Stud. Univ. Babeş-Bolyai Math. 56(2011), No. 3, 53–60

Properties of certain analytic functions defined by a linear operator

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Abstract. In this paper, we study and investigate starlikeness and convexity of a class of multivalent functions defined by a linear operator $L_{p,k}(a,c)f(z)$. As a consequence, a number of sufficient conditions for starlikeness and convexity of analytic functions are also obtained.

Mathematics Subject Classification (2010): 30C45.

Keywords: Multivalent function, starlike function, convex function, linear operator.

1. Introduction

Let $A(p,k)(p,k \in N = \{1,2,3,....\})$ be the class of functions of the form

$$f(z) = z^p + \sum_{m=k}^{\infty} a_{p+m} z^{p+m} \tag{1.1}$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$. We denote $A(p,1) = A_p$ and $A(1,1) = A$.

A function $f(z) \in A(p,k)$ is said to be p-valent starlike of order α $(0 \leq \alpha < p)$ in U if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \qquad z \in U.$$

We denote by $S_p^*(\alpha)$, the class of all such functions.

A function $f(z) \in A(p,k)$ is said to be p-valent convex of order α $(0 \leq \alpha < p)$ in U if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \qquad z \in U.$$

Let $K_p(\alpha)$ denote the class of all those functions $f \in A(p,k)$, which are multivalently convex of order α in U. Note that $S_1^*(\alpha)$ and $K_1(\alpha)$ are, respectively, the usual classes of univalent starlike functions of order α and univalent convex functions of order $\alpha, 0 \leq \alpha < 1$, and will be denoted here by $S^*(\alpha)$ and $K(\alpha)$, respectively. We shall use S^* and K to denote $S^*(0)$ and $K(0)$, respectively, which are the classes of univalent starlike (w.r.t the origin) and univalent convex functions. These classes considered also by S. Singh et. al. [6].
The class $A(p,k)$ is closed under the Hadamard product (or convolution)

$$\begin{aligned} f(z) * g(z) &= (f * g)(z) = z^p + \sum_{m=k}^{\infty} a_{p+m} b_{p+m} z^{p+m} \\ &= (g * f)(z) \qquad (z \in U), \end{aligned}$$

where

$$f(z) = z^p + \sum_{m=k}^{\infty} a_{p+m} z^{p+m}, \quad g(z) = z^p + \sum_{m=k}^{\infty} b_{p+m} z^{p+m}.$$

Let the function $\varphi_{p,k}(a,c)$ be defined by

$$\varphi_{p,k}(a,c;z) = z^p + \sum_{m=k}^{\infty} \frac{(a)_m}{(c)_m} z^{p+m} \quad (z \in U), \tag{1.2}$$

where $c \neq 0, -1, -2, ..., (\lambda)_0 = 1$ and $(\lambda)_m = \lambda(\lambda+1)...(\lambda+m-1)$ for $m \in N$.

Carlson and Shaffer [2] defined a convolution operator on A by

$$L(a,c)f(z) = \varphi_{1,1}(a,c) * f(z) \quad (f(z) \in A). \tag{1.3}$$

Similarly Xu and Aouf [1] define a linear operator $L_{p,k}(a,c)$ on $A(p,k)$ by

$$\begin{aligned} L_{p,k}(a,c)f(z) &= \varphi_{p,k}(a,c) * f(z) \quad (f(z) \in A(p,k)) \\ &= \left(z^p + \sum_{m=k}^{\infty} \frac{(a)_m}{(c)_m} z^{p+m} \right) * \left(z^p + \sum_{m=k}^{\infty} a_{p+m} z^{p+m} \right) \\ &= z^p + \sum_{m=k}^{\infty} \frac{(a)_m}{(c)_m} a_{p+m} z^{p+m}. \end{aligned} \tag{1.4}$$

It is easily seen from (1.4) that

$$z(L_{p,k}(a,c)f(z))' = aL_{p,k}(a+1,c)f(z) - (a-p)L_{p,k}(a,c)f(z). \tag{1.5}$$

Clearly $L_{p,k}(a,c)$ maps $A(p,k)$ into itself and $L_{p,k}(c,c)$ is identity.
If $a \neq 0, -1, -2, ...,$ then $L_{p,k}(a,c)$ has an inverse $L_{p,k}(c,a)$.
We note that

$$L_{p,k}(p+1,p)f(z) = \frac{zf'(z)}{p}.$$

For a real number $\lambda > -p$, we get

$$L_{p,\lambda}(\lambda+p,\lambda+p+1)f(z) = J_{p,\lambda}f(z) = \frac{\lambda+p}{z^{\lambda}}\int_0^z t^{\lambda-1}f(t)dt \qquad (1.6)$$

where $J_{p,\lambda}$ the generalized Libera integral operator (see [4]), and

$$L_{p,k}(\lambda+p,1)f(z) = D^{\lambda+p-1}f(z)$$

where $D^{\lambda+p-1}$ the generalized Ruscheweyh derivative (see [5]).

A function $f(z) \in A(p,k)$ is said to be in the class $S_{p,k}(\alpha,a,c)$ for all z in U if it satisfies

$$\operatorname{Re}\left[\frac{L_{p,k}(a+1,c)f(z)}{L_{p,k}(a,c)f(z)}\right] > \frac{\alpha}{p}\ , \qquad (1.7)$$

for some $\alpha(0 \le \alpha < p, p \in N)$. We note that $S_{p,k}(\alpha,p,p)$ is the usual class $S_p^*(\alpha)$ of p-valent starlike functions of order α.

In the present paper, our aim is to determine sufficient conditions for a function $f \in A(p,k)$ to be a member of the class $S_{p,k}(\alpha,a,c)$. As a consequence of our main result we get a number of sufficient conditions for starlikeness and convexity of analytic functions.

2. Main result

To prove our result, we shall make use of the famous Jack's Lemma which we state below.

Lemma 2.1. (Jack [3]). *Suppose $w(z)$ be a nonconstant analytic function in U with $w(0)=0$. If $|w(z)|$ attains its maximum value at a point $z_0 \in U$ on the circle $|z| = r < 1$, then $z_0w'(z_0) = mw(z_0)$, where m is a real and $m \ge 1$.*

We now state and prove our main result.

Theorem 2.2. *If $f(z) \in A(p,k)$ satisfies*

$$\left|\frac{L_{p,k}(a+1,c)f(z)}{L_{p,k}(a,c)f(z)} - 1\right|^{\gamma}\left|\frac{L_{p,k}(a+2,c)f(z)}{L_{p,k}(a+1,c)f(z)} - 1\right|^{\beta} < M_p(k,a,c,\alpha,\beta,\gamma) \qquad (2.1)$$

$(z \in U)$, for some $\alpha(0 \le \alpha < p), \beta(\beta \ge 0)$ $\gamma \ge 0$ and $\beta+\gamma > 0$, then $f(z) \in S_p(k,a,c,\alpha)$, and

$$M_p(k,a,c,\alpha,\beta,\gamma) = \begin{cases} (1-\frac{\alpha}{p})^{\gamma}(1-\frac{\alpha}{p}+\frac{1}{2a})^{\beta}, & 0 \le \alpha < \frac{p}{2} \\ (1-\frac{\alpha}{p})^{\gamma+\beta}(1+\frac{1}{a})^{\beta}, & \frac{p}{2} \le \alpha < p. \end{cases}$$

Proof. Case (i). Let $0 \le \alpha < \frac{p}{2}$. Writing $\frac{\alpha}{p} = \mu$, we see that $0 \le \mu \le \frac{1}{2}$. Define a function $w(z)$ as**:**

$$\frac{L_{p,k}(a+1,c)f(z)}{L_{p,k}(a,c)f(z)} = \frac{1+(1-2\mu)w(z)}{1-w(z)}\quad , \quad z \in U. \qquad (2.2)$$

Then $w(z)$ is analytic in U, $w(0) = 0$ and $w(z) \neq 1$ in U. By a simple computation, we obtain from (2.2),

$$\frac{z(L_{p,k}(a+1,c)f(z))^{'}}{L_{p,k}(a+1,c)f(z)} - \frac{z(L_{p,k}(a,c)f(z))^{'}}{L_{p,k}(a,c)f(z)} = \frac{2(1-\mu)zw^{'}(z)}{(1-w(z))(1+(1-2\mu)w(z))} \tag{2.3}$$

and from (1.5) we get

$$\frac{L_{p,k}(a+2,c)f(z)}{L_{p,k}(a+1,c)f(z)} = \frac{1+(1-2\mu)w(z)}{1-w(z)} + \frac{2(1-\mu)zw^{'}(z)}{a(1-w(z))(1+(1-2\mu)w(z))}.$$

Thus, we have

$$\left|\frac{L_{p,k}(a+1,c)f(z)}{L_{p,k}(a,c)f(z)} - 1\right|^{\gamma} \left|\frac{L_{p,k}(a+2,c)f(z)}{L_{p,k}(a+1,c)f(z)} - 1\right|^{\beta}$$

$$= \left|\frac{2(1-\mu)w(z)}{1-w(z)}\right|^{\gamma+\beta} \left|1 + \frac{zw^{'}(z)}{aw(z)(1+(1-2\mu))w(z))}\right|^{\beta}.$$

Suppose that there exists a point $z_0 \in U$ such that

$$\max_{|z|\leq|z_0|} |w(z)| = |w(z_0)| = 1,$$

then by Lemma 2.1, we have $w(z_0) = e^{i\theta}, 0 < \theta \leq 2\pi$ and

$$z_0 w^{'}(z_0) = mw(z_0), \ m \geq 1.$$

Therefore, we have

$$\left|\frac{L_{p,k}(a+1,c)f(z_0)}{L_{p,k}(a,c)f(z_0)} - 1\right|^{\gamma} \left|\frac{L_{p,k}(a+2,c)f(z_0)}{L_{p,k}(a+1,c)f(z_0)} - 1\right|^{\beta}$$

$$\begin{aligned} &= \left|\frac{2(1-\mu)w(z_0)}{1-w(z_0)}\right|^{\gamma+\beta} \left|1 + \frac{m}{a(1+(1-2\mu)w(z_0))}\right|^{\beta} \\ &= \frac{2^{\gamma+\beta}(1-\mu)^{\gamma+\beta}}{|1-e^{i\theta}|^{\beta+\gamma}} \left|1 + \frac{m}{a(1+(1-2\mu)e^{i\theta})}\right|^{\beta} \end{aligned}$$

$$\geq (1-\mu)^{\gamma+\beta}\left(1 + \frac{m}{2a(1-\mu)}\right)^{\beta} \geq (1-\mu)^{\beta+\gamma}\left(1 + \frac{1}{2a(1-\mu)}\right)^{\beta}$$

$$= (1-\mu)^{\gamma}\left(1-\mu+\frac{1}{2a}\right)^{\beta}$$

which contradicts (2.1) for $0 \leq \alpha \leq \frac{p}{2}$. Therefore, we must have $|w(z)| < 1$ for all $z \in U$, and hence $f(z) \in S_p(k,a,c,\alpha)$.

Case (ii). When $\frac{p}{2} \leq \alpha < p$. In this case, we must have $\frac{1}{2} \leq \mu < 1$, where $\mu = \frac{\alpha}{p}$. Let w be defined by

$$\frac{L_{p,k}(a+1,c)f(z_0)}{L_{p,k}(a,c)f(z)} = \frac{\mu}{\mu-(1-\mu)w(z)}, \qquad z \in U,$$

where $w(z) \neq \frac{\mu}{1-\mu}$ in U. Then $w(z)$ is analytic in U and $w(0) = 0$. Proceeding as in Case (i) and using identity (1.5), we obtain

$$\left|\frac{L_{p,k}(a+1,c)f(z)}{L_{p,k}(a,c)f(z)} - 1\right|^{\gamma} \left|\frac{L_{p,k}(a+2,c)f(z)}{L_{p,k}(a+1,c)f(z)} - 1\right|^{\beta}$$

$$= \left|\frac{(1-\mu)w(z)}{\mu-(1-\mu)w(z)}\right|^{\gamma} \left|\frac{(1-\mu)w(z)}{\mu-(1-\mu)w(z)} + \frac{(1-\mu)zw^{'}(z)}{a(\mu-(1-\mu)w(z))}\right|^{\beta}$$

$$= \left|\frac{1-\mu}{\mu-(1-\mu)w(z)}\right|^{\gamma+\beta} |w(z)|^{\gamma} \left|w(z) + \frac{zw^{'}(z)}{a}\right|^{\beta}.$$

Suppose that there exists a point $z_0 \in U$ such that $\max\limits_{|z|\leq|z_0|} |w(z)| = |w(z_0)| = 1$, then by Lemma 2.1, we obtain $w(z_0) = e^{i\theta}$ and $z_0 w^{'}(z_0) = mw(z_0), m \geq 1$. Therefore

$$\left|\frac{L_{p,k}(a+1,c)f(z_0)}{L_{p,k}(a,c)f(z_0)} - 1\right|^{\gamma} \left|\frac{L_{p,k}(a+2,c)f(z_0)}{L_{p,k}(a+1,c)f(z_0)} - 1\right|^{\beta}$$

$$= \frac{(1-\mu)^{\gamma+\beta}\left(1+\frac{m}{a}\right)^{\gamma+\beta}}{|\mu-(1-\mu)e^{i\theta}|^{\gamma+\beta}} \geq \left(1-\frac{\alpha}{p}\right)^{\gamma+\beta}\left(1+\frac{1}{a}\right)^{\beta}$$

which contradicts (2.1) for $\frac{p}{2} \leq \alpha < p$. Therefore, we must have $|w(z)| < 1$ for all $z \in U$, and hence $f(z) \in S_p(k, a, c, \alpha)$. This completes the proof of our theorem.

3. Deductions

For $p = 1$, Theorem 2.2 reduces to the following results:

Corollary 3.1. *If, for all $z \in U$, a function $f(z) \in A$ satisfies*

$$\left|\frac{L(a+1,c)f(z)}{L(a,c)f(z)} - 1\right|^{\gamma} \left|\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)} - 1\right|^{\beta} < \begin{cases} (1-\alpha)^{\gamma}(1-\alpha+\frac{1}{2a})^{\beta}, & 0 \leq \alpha < \frac{1}{2} \\ (1-\alpha)^{\gamma+\beta}(1+\frac{1}{a})^{\beta} & \frac{1}{2} \leq \alpha < 1, \end{cases}$$

for some real $\alpha(0 \leq \alpha < 1), \beta \geq 0$ and $\gamma \geq 0$ with $\beta + \gamma > 0$, then $f(z) \in S_1(k, a, c, \alpha)$.

For $\dot{\gamma} = 0$ and $\beta = 1$ in Theorem 2.2, we obtain

Corollary 3.2. *If, for all $z \in U$, a function $f(z) \in A(p,k)$ satisfies*

$$\left|\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)} - 1\right| < \begin{cases} 1-\frac{\alpha}{p}+\frac{1}{2a}, & 0 \leq \alpha < \frac{p}{2} \\ (1-\frac{\alpha}{p})(1+\frac{1}{a}) & \frac{p}{2} \leq \alpha < p, \end{cases}$$

then

$$\left(\frac{L_{p,k}(a+1,c)f(z)}{L_{p,k}(a,c)f(z)}\right) > \frac{\alpha}{p}, \quad (z \in U).$$

Setting $p = a = c = 1$ in Theorem 2.2, we obtain the following result:

Corollary 3.3. *If $f(z) \in A$ satisfies*

$$\left|\frac{zf'(z)}{f(z)}-1\right|^{\gamma}\left|\frac{zf''(z)}{f'(z)}\right|^{\beta}<2^{2\beta}(1-\alpha)^{\gamma+\beta}, 0\leq\alpha<1(z\in U)$$

for some $\beta \geq 0$ and $\gamma \geq 0$ with $\beta+\gamma>0$, then

$$f(z) \in S^*(\alpha).$$

Setting $\alpha = 0$ in Corollary 3.1, we obtain the following criterion for starlikeness:

Corollary 3.4. *For some non-negative real numbers β and γ with $\beta+\gamma>0$, if $f(z) \in A$ satisfies*

$$\left|\frac{zf'(z)}{f(z)}-1\right|^{\gamma}\left|\frac{zf''(z)}{f'(z)}\right|^{\beta}<2^{2\beta} \qquad (z\in U),$$

then $f(z) \in S^*$.

In particular, for $\beta = 1$ and $\gamma = 1$, we obtain the following interesting criterion for starlikeness:

Corollary 3.5. *If $f(z) \in A$ satisfies*

$$\left|\frac{zf'(z)}{f(z)}-1\right|\left|\frac{zf''(z)}{2f'(z)}\right|<2^{2} \qquad (z\in U),$$

then $f(z) \in S^*$.

Setting $a = c = p$ in Theorem 2.2, we obtain the following sufficient condition for a function $f(z) \in A(p,k)$ to be a p-valent starlike function of order α.

Corollary 3.6. *For all $z \in U$, if $f(z) \in A(p,k)$ satisfies the following condition*

$$\left|\frac{zf'(z)}{pf(z)}-1\right|^{\gamma}\left(\frac{p}{p+1}\right)^{\beta}\left|\frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)-1\right|^{\beta}$$

$$<\begin{cases}\left(1-\frac{\alpha}{p}\right)^{\gamma}\left(1-\frac{\alpha}{p}+\frac{1}{2p}\right)^{\beta}, & 0\leq\alpha\leq\frac{p}{2}\\ \left(1-\frac{\alpha}{p}\right)^{\gamma+\beta}\left(1+\frac{1}{p}\right)^{\beta}, & \frac{p}{2}\leq\alpha<0,\end{cases}$$

for some real numbers α, β and γ with $0 \leq \alpha < p, \beta \geq 0, \gamma \geq 0, \beta+\gamma>0$ then $f(z) \in S_p^*(\alpha)$.

The substition $p = 1$ in Corollary 3.6, yields the following result:

Corollary 3.7. *If $f(z) \in A$ satisfies*

$$\left|\frac{zf'(z)}{f(z)}-1\right|^{\gamma}\left|\frac{zf''(z)}{f'(z)}\right|^{\beta}<\begin{cases}(1-\alpha)^{\gamma}(3-2\alpha)^{\beta}, & 0\leq\alpha<\frac{1}{2}\\ (1-\alpha)^{\gamma+\beta}(4)^{\beta}, & \frac{1}{2}\leq\alpha<1\end{cases}$$

where $z \in U$ and α, β, γ are real numbers with $0 \leq \alpha < 1$, $\beta \geq 0$, $\gamma \geq 0$, $\beta+\gamma>0$, then $f(z) \in S^*(\alpha)$.

In particular, writing $\beta = 1, \gamma = 1$ and $\alpha = 0$ in Corollary 3.7, we obtain the following result:

Corollary 3.8. *If* $f(z) \in A$ *satisfies*

$$\left| \frac{zf''(z)}{f'(z)} \left(\frac{zf'(z)}{f(z)} - 1 \right) \right| < 3, \quad z \in U$$

then $f \in S^*$.

Taking $a = p + 1, c = p$ in Theorem 2.2, we get the following interesting criterion for convexity of multivalent functions:

Corollary 3.9. *If, for all* $z \in U$, *a function* $f(z) \in A(p,k)$ *satisfies*

$$\left| \frac{L_{p,k}(p+2,p)f(z)}{L_{p,k}(p+1,p)f(z)} - 1 \right|^{\gamma} \left| \frac{L_{p,k}(p+3,p)f(z)}{L_{p,k}(p+2,p)f(z)} - 1 \right|^{\beta}$$

$$= \left(\frac{p}{p+1} \right)^{\gamma} \left| \frac{1}{p} \left(1 + \frac{zf''(z)}{f'(z)} \right) - 1 \right|^{\gamma} \left| \frac{z^2 f'''(z) + 6zf'' + 6f'}{(p+2)\left(zf''(z) + 2f'(z)\right)} - 1 \right|^{\beta}$$

$$< \begin{cases} (1 - \frac{\alpha}{p})^{\gamma} \left(1 - \frac{\alpha}{p} + \frac{1}{2p} \right)^{\beta}, & 0 \le \alpha < \frac{p}{2} \\ (1 - \frac{\alpha}{p})^{\gamma+\beta} (1 + \frac{1}{p})^{\beta}, & \frac{p}{2} \le \alpha < p, \end{cases}$$

for some real numbers α, β *and* γ *with* $0 \le \alpha < p, \beta \ge 0, \gamma \ge 0, \beta + \gamma > 0$, *then* $f(z) \in K_p(\alpha)$.

Taking $p = 1$ in Corollary 3.9, we obtain the following sufficient condition for convexity of univalent functions.

Corollary 3.10. *For some non-negative real numbers* α, β *and* γ *with* $\beta + \gamma > 0$ *and* $\alpha < 1$, *if* $f(z)$ *satisfies*

$$\left| \frac{zf''(z)}{f'(z)} \right|^{\gamma} \left| \frac{z^2 f'''(z) + 6z^2 f''(z) + 6f'(z)}{zf''(z) + 2f'(z)} - 1 \right|^{\beta}$$

$$< \begin{cases} (2)^{\gamma} (1-\alpha)^{\gamma} \left(\frac{9}{2} - 3\alpha \right)^{\beta}, & 0 \le \alpha < \frac{1}{2} \\ (2)^{\gamma} (1-\alpha)^{\gamma+\beta} (6)^{\beta}, & \frac{1}{2} \le \alpha < 1, \end{cases}$$

for all $z \in U$, *then* $f \in K(\alpha)$.

In particular, writing $\beta = 1, \gamma = 1$, and $\alpha = 0$ in Corollary 3.10, we obtain the following sufficient condition for convexity of analytic functions:

Corollary 3.11. *If* $f \in A$ *satisfies*

$$\left| \frac{zf''(z)}{f'(z)} \left(\frac{z^2 f'''(z) + 5z^2 f''(z) + 4f'(z)}{zf''(z) + 2f'(z)} \right) \right| < 9, \quad (z \in U)$$

then $f(z) \in K$.

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