delta)}\right\}^{\frac{1}{k-1}}.$$

Each of these results are sharp for the extremal function $f(z)$ given by (2.4).

Proof. Given $f \in T$ and f is starlike of order δ, we have

$$\left\|\frac{\mathbb{P}f'(\mathbb{P})}{f(\mathbb{P})} - 1\right\| < 1-\delta \quad (\mathbb{P} = r_2\mathbb{I}(0 < r_1 < 1)). \tag{4.3}$$

For the left hand side of (4.3) we have

$$\left\|\frac{\mathbb{P}f'(\mathbb{P})}{f(\mathbb{P})} - 1\right\| \leq \frac{\sum\limits_{k=2}^{\infty}(k-1)a_k\ \|\mathbb{P}\|^{k-1}}{1-\sum\limits_{k=2}^{\infty} a_k\ \|\mathbb{P}\|^{k-1}}.$$

The last expression is less than $1-\delta$ if

$$\sum_{k=2}^{\infty}\frac{k-\delta}{1-\delta}a_k\ \|\mathbb{P}\|^{k-1} < 1.$$

Using the fact, that $f \in \mathcal{WT}(\alpha, \beta)$ if and only if

$$\sum_{k=2}^{\infty} \frac{[1+\lambda(k-1)][1+\beta]a_k\ \sigma_k(\alpha_1)}{2\beta(1-\alpha)} < 1.$$

We can say that (4.3) is true if

$$\frac{k-\delta}{1-\delta}\|\mathbb{P}\|^{k-1} < \frac{[1+\lambda(k-1)][1+\beta]\ \sigma_k(\alpha_1)}{2\beta(1-\alpha)}.$$

Or, equivalently,

$$\|\mathbb{P}\|^{k-1} = r_2 < \frac{(1-\delta)[1+\lambda(k-1)][1+\beta]\ \sigma_k(\alpha_1)}{2\beta(1-\alpha)(k-\delta)},$$

which yields the starlikeness of the family.

(ii) Using the fact that f is convex if and only if zf' is starlike, we can prove (ii), similar the to proof of (i). □

5. Modified Hadamard products

Let the functions $f_j(z)(j = 1,2)$ be defined by (1.12). The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) \ = \ z - \sum_{n=2}^{\infty} a_{k,1}a_{k,2}\ z^k.$$

Using the techniques of Schild and Silverman [13], we prove the following results.

Theorem 5.1. *For functions $f_j(z)(j = 1,2)$ defined by (1.12), let $f_1 \in \mathcal{WT}(\alpha,\beta)$ and $f_2 \in \mathcal{WT}(\gamma,\beta)$. Then $(f_1 * f_2) \in\in \mathcal{WT}(\xi,\beta)$ where*

$$\xi = 1 - \frac{2\beta(1-\alpha)(1-\gamma)}{\sigma_2(\alpha_1)}, \tag{5.1}$$

where $\sigma_k(\alpha_1)$ is given by (1.7).

Proof. In view of Theorem 2.1, it suffice to prove that

$$\sum_{k=2}^{\infty}\frac{(1+\lambda(k-1))[1+\beta]\sigma_k(\alpha_1)}{2\beta(1-\alpha)(1-\xi)}a_{k,1}a_{k,2} \le 1, \ \ (0 \le \xi < 1)$$

where ξ is defined by (5.1). On the other hand, under the hypothesis, it follows from (2.1) and the Cauchy's-Schwarz inequality that

$$\sum_{k=2}^{\infty}\frac{(1+\lambda(k-1))[1+\beta]\sigma_k(\alpha_1)}{2\beta\sqrt{(1-\alpha)(1-\gamma)}}\sqrt{a_{k,1}a_{k,2}} \ \le 1. \tag{5.2}$$

We need to find the largest ξ such that

$$\begin{aligned}&\sum_{k=2}^{\infty}\frac{(1+\lambda(k-1))[1+\beta]\sigma_k(\alpha_1)}{2\beta(1-\xi)(1-\alpha)}a_{k,1}a_{k,2}\\ \le\ &\sum_{k=2}^{\infty}\frac{(1+\lambda(k-1))[1+\beta]\sigma_k(\alpha_1)}{\sqrt{(1-\alpha)(1-\gamma)}}\sqrt{a_{k,1}a_{k,2}}\end{aligned}$$

or, equivalently that

$$\sqrt{a_{k,1}a_{k,2}} \quad \le \quad \frac{1-\xi}{\sqrt{(1-\alpha)((1-\gamma))}}, \ \ (k \ge 2).$$

By view of (5.2) it is sufficient to find the largest ξ such that

$$2\beta\sqrt{(1-\alpha)(1-\gamma)}(\sigma_n(\alpha_1))^{-1} \leq \frac{1-\xi}{\sqrt{(1-\alpha)((1-\gamma))}}$$

which yields

$$\xi = 1 - \frac{2\beta(1-\alpha)(1-\gamma)}{\sigma_k(\alpha_1)} \text{ for } k \geq 2 \tag{5.3}$$

is an increasing function of k and letting $k = 2$ in (5.3), we have

$$\xi = 1 - \frac{2\beta(1-\alpha)(1-\gamma)}{\sigma_2(\alpha_1)}$$

where $\sigma_2(\alpha_1)$ is given by (1.7). $\square$

Theorem 5.2. *Let the function $f(z)$ defined by (1.12) be in the class $\mathcal{WT}(\alpha,\beta)$. Also let $g(z) = z - \sum\limits_{k=2}^{\infty} b_k z^k$ for $|b_k| \leq 1$. Then $(f*g) \in \mathcal{WT}(\alpha,\beta)$.*

Proof. Since

$$\begin{aligned}
& \sum_{k=2}^{\infty}(1+\lambda(k-1))[1+\beta]\sigma_k(\alpha_1)|a_k b_k| \\
\leq\ & \sum_{k=2}^{\infty}(1+\lambda(k-1))[1+\beta]\sigma_k(\alpha_1)a_k|b_k| \\
\leq\ & \sum_{k=2}^{\infty}(1+\lambda(k-1))[1+\beta]\sigma_k(\alpha_1)a_k \\
\leq\ & 2\beta(1-\alpha)
\end{aligned}$$

it follows that $(f*g) \in \mathcal{WT}(\alpha,\beta)$, by the view of Theorem 2.1. $\square$

Theorem 5.3. *Let the functions $f_j(z)(j = 1,2)$ defined by (1.12) be in the class $\in \mathcal{WT}(\alpha,\beta)$. Then the function $h(z)$ defined by $h(z) = z - \sum\limits_{k=2}^{\infty}(a_{n,1}^2 + a_{n,2}^2)z^n$ is in the class $\in \mathcal{WT}(\xi,\beta)$, where*

$$\xi = 1 - \frac{4\beta(1-\alpha)^2}{\sigma_2(\alpha_1)(1+\lambda)(1+\beta)}$$

and $\sigma_2(\alpha_1)$ is given by (1.7).

Proof. By virtue of Theorem 2.1, it is sufficient to prove that

$$\sum_{k=2}^{\infty}\frac{(1+\lambda(k-1))[1+\beta]\sigma_k(\alpha_1)}{2\beta(1-\xi)}(a_{n,1}^2 + a_{n,2}^2) \leq 1, \tag{5.4}$$

where $f_j \in \mathcal{WT}(\xi, \beta)$ we find from (2.1) and Theorem 2.1, that

$$\begin{aligned} &\sum_{k=2}^{\infty} \left[\frac{(1+\lambda(k-1))[1+\beta]\sigma_k(\alpha_1)}{2\beta(1-\alpha)} \right]^2 a_{n,j}^2 \\ \leq &\sum_{k=2}^{\infty} \left[\frac{(1+\lambda(k-1))[1+\beta]\sigma_k(\alpha_1)}{2\beta(1-\alpha)} a_{n,j} \right]^2, \end{aligned} \tag{5.5}$$

which yields

$$\sum_{k=2}^{\infty} \frac{1}{2} \left[\frac{(1+\lambda(k-1))[1+\beta]\sigma_k(\alpha_1)}{2\beta(1-\alpha)} \right]^2 (a_{n,1}^2 + a_{n,2}^2) \leq 1. \tag{5.6}$$

On comparing (5.5) and (5.6), it is easily seen that the inequality (5.4) will be satisfied if

$$\begin{aligned} &\frac{(1+\lambda(k-1))[1+\beta]\sigma_k(\alpha_1)}{2\beta(1-\xi)} \\ \leq &\frac{1}{2} \left[\frac{(1+\lambda(k-1))[1+\beta]\sigma_k(\alpha_1)}{2\beta(1-\alpha)} \right]^2, \quad \text{for } k \geq 2. \end{aligned}$$

That is an increasing function of k $(k \geq 2)$. Taking $k = 2$ in (5.7), we have

$$\xi = 1 - \frac{4\beta(1-\alpha)^2}{\sigma_k(\alpha_1)(1+\lambda)(1+\beta)} \tag{5.7}$$

which completes the proof. □

6. Integral means inequalities

Lemma 6.1. [8] *If the functions f and g are analytic in Δ with $g \prec f$, then for $\kappa > 0$, and $0 < r < 1$,*

$$\int_0^{2\pi} \left| g(re^{i\theta}) \right|^{\kappa} d\theta \leq \int_0^{2\pi} \left| f(re^{i\theta}) \right|^{\kappa} d\theta. \tag{6.1}$$

In [10], Silverman found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal over the family T. He applied this function to resolve his integral means inequality, conjectured in [11] and settled in [12], that

$$\int_0^{2\pi} \left| f(re^{i\theta}) \right|^{\kappa} d\theta \leq \int_0^{2\pi} \left| f_2(re^{i\theta}) \right|^{\kappa} d\theta,$$

for all $f \in T$, $\kappa > 0$ and $0 < r < 1$. In [12], he also proved his conjecture for the subclasses of starlike functions of order α and convex functions of order α.

In this section, we obtain integral means inequalities for the functions in the family $\mathcal{WT}(\alpha, \beta)$.

Applying Lemma 6.1, Theorem 2.1 and Theorem 2.3, we prove the following result.

Theorem 6.2. *Suppose $f(z) \in \mathcal{WT}(\alpha, \beta)$ and $f_2(z)$ is defined by*

$$f_2(z) = z - \frac{2\beta(1-\alpha)}{[1+\lambda][1+\beta]\sigma_2(\alpha_1)} z^2.$$

Then for $z = re^{i\theta}$, $0 < r < 1$, we have

$$\int_0^{2\pi} \|f(z)\|^{\kappa}\, d\theta \leq \int_0^{2\pi} \|f_2(z)\|^{\kappa}\, d\theta. \tag{6.2}$$

Proof. For $f(z) = z - \sum_{k=2}^{\infty} a_k z^n$, (6.2) is equivalent to proving that

$$\int_0^{2\pi} \left\| 1 - \sum_{k=2}^{\infty} a_k z^{n-1} \right\|^{\kappa} d\theta \leq \int_0^{2\pi} \left\| 1 - \frac{2\beta(1-\alpha)}{[1+\lambda][1+\beta]\sigma_2(\alpha_1)} z \right\|^{\kappa} d\theta.$$

By Lemma 6.1, it suffices to show that

$$1 - \sum_{k=2}^{\infty} a_k \|\mathbb{P}\|^{n-1} \prec 1 - \frac{2\beta(1-\alpha)}{[1+\lambda][1+\beta]\sigma_2(\alpha_1)} \|\mathbb{P}\|.$$

Setting

$$1 - \sum_{k=2}^{\infty} a_k \|\mathbb{P}\|^{n-1} = 1 - \frac{2\beta(1-\alpha)}{[1+\lambda][1+\beta]\sigma_2(\alpha_1)} w(z), \tag{6.3}$$

and using (2.1), we obtain

$$\begin{aligned} \|w(z)\| &= \left\| \sum_{k=2}^{\infty} \frac{2\beta(1-\alpha)}{(1+k\lambda-\lambda)[1+\beta]\sigma_k(\alpha_1)} a_k z^{n-1} \right\| \\ &\leq \|\mathbb{P}\| \sum_{k=2}^{\infty} \frac{2\beta(1-\alpha)}{(1+k\lambda-\lambda)[1+\beta]\sigma_k(\alpha_1)} |a_k| \\ &\leq \|\mathbb{P}\|. \end{aligned}$$

This completes the proof. □

Remark 6.3. In view of the relationship (1.4) the linear operator (1.6) and by setting $A_t = 1(t = 1, ..., l)$ and $B_t = 1(t = 1, ..., m)$ and specific choices of parameters l, m, α_1, β_1 the various results presented in this paper would provide interesting extensions and generalizations of those considered earlier for simpler analytic function classes.

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