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Some multivalent functions with negative coefficients defined by using a certain fractional derivative operator

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Abstract. In this paper we investigate the various important properties and characteristics of the subclasses $S_n(p,q,\alpha,\lambda)$ and $C_n(p,q,\alpha,\lambda)$ of multivalent functions with negative coefficients defined by using a certain operator of fractional derivatives. We also derive many results for the modified Hadamard products of functions belonging to the classes $S_n(p,q,\alpha,\lambda)$ and $C_n(p,q,\alpha,\lambda)$. Finally, several applications involving an integral operator and certain fractional calculus operators are also considered.

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1. Introduction

Let $T(n,p)$ denote the class of functions of the form :

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k \quad (a_k \geq 0; p, n \in N = \{1, 2, ...\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disc $U = \{z : |z| < 1\}$. A function $f(z) \in T(n,p)$ is said to be p-valently starlike of order α if it satisfies the inequality :

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in U;\ 0 \leq \alpha < p; p \in N). \tag{1.2}$$

We denote by $T_n^*(p,\alpha)$ the class of all p-valently starlike functions of order α. Also a function $f(z) \in T(n,p)$ is said to be p-valently convex of order α if it satisfies the inequality:

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha \quad (z\in U;\ 0\leq\alpha<p; p\in N). \tag{1.3}$$

We denote by $C_n(p,\alpha)$ the class of all p-valently convex functions of order α. We note that (see for example Duren [4] and Goodman [5])

$$f(z)\in C_n(p,\alpha) \Longleftrightarrow \frac{zf'(z)}{p}\in T_n^*(p,\alpha) \quad (0\leq\alpha<p; p\in N). \tag{1.4}$$

The classes $T_n^*(p,\alpha)$ and $C_n(p,\alpha)$ are studied by Owa [12].

Various operators of fractional calculus (that is, fractional integral and fractional derivative) have been studied in the literature rather extensively (cf., e.g. [3], [10], [15] and [16]) see also the various references cited therein). For our present investigations, we recall the following definitions.

Definition 1.1. (Fractional Integral Operator). *The fractional integral operator of order λ is defined, for a function $f(z)$, by*

$$D_z^{-\lambda}f(z)=\frac{1}{\Gamma(\lambda)}\int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}}d\zeta \quad (\lambda>0)\,, \tag{1.5}$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containg the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta>0$.

Definition 1.2. (Fractional Derivative Operator). *The fractional derivative of order λ is defined, for a function $f(z)$, by*

$$D_z^{\lambda}f(z)=\frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}}d\zeta \quad (0\leq\lambda<1)\,, \tag{1.6}$$

where $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed, as in Definition 1.1.

Definition 1.3. (Extended Fractional Derivative Operator). *Under the hypotheses of Definition 1.2, the fractional derivative of order $n+\lambda$ is defined, for a function $f(z)$, by*

$$D_z^{n+\lambda}f(z)=\frac{d^n}{dz^n}D_z^{\lambda}f(z) \quad (0\leq\lambda<1; n\in N_0=N\cup\{0\})\,. \tag{1.7}$$

Srivastava and Aouf [15] defined and studied the operator :

$$\Omega_z^{(\lambda,p)}f(z)=\frac{\Gamma(p+1-\lambda)}{\Gamma(p+1)}z^{\lambda}D_z^{\lambda}f(z) \quad (0\leq\lambda\leq1; p\in N)\,. \tag{1.8}$$

For each $f(z)\in T(n,p)\,,$ we have

$$\text{(i)} \qquad \Omega_z^{(\lambda,p)}f(z)=z^p-\sum_{k=n+p}^{\infty}\frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)}a_k z^k\,, \tag{1.9}$$

(ii) $$\left(\Omega_z^{(\lambda,p)} f(z)\right)^{(q)} = \delta(p,q) z^{p-q} - \sum_{k=n+p}^{\infty} \frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)} \delta(k,q) a_k z^{k-q}$$

$$(q \in N_0 = N \cup \{0\}) \tag{1.10}$$

where

$$\delta(p,q) = \begin{cases} 1 & (q = 0) \\ p(p-1)...(p-q+1) & (q \neq 0), \end{cases} \tag{1.11}$$

and

(iii) $$\Omega_z^{(0,p)} f(z) = f(z) \quad \text{and} \quad \Omega_z^{(1,p)} f(z) = \frac{zf'(z)}{p}. \tag{1.12}$$

In this paper we investigate various interesting properties and characteristics of functions belonging to two subclasses $S_n(p,q,\alpha,\lambda)$ and $C_n(p,q,\alpha,\lambda)$ of the class $T(n,p)$, which consist (respectively) of p-valently starlike and p-valently convex functions of order $\alpha\,(0 \leq \alpha < p; p \in N)$. Indeed we have

$$S_n(p,q,\alpha,\lambda) = \left\{ f(z) \in T(n,p) : \text{Re}\left(\frac{z\left(\Omega_z^{(\lambda,p)} f(z)\right)^{(1+q)}}{\left(\Omega_z^{(\lambda,p)} f(z)\right)^{(q)}} \right) > \alpha \right.$$

$$\left. (z \in U; 0 \leq \alpha < p-q; p,n \in N; q \in N_0; p > q) \right\} \tag{1.13}$$

and

$$C_n(p,q,\alpha,\lambda) = \left\{ f(z) \in T(n,p) : \text{Re}\left(1 + \frac{z\left(\Omega_z^{(\lambda,p)} f(z)\right)^{(2+q)}}{\left(\Omega_z^{(\lambda,p)} f(z)\right)^{(1+q)}} \right) > \alpha \right.$$

$$\left. (z \in U; 0 \leq \alpha < p-q; p,n \in N; q \in N_0; p > q) \right\}. \tag{1.14}$$

We note that, by specializing the parameters n,p,q,α and λ, we obtain the following subclasses studied by various authors :

(i) $S_n(p,q,\alpha,0) = S_n(p,q,\alpha)$ and $C_n(p,q,\alpha,0) = C_n(p,q,\alpha)$ (Chen et al. [2]);

(ii) $S_n(p,0,\alpha,0)$ $(0 \leq \alpha < p; p,n \in N)$

$$= \begin{cases} T_n^*(p,\alpha) & \text{(Owa [12])} \\ T_\alpha(p,n) & \text{(Yamakawa [19])}; \end{cases}$$

(iii) $S_n(p,0,\alpha,1) = C_n(p,0,\alpha,0)$ $(0 \leq \alpha < p; p,n \in N)$

$$= \begin{cases} C_n(p,\alpha) & \text{(Owa [12])} \\ CT_\alpha(p,n) & \text{(Yamakawa [19])}; \end{cases}$$

(iv) $S_1(p,0,\alpha,0) = T^*(p,\alpha)$ and $S_n(1,0,\alpha,1) = C_1\ (p,0,\alpha,0) = C(p,\alpha)$, $(0 \leq \alpha < p; p \in N)$ (Owa [11] and Salagean et al. [13]);

(v) $S_1(p,0,\alpha,\beta) = S^*(p,\alpha,\beta)$ and $C_1(p,0,\alpha,\beta) = C^*\ (p,\alpha,\beta)$ $(0 \leq \alpha < p;$ $p \in N;\ 0 \leq \beta < 1)$ (Hossen [7]);

(vi) $S_1(1,0,\alpha,\beta) = T^*(\alpha,\beta)$ and $C_1(1,0,\alpha,\beta) = C(\alpha,\beta)$ $(0 \leq \alpha < 1;\ 0 < \beta \leq 1)$ (Gupta and Jain [6]);

(vii) $S_n(1,0,\alpha,1) = T_\alpha(n)$ and $C_n(1,0,\alpha,1) = C_\alpha(n)\,(0 \le \alpha < 1;\, n \in N)$ (Srivastava et al. [18]);

In our present paper, we shall make use of the familiar operator $J_{c,p}$ defined by (cf. [1], [8] and [9]; see also [17])

$$(J_{c,p}f)(z) = \frac{c+p}{z^c}\int_0^z t^{c-1} f(t)dt\,. \tag{1.15}$$

$$(f(z) \in T(n,p); c > -p; p \in N)$$

as well as the fractional calculus operator D_z^μ for which it is well known that (see, for details, [10] and [15]; see also Section 6 below)

$$D_z^\mu\{z^\rho\} = \frac{\Gamma(\rho+1)}{\Gamma(\rho+1-\mu)} z^{\rho-\mu} \quad (\rho > -1;\ \mu \in R) \tag{1.16}$$

in terms of Gamma functions.

2. Coefficients estimates

Theorem 2.1. *Let the function $f(z)$ be defined by (1.1).*

Then $f(z) \in S_n(p,q,\alpha,\lambda)$ if and only if

$$\sum_{k=n+p}^{\infty} \frac{(k-q-\alpha)\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)} \delta(k,q)a_k \le (p-q-\alpha)\delta(p,q)\,, \tag{2.1}$$

where $\delta(p,q)$ is given by (1.9).

Proof. Assume that the inequality (2.1) holds true. Thus we find that

$$\begin{aligned}
&\left| \frac{z\left(\Omega_z^{(\lambda,p)} f(z)\right)^{(1+q)}}{\left(\Omega_z^{(\lambda,p)} f(z)\right)^{(q)}} - (p-q) \right| \\
&\le \frac{\sum_{k=n+p}^{\infty} \frac{(k-p)\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)} \delta(k,q)a_k\,|z|^{k-p}}{\delta(p,q) - \sum_{k=n+p}^{\infty} \frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)} \delta(k,q)a_k\,|z|^{k-p}} \\
&\le \frac{\sum_{k=n+p}^{\infty} \frac{(k-p)\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)} \delta(k,q)a_k}{\delta(p,q) - \sum_{k=n+p}^{\infty} \frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)} \delta(k,q)a_k} \\
&\le p-q-\alpha\,.
\end{aligned}$$

This shows that the values of the function

$$\Phi(z) = \frac{z\left(\Omega_z^{(\lambda,p)} f(z)\right)^{(1+q)}}{\left(\Omega_z^{(\lambda,p)} f(z)\right)^{(q)}} \tag{2.2}$$

lie in a circle which is centered at $w = (p-q)$ and whose radius is $(p-q-\alpha)$. Hence $f(z)$ satisfies the condition (1.11).

Conversely, assume that the function $f(z)$ defined by (1.1) is in the class $S_n(p,q,\alpha,\lambda)$. Then we have

$$\operatorname{Re}\left\{\frac{z\left(\Omega_z^{(\lambda,p)}f(z)\right)^{(1+q)}}{\left(\Omega_z^{(\lambda,p)}f(z)\right)^{(q)}}\right\}$$

$$= \operatorname{Re}\left\{\frac{(p-q)\delta(p,q) - \sum_{k=n+p}^{\infty}\frac{(k-q)\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)}\delta(k,q)a_k z^{k-p}}{\delta(p,q) - \sum_{k=n+p}^{\infty}\frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)}\delta(k,q)a_k z^{k-p}}\right\} > \alpha \tag{2.3}$$

for some $\alpha\,(0 \le \alpha < p-q)$, $0 \le \lambda \le 1$, $p, n \in N$, $q \in N_0$, $p > q$ and $z \in U$. Choose values of z on the real axis so that $\Phi(z)$ given by (2.2) is real. Upon clearing the denominator in (2.3) and letting $z \to 1^-$ through real values, we can see that

$$(p-q)\delta(p,q) - \sum_{k=n+p}^{\infty}\frac{(k-q)\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)}\delta(k,q)a_k$$

$$\ge \alpha\left(\delta(p,q) - \sum_{k=n+p}^{\infty}\frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)}\delta(k,q)a_k\right). \tag{2.4}$$

Thus we have the inequality (2.1). □

Corollary 2.2. *Let the function $f(z)$ defined by (1.1) be in the class $S_n(p,q,\alpha,\lambda)$. Then*

$$a_k \le \frac{(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(k+1-\lambda)}{(k-q-\alpha)\delta(k,q)\Gamma(k+1)\Gamma(p+1-\lambda)}$$

$$(k \ge n+p; p, n \in N; q \in N_0; p > q). \tag{2.5}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(k+1-\lambda)}{(k-q-\alpha)\delta(k,q)\Gamma(k+1)\Gamma(p+1-\lambda)}z^k$$

$$(k \ge n+p; p, n \in N; q \in N_0; p > q). \tag{2.6}$$

Theorem 2.3. *Let the function $f(z)$ defined by (1.1). Then $f(z) \in C_n(p,q,\alpha,\lambda)$ if and only if*

$$\sum_{k=n+p}^{\infty}\frac{(k-q-\alpha)\Gamma(k+1)\Gamma(p+1-\lambda)}{(p-q-\alpha)\Gamma(p+1)\Gamma(k+1-\lambda)}\delta(k,q+1)a_k \le (p-q-\alpha)\delta(p,q+1). \tag{2.7}$$

Corollary 2.4. *Let the function $f(z)$ defined by (1.1) be in the class $C_n(p,q,\alpha,\lambda)$. Then*

$$a_k \le \frac{(p-q-\alpha)\delta(p,q+1)\Gamma(p+1)\Gamma(k+1-\lambda)}{(k-q-\alpha)\delta(k,q+1)\Gamma(k+1)\Gamma(p+1-\lambda)} \quad (k \ge n+p; p,n \in N; q \in N_0; p > q). \tag{2.8}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{(p-q-\alpha)\delta(p,q+1)\Gamma(p+1)\Gamma(k+1-\lambda)}{(k-q-\alpha)\delta(k,q+1)\Gamma(k+1)\Gamma(p+1-\lambda)} z^k \quad (k \ge n+p; p,n \in N; q \in N_0; p > q). \tag{2.9}$$

3. Distortion theorems

Theorem 3.1. *If a function $f(z)$ defined by (1.1) is in the class $S_n(p,q,\alpha,\lambda)$, then*

$$\left\{\frac{p!}{(p-j)!} - \frac{(p-q-\alpha)\delta(p,q)(n+p-q)!\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(n+p-q-\alpha)(n+p-j)!\Gamma(n+p+1)\Gamma(p+1-\lambda)}|z|^n\right\}|z|^{p-j}$$

$$\le \left|f^{(j)}(z)\right| \le \left\{\frac{p!}{(p-j)!} + \frac{(p-q-\alpha)\delta(p,q)(n+p-q)!\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(n+p-q-\alpha)(n+p-j)!\Gamma(n+p+1)\Gamma(p+1-\lambda)}|z|^n\right\}|z|^{p-j}$$

$$(z \in U; 0 \le \alpha < p-q;\ p,n \in N;\ q,j \in N_0;\ p > \max\{q,j\}). \tag{3.1}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(n+p-q-\alpha)\delta(n+p,q)\Gamma(n+p+1)\Gamma(p+1-\lambda)} z^{n+p}\ (p,n \in N). \tag{3.2}$$

Proof. In view of Theorem 2.1, we have

$$\frac{(n+p-q-\alpha)\delta(n+p,q)\Gamma(n+p+1)\Gamma(p+1-\lambda)}{(p-q-\alpha)\delta(p,q)(n+p)!\Gamma(p+1)\Gamma(n+p+1-\lambda)} \sum_{k=n+p}^{\infty} k!a_k$$

$$\le \sum_{k=n+p}^{\infty} \frac{(k-q-\alpha)\delta(k,q)\Gamma(k+1)\Gamma(p+1-\lambda)}{(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(k+1-\lambda)} a_k \le 1$$

which readily yields

$$\sum_{k=n+p}^{\infty} k!a_k \le \frac{(p-q-\alpha)\delta(p,q)(n+p-q)!\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(n+p-q-\alpha)\Gamma(n+p+1)\Gamma(p+1-\lambda)}. \tag{3.3}$$

Now, by differentiating both sides of (1.1) j times, we obtain

$$f^{(j)}(z) = \frac{p!}{(p-j)!} z^{p-j} - \sum_{k=n+p}^{\infty} \frac{k!}{(k-j)!} a_k z^{k-j} \tag{3.4}$$

$$(k \ge n+p; p,n \in N; q,j \in N_0 = N \cup \{0\}; p > \max\{q,j\}).$$

Theorem 3.1 follows readily from (3.3) and (3.4).

Finally, it is easy to see that the bounds in (3.1) are attained for the function $f(z)$ given by (3.2). □

Theorem 3.2. *If a function $f(z)$ defined by (1.1) is in the class $C_n(p,q,\alpha,\lambda)$, then*

$$\left\{\frac{1}{(p-j)!}-\frac{(p-q-\alpha)(n+p-q-1)!\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(n+p-q-\alpha)(p-q-1)!(n+p-j)!\Gamma(n+p+1)\Gamma(p+1-\lambda)}\,|z|^n\right\}p!\,|z|^{p-j}$$

$$\leq \left|f^{(j)}(z)\right|$$

$$\leq \left\{\frac{1}{(p-j)!}+\frac{(p-q-\alpha)(n+p-q-1)!\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(n+p-q-\alpha)(p-q-1)!(n+p-j)!\Gamma(n+p+1)\Gamma(p+1-\lambda)}\,|z|^n\right\}p!\,|z|^{p-j}$$

$$\left(z\in U; 0\leq\alpha<p-q;\ p,n\in N;\ q,j\in N_0;\ p>\max\{q,j\}\right). \tag{3.5}$$

The result is sharp for the function $f(z)$ given by

$$f(z)=z^p-\frac{(p-q-\alpha)\delta(p,q+1)}{(n+p-q-\alpha)\delta(n+p,q+1)}z^{n+p}\ \ (p,n\in N; q\in N_0; p>q). \tag{3.6}$$

Radii of close-to-convexity, starlikeness and convexity

Theorem 3.3. *Let the function $f(z)$ defined by (1.1) be in the class $S_n(p,q,\alpha,\lambda)$, then*

(i) $f(z)$ is p-valently close-to-convex of order $\varphi\,(0\leq\varphi<p)$ in $|z|<r_1$, where

$$r_1=\inf_k\left\{\frac{(k-q-\alpha)\delta(k,q)\Gamma(k+1)\Gamma(p+1-\lambda)}{(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(k+1-\lambda)}\left(\frac{p-\varphi}{k}\right)\right\}^{\frac{1}{k-p}}$$

$$(k\geq n+p; p,n\in N; q\in N_0; p>q), \tag{4.1}$$

(ii) $f(z)$ is p-valently starlike of order $\varphi\,(0\leq\varphi<p)$ in $|z|<r_2$, where

$$r_2=\inf_k\left\{\frac{(k-q-\alpha)\delta(k,q)\Gamma(k+1)\Gamma(p+1-\lambda)}{(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(k+1-\lambda)}\left(\frac{p-\varphi}{k-\varphi}\right)\right\}^{\frac{1}{k-p}}$$

$$(k\geq n+p; p,n\in N; q\in N_0; p>q), \tag{4.2}$$

(iii) $f(z)$ is p-valently convex of order $\varphi\,(0\leq\varphi<p)$ in $|z|<r_3$, where

$$r_3=\inf_k\left\{\frac{(k-q-\alpha)\delta(k,q)\Gamma(k+1)\Gamma(p+1-\lambda)}{(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(k+1-\lambda)}\cdot\frac{p(p-\varphi)}{k(k-\varphi)}\right\}^{\frac{1}{k-p}}$$

$$(k\geq n+p; p,n\in N; q\in N_0; p>q). \tag{4.3}$$

Each of these results is sharp for the function $f(z)$ given by (2.6).

Proof. It is sufficient to show that

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|\leq p-\varphi\ \ (|z|<r_1; 0\leq\varphi<p; p\in N), \tag{4.4}$$

$$\left|\frac{zf'(z)}{f(z)}-p\right|\leq p-\varphi\ \ (|z|<r_2; 0\leq\varphi<p; p\in N), \tag{4.5}$$

and that

$$\left|1+\frac{zf''(z)}{f'(z)}-p\right| \le p-\varphi \quad (|z|<r_3; 0\le\varphi<p; p\in N) \tag{4.6}$$

for a function $f(z)\in S_n(p,q,\alpha,\lambda)$, where r_1, r_2 and r_3 are defined by (4.1), (4.2) and (4.3), respectively. □

4. Modified Hadamard products

For the functions $f_\nu(z)\,(\nu=1,2)$ given by

$$f_\nu(z)=z^p-\sum_{k=n+p}^{\infty} a_{k,\nu}z^k \quad (a_{k,\nu}\ge 0; \nu=1,2) \tag{5.1}$$

we denote that $(f_1\otimes f_2)(z)$ the modified Hadamard product (or convolution) of the functions $f_1(z)$ and $f_2(z)$, defined by

$$(f_1\otimes f_2)(z)=z^p-\sum_{k=n+p}^{\infty} a_{k,1}.a_{k,2}z^k\,. \tag{5.2}$$

Theorem 4.1. *Let the functions $f_\nu(z)\,(\nu=1,2)$ defined by (5.1) be in the class $S_n(p,q,\alpha,\lambda)$. Then $(f_1\otimes f_2)(z)\in S_n(p,q,\gamma,\lambda)$ where*

$$\gamma=(p-q)-$$

$$\frac{n(p-q-\alpha)^2\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(n+p-q-\alpha)^2\delta(n+p,q)\Gamma(n+p+1)\Gamma(p+1-\lambda)-(p-q-\alpha)^2\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}\,. \tag{5.3}$$

The result is sharp for the functions $f_\nu(z)\,(\nu=1,2)$ given by

$$f_\nu(z)=z^p-\frac{(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(n+p-q-\alpha)\delta(n+p,q)\Gamma(n+p+1)\Gamma(p+1-\lambda)}z^{n+p}$$

$$(p,n\in N; \nu=1,2)\,. \tag{5.4}$$

Proof. Emploing the technique used earlier by Schild and Silverman [14], we need to find the largest γ such that

$$\sum_{k=n+p}^{\infty}\frac{(k-q-\gamma)\delta(k,q)\Gamma(k+1)\Gamma(k+1)\Gamma(p+1-\lambda)}{(p-q-\gamma)\delta(p,q)\Gamma(p+1)\Gamma(k+1-\lambda)}a_{k,1};a_{k,2}\le 1$$

$$(f_\nu(z)\in S_n(p,\alpha,\beta,\lambda) \quad (\nu=1,2))\,. \tag{5.5}$$

Since $f_\nu(z)\in S_n(p,\alpha,\beta,\lambda)$ $(\nu=1,2)$, we readily see that

$$\sum_{k=n+p}^{\infty}\frac{(k-q-\alpha)\delta(k,q)\Gamma(k+1)\Gamma(p+1-\lambda)}{(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(k+1-\lambda)}a_{k,\nu}\le 1 \quad (\nu=1,2). \tag{5.6}$$

Therefore, by the Cauchy-Schwarz inequality, we obtain

$$\sum_{k=n+p}^{\infty}\frac{(k-q-\alpha)\delta(k,q)\Gamma(k+1)\Gamma(p+1-\lambda)}{(p-q-\gamma)\delta(p,q)\Gamma(p+1)\Gamma(k+1-\lambda)}\sqrt{a_{k,1};a_{k,2}}\le 1\,. \tag{5.7}$$

Thus we only need to show that

$$\frac{(k-q-\gamma)}{(p-q-\gamma)}a_{k,1}.a_{k,2} \leq \frac{(k-q-\alpha)}{(p-q-\alpha)}\sqrt{a_{k,1}.a_{k,2}} \;\; (k \geq n+p; p,n \in N)\,, \quad (5.8)$$

or, equivalently, that

$$\sqrt{a_{k,1}.a_{k,2}} \leq \frac{(p-q-\gamma)(k-q-\alpha)}{(p-\alpha)(k-q-\gamma)} \quad (k \geq n+p; p,n \in N)\,. \qquad (5.9)$$

Hence, in light of the inequality (5.7), it is sufficient to prove that

$$\frac{(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(k+1-\lambda)}{(k-q-\alpha)\delta(k,q)\Gamma(k+1)\Gamma(p+1-\lambda)} \leq$$

$$\frac{(p-\gamma)(k-q-\alpha)}{(p-\alpha)(k-q-\gamma)} \quad (k \geq n+p; p,n \in N)\,. \qquad (5.10)$$

It follows from (5.10) that

$$\gamma \leq (p-q)-$$

$$\frac{(k-p)(p-q-\alpha)^2\delta(p,q)\Gamma(p+1)\Gamma(k+1-\lambda)}{(k-q-\alpha)^2\delta(k,q)\Gamma(k+1)\Gamma(p+1-\lambda)-(p-q-\alpha)^2\delta(p,q)\Gamma(p+1)\Gamma(k+1-\lambda)}$$

$$(k \geq n+p;\; p,n \in N;\; q \in N_0;\; p > q)\,. \qquad (5.11)$$

Now, defining the function $\theta(k)$ by

$$\theta(k) = (p-q)-$$

$$\frac{(k-p)(p-q-\alpha)^2\delta(p,q)\Gamma(p+1)\Gamma(k+1-\lambda)}{(k-q-\alpha)^2\delta(k,q)\Gamma(k+1)\Gamma(p+1-\lambda)-(p-q-\alpha)^2\delta(p,q)\Gamma(p+1)\Gamma(k+1-\lambda)}$$

$$(k \geq n+p;\; p,n \in N;\; q \in N_0;\; p > q)\,, \qquad (5.12)$$

we see that $\theta(k)$ is an increasing function of k. Therefore, we conclude that

$$\gamma \leq \theta(n+p) = (p-q)-$$

$$\frac{n(p-q-\alpha)^2\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(n+p-q-\alpha)^2\delta(n+p,q)\Gamma(n+p+1)\Gamma(p+1-\lambda)-(p-q-\alpha)^2\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)} \quad (5.13)$$

which evidently completes the proof of Theorem 4.1. □

Putting $\lambda = 0$ in Theorem 4.1, we obtain

Corollary 4.2. *Let the functions $f_\nu(z)\,(\nu = 1,2)$ defined by (5.1) be in the class $S_n(p,q,\alpha)$. Then $(f_1 \otimes f_2)(z) \in S_n(p,q,\gamma)$, where*

$$\gamma = (p-q) - \frac{n(p-q-\alpha)^2\delta(p,q)}{(n+p-q-\alpha)^2\delta(n+p,q)-(p-q-\alpha)^2\delta(p,q)}\,. \qquad (5.14)$$

The result is sharp.

Remark 4.3. We note that the result obtained by Chen et al. [2, Theorem 5] is not correct. The correct result is given by (5.14).

Using arguments similar to those in the proof of Theorem 4.1, we obtain the following results.

Theorem 4.4. *Let the functions $f_\nu(z)\,(\nu=1,2)$ defined by (5.1) be in the class $C_n(p,q,\alpha,\lambda)$. Then $(f_1\otimes f_2)(z)\in C_n(p,q,\gamma,\lambda)$ where*

$$\gamma=(p-q)-\frac{n(p-q-\alpha)^2\delta(p,q+1)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(n+p-q-\alpha)^2\delta(n+p,q+1)\Gamma(n+p+1)\Gamma(p+1-\lambda)-(p-q-\alpha)^2\delta(p,q+1)\Gamma(p+1)\Gamma(n+p+1-\lambda)}. \tag{5.15}$$

The result is sharp for the functions $f_\nu(z)\,(\nu=1,2)$ given by

$$f(z)=z^p-\frac{(p-q-\alpha)\delta(p,q+1)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(n+p-q-\alpha)\delta(n+p,q+1)\Gamma(n+p+1)\Gamma(p+1-\lambda)}z^{n+p}$$
$$(p,n\in N;\ q\in N_0;\ p>q;\ \nu=1,2). \tag{5.16}$$

Putting $\lambda=0$ in Theorem 4.4, we obtain

Corollary 4.5. *Let the functions $f_\nu(z)\,(\nu=1,2)$ defined by (5.1) be in the class $C_n(p,q,\alpha)$. Then $(f_1\otimes f_2)(z)\in C_n(p,q,\gamma)$, where*

$$\gamma=(p-q)-\frac{n(p-q-\alpha)^2\delta(p,q+1)}{(n+p-q-\alpha)^2\delta(n+p,q+1)-(p-q-\alpha)^2\delta(p,q+1)}. \tag{5.17}$$

The result is sharp.

Remark 4.6. We note that the result obtained by Chen et al. [2, Theorem 6] is not correct. The correct result is given by (5.17).

Theorem 4.7. *Let the function $f_1(z)$ defined by (5.1) be in the class $S_n(p,q,\alpha,\lambda)$. Suppose also that the function $f_2(z)$ defined by (5.2) be in the class $S_n(p,q,\gamma,\lambda)$. Then $(f_1\otimes f_2)(z)\in S_n(p,q,\zeta,\lambda)$, where $\zeta=(p-q)-$*

$$\frac{n(p-q-\alpha)(p-q-\gamma)\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(n+p-q-\alpha)(n+p-q-\gamma)\delta(n+p,q)\Gamma(n+p+1)\Gamma(p+1-\lambda)-\Omega}.$$
$$(\Omega=(p-q-\alpha)(p-q-\gamma)\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)). \tag{5.18}$$

The result is sharp for the functions $f_\nu(z)\,(\nu=1,2)$ given by

$$f_1(z)=z^p-\frac{(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(n+p-q-\alpha)\delta(n+p,q)\Gamma(n+p+1)\Gamma(p+1-\lambda)}z^{n+p}\quad(p,n\in N) \tag{5.19}$$

and

$$f_2(z)=z^p-\frac{(p-q-\gamma)\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(n+p-q-\gamma)\delta(n+p,q)\Gamma(n+p+1)\Gamma(p+1-\lambda)}z^{n+p}\quad(p,n\in N). \tag{5.20}$$

Theorem 4.8. *Let the functions $f_\nu(z)\,(\nu=1,2)$ defined by (5.1) be in the class $S_n(p,q,\alpha,\lambda)$. Then the function*

$$h(z)=z^p-\sum_{k=n+p}^{\infty}\left(a_{k,1}^2+a_{k,2}^2\right)z^k \tag{5.21}$$

belongs to the class $S_n(p,q,\xi,\lambda)$, where

$$\xi=(p-q)-\frac{2n(p-q-\alpha)^2\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(n+p-q-\alpha)^2\delta(n+p,q)\Gamma(n+p+1)\Gamma(p+1-\lambda)-2(p-q-\alpha)^2\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}. \tag{5.22}$$

The result is sharp for the functions $f_\nu(z)\,(\nu=1,2)$ defined by (5.4).

Theorem 4.9. *Let the functions* $f_\nu(z)\,(\nu = 1,2)$ *defined by (5.1) be in the class* $C_n(p,q,\alpha,\lambda)$. *Then the function* $h(z)$ *defined by (5.21) belongs to the class* $C_n(p,q,\eta,\lambda)$, *where*

$$\eta = (p-q)- \frac{2n(p-q-\alpha)^2\delta(p,q+1)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(n+p-q-\alpha)^2\delta(n+p,q+1)\Gamma(n+p+1)\Gamma(p+1-\lambda)-2(p-q-\alpha)^2\delta(p,q+1)\Gamma(p+1)\Gamma(n+p+1-\lambda)}. \tag{5.23}$$

The result is sharp for the functions $f_\nu(z)\,(\nu = 1,2)$ *given by (5.16).*

5. Applications of fractional calculus

In this section, we shall investigate the growth and distortion properties of functions in the classes $S_n(p,q,\alpha,\beta)$ and $C_n(p,q,\alpha,\beta)$, involving the operators $J_{c,p}$ and D_z^μ. In order to derive our results, we need the following lemma given by Chen et al. [3].

Lemma 5.1. (see Chen et al. [3]). *Let the function* $f(z)$ *defined by (1.1). Then*

$$D_z^\mu\{(J_{c,p})(z)\} = \frac{\Gamma(p+1)}{\Gamma(p+1-\mu} z^{p-\mu} - \sum_{k=n+p}^{\infty} \frac{(c+p)\Gamma(k+1)}{(c+k)\Gamma(k+1-\mu)} a_k z^{k-\mu}$$

$$(\mu \in R; c > -p; p,n \in N) \tag{6.1}$$

and

$$J_{c,p}(D_z^\mu\{f(z)\}) = \frac{(c+p)\Gamma(p+1)}{(c+p-\mu)\Gamma(p+1-\mu)} z^{p-\mu} - \sum_{k=n+p}^{\infty} \frac{(c+p)\Gamma(k+1)}{(c+k-\mu)\Gamma(k+1-\mu)} a_k z^{k-\mu}$$

$$(\mu \in R; c > -p; p,n \in N), \tag{6.2}$$

provided that no zeros appear in the denominators in (6.1) and (6.2).

Theorem 5.2. *Let the function* $f(z)$ *defined by (1.1) be in the class* $S_n(p,q,\alpha,\lambda)$. *Then*

$$\left|D_z^{-\mu}\{(J_{c,p}f)(z)\}\right| \geq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1+\mu)} - \frac{(c+p)(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(c+n+p)(n+p-q-\alpha)\delta(n+p,q)\Gamma(n+p+1+\mu)\Gamma(p+1-\lambda)}|z|^n\right\}|z|^{p+\mu}$$

$$(z \in U; 0 \leq \alpha < p-q; 0 \leq \lambda \leq 1; \mu > 0; c > -p; p,n \in N; q \in N_0; p > q) \tag{6.3}$$

and

$$\left|D_z^{-\mu}\{(J_{c,p}f)(z)\}\right| \leq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1+\mu)} + \frac{(c+p)(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(c+n+p)(n+p-q-\alpha)\delta(n+p,q)\Gamma(n+p+1+\mu)\Gamma(p+1-\lambda)}|z|^n\right\}|z|^{p+\mu}$$

$$(z \in U; 0 \leq \alpha < p-q; 0 \leq \lambda \leq 1; \mu > 0; c > -p; p,n \in N; q \in N_0; p > q). \tag{6.4}$$

Each of the assertions (6.3) and (6.4) is sharp.

Proof. In view of Theorem 2.1, we have

$$\frac{(n+p-q-\alpha)\delta(n+p,q)\Gamma(n+p+1)\Gamma(p+1-\lambda)}{(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}\sum_{k=n+p}^{\infty}a_k$$

$$\leq \sum_{k=n}^{\infty}\frac{(k-q-\alpha)\delta(k,q)\Gamma(k+1)\Gamma(p+1-\lambda)}{(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(k+1-\lambda)}a_k \leq 1\,,$$

which readily yields

$$\sum_{k=n+p}^{\infty}a_k \leq \frac{(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(n+p-q-\alpha)\delta(n+p,q)\Gamma(n+p+1)\Gamma(p+1-\lambda)}\,. \tag{6.5}$$

Consider the function $F(z)$ defined in U by

$$\begin{aligned} F(z) &= \frac{\Gamma(p+1+\mu)}{\Gamma(p+1)}z^{-\mu}D_z^{-\mu}\{(J_{c,p}f)(z)\} \\ &= z^p - \sum_{k=n+p}^{\infty}\frac{(c+p)\Gamma(k+1)\Gamma(p+1+\mu)}{(c+k)\Gamma(p+1)\Gamma(k+1+\mu)}a_k z^k \\ &= z^p - \sum_{k=n+p}^{\infty}\Phi(k)a_k z^k \qquad (z\in U)\,, \end{aligned}$$

where

$$\Phi(k)=\frac{(c+p)\Gamma(k+1)\Gamma(p+1+\mu)}{(c+k)\Gamma(p+1)\Gamma(k+1+\mu)} \quad (k\geq n+p; p,n\in N; \mu>0)\,. \tag{6.6}$$

Since $\Phi(k)$ is a decreasing function of k when $\mu > 0$, we get

$$0<\Phi(k)\leq\Phi(n+p)=\frac{(c+p)\Gamma(n+p+1)\Gamma(p+1+\mu)}{(c+n+p)\Gamma(p+1)\Gamma(n+p+1+\mu)}$$
$$(c>-p; p,n\in N; \mu>0)\,. \tag{6.7}$$

Thus, by using (6.5) and (6.7), we deduce that

$$|F(z)|\geq |z|^p-\Phi(n+p)\,|z|^{n+p}\sum_{k=n+p}^{\infty}a_k \geq |z|^p -$$
$$\frac{(c+p)(p-q-\alpha)\delta(p,q)\Gamma(p+1+\mu)\Gamma(n+p+1-\lambda)}{(c+n+p)(n+p-q-\alpha)\delta(n+p,q)\Gamma(n+p+1+\mu)\Gamma(p+1-\lambda)}\,|z|^{n+p} \quad (z\in U)$$

and

$$|F(z)|\leq |z|^p+\Phi(n+p)\,|z|^{n+p}\sum_{k=n+p}^{\infty}a_k \leq |z|^p +$$
$$\frac{(c+p)(p-q-\alpha)\delta(p,q)\Gamma(p+1+\mu)\Gamma(n+p+1-\lambda)}{(c+n+p)(n+p-q-\alpha)\delta(n+p,q)\Gamma(n+p+1+\mu)\Gamma(p+1-\lambda)}\,|z|^{n+p} \quad (z\in U),$$

which yield the inequalities (6.3) and (6.4) of Theorem 5.2. The equalities in (6.3) and (6.4) are attained for the function $f(z)$ given by

$$D_z^{-\mu}\{(J_{c,p}f)(z)\}\leq\left\{\frac{\Gamma(p+1)}{\Gamma(p+1+\mu)}-\right.$$

$$\left.\frac{(c+p)(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(c+n+p)(n+p-q-\alpha)\delta(n+p,q)\Gamma(n+p+1)\Gamma(p+1-\lambda)}z^{n}\right\} z^{p+\mu} \tag{6.8}$$

or, equivalently, by

$$(J_{c,p}f)(z) = z^{p} - \frac{(c+p)(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(c+n+p)(n+p-q-\alpha)\delta(n+p,q)\Gamma(n+p+1)\Gamma(p+1-\lambda)}z^{n+p}. \tag{6.9}$$

Thus we complete the proof of Theorem 5.2. □

Theorem 5.3. *Let the function $f(z)$ defined by (1.1) be in the class $S_n(p,q,\alpha,\lambda)$. Then*

$$|D_z^{\mu}\{(J_{c,p}f)(z)\}| \geq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1-\mu)} - \frac{(c+p)(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(c+n+p)(n+p-q-\alpha)\delta(n+p,q)\Gamma(n+p+1-\mu)\Gamma(p+1-\lambda)}|z|^{n}\right\}|z|^{p-\mu}$$

$$(z \in U; 0 \leq \alpha < p-q; 0 \leq \lambda \leq 1; 0 \leq \mu < 1; c > -p; p,n \in N; q \in N_0; p > q) \tag{6.10}$$

and

$$|D_z^{\mu}\{(J_{c,p}f)(z)\}| \leq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1-\mu)} + \frac{(c+p)(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(c+n+p)(n+p-q-\alpha)\delta(n+p,q)\Gamma(n+p+1-\mu)\Gamma(p+1-\lambda)}|z|^{n}\right\}|z|^{p-\mu}$$

$$(z \in U; 0 \leq \alpha < p-q; 0 \leq \lambda \leq 1; 0 \leq \mu < 1; c > -p; p,n \in N; q \in N_0; p > q). \tag{6.11}$$

Each of the assertions (6.10) and (6.11) is sharp.

Proof. It follows from Theorem 2.1, that

$$\sum_{k=n+p}^{\infty} k a_k \leq \frac{(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(n+p-q-\alpha)\delta(n+p,q)\Gamma(n+p)\Gamma(p+1-\lambda)}. \tag{6.12}$$

We consider the function $H(z)$ defined in U by

$$\begin{aligned} H(z) &= \frac{\Gamma(p+1-\mu)}{\Gamma(p+1)} z^{\mu} D_z^{\mu}\{(J_{c,p}f)(z)\} \\ &= z^{p} - \sum_{k=n+p}^{\infty} \Psi(k) k a_k z^{k} \quad (z \in U), \end{aligned}$$

where, for convience,

$$\Psi(k) = \frac{(c+p)\Gamma(k)\Gamma(p+1-\mu)}{(c+k)\Gamma(p+1)\Gamma(k+1-\mu)} \quad (k \geq n+p; p,n \in N; 0 \leq \mu < 1).$$

Since $\Psi(k)$ is a decreasing function of k when $\mu < 1$, we find that

$$0 < \Psi(k) \leq \Psi(n+p) = \frac{(c+p)\Gamma(n+p)\Gamma(p+1-\mu)}{(c+n+p)\Gamma(p+1)\Gamma(n+p+1-\mu)}$$

$$(c > -p; p,n \in N; 0 \leq \mu < 1). \tag{6.13}$$

Consequently, with the aid of (6.12) and (6.13), we find that

$$|H(z)| \geq |z|^p - \Psi(n+p)\,|z|^{n+p} \sum_{k=n+p}^{\infty} k a_k \geq |z|^p - \frac{(c+p)(p-q-\alpha)\delta(p,q)\Gamma(p+1-\mu)\Gamma(n+p+1-\lambda)}{(c+n+p)(n+p-q-\alpha)\delta(n+p,q)\Gamma(n+p+1-\mu)\Gamma(p+1-\lambda)}\,|z|^{n+p} \quad (z \in U)$$

and

$$|H(z)| \leq |z|^p + \Psi(n+p)\,|z|^{n+p} \sum_{k=n+p}^{\infty} k a_k \leq |z|^p + \frac{(c+p)(p-q-\alpha)\delta(p,q)\Gamma(p+1-\mu)\Gamma(n+p+1-\lambda)}{(c+n+p)(n+p-q-\alpha)\delta(n+p,q)\Gamma(n+p+1-\mu)\Gamma(p+1-\lambda)}\,|z|^{n+p} \quad (z \in U)$$

which yield the inequalities (6.10) and (6.11) of Theorem 5.3. The equalities in (6.10) and (6.11) are attained for the function $f(z)$ given by

$$D_z^{\mu}\{(J_{c,p}f)(z)\} = \left\{\frac{\Gamma(p+1)}{\Gamma(p+1-\mu)} - \frac{(c+p)(p-q-\alpha)\delta(p,q)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(c+n+p)(n+p-q-\alpha)\delta(n+p,q)\Gamma(n+p+1-\mu)\Gamma(p+1-\lambda)} z^n\right\} z^{p-\mu} \tag{6.14}$$

or for the function $(J_{c,p}f)(z)$ given by (6.9). The proof of Theorem 5.3 is thus completed. □

Theorem 5.4. *Let the function $f(z)$ defined by (1.1) be in the class $C_n(p,q,\alpha,\lambda)$. Then for $z \in U$; $0 \leq \alpha < p-q$; $0 \leq \lambda \leq 1$; $\mu > 0$; $c > 0$; $p, n \in N$; $q \in N_0$ and $p > q$, we have*

$$\left|D_z^{-\mu}\{(J_{c,p}f)(z)\}\right| \geq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1+\mu)} - \frac{(c+p)(p-q-\alpha)\delta(p,q+1)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(c+n+p)(n+p-q-\alpha)\delta(n+p,q+1)\Gamma(n+p+1+\mu)\Gamma(p+1-\lambda)}\,|z|^n\right\} |z|^{p-\mu} \tag{6.15}$$

and

$$\left|D_z^{-\mu}\{(J_{c,p}f)(z)\}\right| \leq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1-\mu)} + \frac{(c+p)(p-q-\alpha)\delta(p,q+1)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(c+n+p)(n+p-q-\alpha)\delta(n+p,q+1)\Gamma(n+p+1+\mu)\Gamma(p+1-\lambda)}\,|z|^n\right\} |z|^{p-\mu}. \tag{6.16}$$

Also for $z \in U$; $0 \leq \alpha < p-q$; $0 \leq \lambda \leq 1$; $0 \leq \mu < 1$; $c > -p$; $p, n \in N$; $q \in N_0$ and $p > q$, we have

$$\left|D_z^{\mu}\{(J_{c,p}f)(z)\}\right| \geq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1-\mu)} - \frac{(c+p)(p-q-\alpha)\delta(p,q+1)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(c+n+p)(n+p-q-\alpha)\delta(n+p,q+1)\Gamma(n+p+1-\mu)\Gamma(p+1-\lambda)}\,|z|^n\right\} |z|^{p-\mu} \tag{6.17}$$

and

$$\left|D_z^{\mu}\{(J_{c,p}f)(z)\}\right| \leq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1-\mu)} + \frac{(c+p)(p-q-\alpha)\delta(p,q+1)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(c+n+p)(n+p-q-\alpha)\delta(n+p,q+1)\Gamma(n+p+1-\mu)\Gamma(p+1-\lambda)}\,|z|^n\right\} |z|^{p-\mu}. \tag{6.18}$$

The equalities in (6.15), (6.16), (6.17) and (6.18) are attained for the function $f(z)$ given by

$$(J_{c,p}f)(z) = z^p - \frac{(c+p)(p-q-\alpha)\delta(p,q+1)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(c+n+p)(n+p-q-\alpha)\delta(n+p,q+1)\Gamma(n+p+1)\Gamma(p+1-\lambda)} z^{n+p}. \quad (6.19)$$

Remark 5.5. Putting $\lambda = 0$ in Theorems 5.2, 5.3, and 5.4, we obtain the corresponding results for the classes $S_n(p, q, \alpha)$ and $C_n(p, q, \alpha)$, respectively.

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