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Periodic solutions in totally nonlinear difference equations with functional delay

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Abstract. We use the modification of Krasnoselskii's fixed point theorem due to T. A. Burton ([1] Theorem 3) to show that the totally nonlinear difference equation with functional delay

$$\triangle x(t) = -a(t) x^3(t+1) + G\left(t, x^3(t), x^3(t-g(t))\right), \ \forall t \in \mathbb{Z},$$

has periodic solutions. We invert this equation to construct a sum of a compact map and a large contraction which is suitable for applying Krasnoselskii-Burton theorem. Finally, an example is given to illustrate our result.

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1. Introduction

In this paper, we are interested in the analysis of qualitative theory of periodic solutions of difference equation. Motivated by the papers [3], [5]-[7] and the references therein, we consider the following totally nonlinear difference equation with functional delay

$$\triangle x(t) = -a(t) x^3(t+1) + G\left(t, x^3(t), x^3(t-g(t))\right), \ \forall t \in \mathbb{Z}, \tag{1.1}$$

where

$$G : \mathbb{Z} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},$$

with $\mathbb{Z}$ is the set of integers and $\mathbb{R}$ is the set of real numbers. Throughout this paper $\triangle$ denotes the forward difference operator $\triangle x(t) = x(t+1) - x(t)$ for any sequence $\{x(t), \ t \in \mathbb{Z}\}$. For more on the calculus of difference equations, we refer the reader to [4]. The equation (1.1) is totally nonlinear and we have to add a linear term to both sides of the equation. Although the added term destroys a contraction already present but it will be replaced it with the so called large contraction which is suitable in the fixed point theory. Our purpose here is to use a modification of Krasnoselskii's fixed point theorem due T. A. Burton (see [1] Theorem 3) to show the existence of periodic solutions for equation (1.1). To apply this variant of Krasnoselskii's fixed point theorem we have to invert equation (1.1) to construct two mappings; one is large contraction and the other is compact. For details on Krasnoselskii's theorem we refer the reader to [8]. In Section 2, we present the inversion of difference equations (1.1) and the modification of Krasnoselskii's fixed point theorem. We present our main results on periodicity in Section 3 and at the end we provide an example to illustrate this work.

2. Inversion of the equation

Let T be an integer such that $T \geq 1$. Define

$$C_T = \{\varphi \in C(\mathbb{Z}, \mathbb{R}) : \varphi(t+T) = \varphi(t)\}$$

where $C(\mathbb{Z}, \mathbb{R})$ is the space of all real valued functions. Then $(C_T, \|.\|)$ is a Banach space with the maximum norm

$$\|\varphi\| = \max_{t \in [0, T-1]} |\varphi(t)|.$$

In this paper we assume the periodicity conditions

$$a(t+T) = a(t),\ g(t+T) = g(t),\ g(t) \geq g^* > 0, \tag{2.1}$$

for some constant g^*. Also, we assume that

$$a(t) > 0. \tag{2.2}$$

We also require that $G(t, x, y)$ is periodic in t and Lipschitz continuous in x and y. That is

$$G(t+T, x, y) = G(t, x, y), \tag{2.3}$$

and there are positive constants k_1, k_2 such that

$$|G(t, x, y) - G(t, z, w)| \leq k_1 |x - z| + k_2 |y - w|, \text{ for } x, y, z, w \in \mathbb{R}. \tag{2.4}$$

Lemma 2.1. *Suppose (2.1) and (2.3) hold. If $x \in C_T$, then x is a solution of equation (1.1) if and only if*

$$\begin{aligned} x(t) = {} & \left(1 - \prod_{s=t-T}^{t-1} (1 + a(s))^{-1}\right)^{-1} \\ & \times \left[\sum_{r=t-T}^{t-1} a(r) \left(x(r+1) - x^3(r+1)\right) \prod_{s=r}^{t-1} (1 + a(s))^{-1} \right. \\ & \left. + \sum_{r=t-T}^{t-1} G\left(r, x^3(r), x^3(r - g(r))\right) \prod_{s=r}^{t-1} (1 + a(s))^{-1}\right]. \end{aligned} \tag{2.5}$$

Proof. Let $x \in C_T$ be a solution of (1.1). First we write this equation as

$$\begin{aligned} \triangle x(t) + a(t)\,x(t+1) &= a(t)\,x(t+1) - a(t)\,x^3(t+1) \\ &\quad + G\left(t, x^3(t), x^3(t-g(t))\right). \end{aligned}$$

Multiplying both sides of the above equation by $\prod_{s=0}^{t-1}(1+a(s))$ and then summing from $t-T$ to $t-1$ to obtain

$$\begin{aligned} &\sum_{r=t-T}^{t-1} \triangle \left[\prod_{s=0}^{r-1}(1+a(s))\,x(r)\right] \\ &= \sum_{r=t-T}^{t-1} \left[a(r)\left\{x(r+1) - x^3(r+1)\right\}\right. \\ &\left. + G\left(r, x^3(r), x^3(r-g(r))\right)\right] \prod_{s=0}^{r-1}(1+a(s)). \end{aligned}$$

As a consequence, we arrive at

$$\begin{aligned} &\prod_{s=0}^{t-1}(1+a(s))\,x(t) - \prod_{s=0}^{t-T-1}(1+a(s))\,x(t-T) \\ &= \sum_{r=t-T}^{t-1} \left[a(r)\left\{x(r+1) - x^3(r+1)\right\}\right. \\ &\left. + G\left(r, x^3(r), x^3(r-g(r))\right)\right] \prod_{s=0}^{r-1}(1+a(s)). \end{aligned}$$

Now, the lemma follows by dividing both sides of the above equation by $\prod_{s=0}^{t-1}(1+a(s))$ and using the fact that $x(t) = x(t-T)$. □

In the analysis, we employ a fixed point theorem in which the notion of a large contraction is required as one of the sufficient conditions. First, we give the following definition which can be found in [1] or [2].

Definition 2.2. (**Large Contraction**) *Let (M, d) be a metric space and $B : M \to M$. B is said to be a large contraction if $\phi, \varphi \in M$, with $\phi \neq \varphi$ then $d(B\phi, B\varphi) \leq d(\phi, \varphi)$ and if for all $\epsilon > 0$, there exists a $\delta < 1$ such that*

$$[\phi, \varphi \in M, d(\phi, \varphi) \geq \epsilon] \Rightarrow d(B\phi, B\varphi) \leq \delta d(\phi, \varphi).$$

The next theorem, which constitutes a basis for our main result, is a reformulated version of Krasnoselskii's fixed point theorem due to T. A. Burton (see [1], [2]).

Theorem 2.3. (**Krasnoselskii-Burton**) *Let M be a bounded convex nonempty subset of a Banach space $(\mathbb{B}, \|.\|)$. Suppose that A and B map M into $\mathbb{B}$ such that*

i. $x, y \in M$, implies $Ax + By \in M$;

ii. A is continuous and AM is contained in a compact subset of M;
iii. B is a large contraction mapping.
Then there exists $z \in M$ with $z = Az + Bz$.

We will use this theorem to prove the existence of periodic solutions for equation (1.1). We begin with the following proposition.

Proposition 2.4. *If $\|.\|$ is the maximum norm,*

$$M = \left\{ \varphi \in C(\mathbb{Z}, \mathbb{R}) : \|\varphi\| \leq \sqrt{3}/3 \right\},$$

and $(B\varphi)(t) = \varphi(t+1) - \varphi^3(t+1)$, then B is a large contraction of the set M.

Proof. For each $t \in \mathbb{Z}$ we have for the real functions φ, ψ

$$\begin{aligned} &|(B\varphi)(t) - (B\psi)(t)| \\ &= |\varphi(t+1) - \psi(t+1)| \\ &\times \left|1 - \left(\varphi^2(t+1) + \varphi(t+1)\psi(t+1) + \psi^2(t+1)\right)\right|. \end{aligned}$$

On the other hand,

$$\begin{aligned} |\varphi(t+1) - \psi(t+1)|^2 &= \varphi^2(t+1) - 2\varphi(t+1)\psi(t+1) + \psi^2(t+1) \\ &\leq 2\left(\varphi^2(t+1) + \psi^2(t+1)\right). \end{aligned}$$

Using $\varphi^2(t+1) + \psi^2(t+1) < 1$ we have

$$\begin{aligned} &|(B\varphi)(t) - (B\psi)(t)| \\ &\leq |\varphi(t+1) - \psi(t+1)| \\ &\times \left[1 - \left(\varphi^2(t+1) + \psi^2(t+1)\right) + |\varphi(t+1)\psi(t+1)|\right] \\ &\leq |\varphi(t+1) - \psi(t+1)| \\ &\times \left[1 - \left(\varphi^2(t+1) + \psi^2(t+1)\right) + \frac{\varphi^2(t+1) + \psi^2(t+1)}{2}\right] \\ &\leq |\varphi(t+1) - \psi(t+1)| \left[1 - \frac{\varphi^2(t+1) + \psi^2(t+1)}{2}\right] \\ &\leq \|\varphi - \psi\|. \end{aligned}$$

Consequently we get

$$\|B\varphi - B\psi\| \leq \|\varphi - \psi\|.$$

Thus B is a large pointwise contraction. But B is still a large contraction for the maximum norm. To show this, let $\epsilon \in (0,1)$ be given and let $\varphi, \psi \in M$ with $\|\varphi - \psi\| \geq \epsilon$.

a) Suppose that for some t we have

$$\epsilon/2 \leq |\varphi(t+1) - \psi(t+1)|.$$

Then

$$(\epsilon/2)^2 \leq |\varphi(t+1) - \psi(t+1)|^2 \leq 2\left(\varphi^2(t+1) + \psi^2(t+1)\right),$$

that is

$$\varphi^2(t+1) + \psi^2(t+1) \geq \epsilon^2/8.$$

For all such t we have

$$\begin{aligned}|(B\varphi)(t) - (B\psi)(t)| &\leq |\varphi(t+1) - \psi(t+1)| \left[1 - \frac{\epsilon^2}{16}\right] \\ &\leq \left[1 - \frac{\epsilon^2}{16}\right] \|\varphi - \psi\|.\end{aligned}$$

b) Suppose that for some t we have

$$|\varphi(t+1) - \psi(t+1)| \leq \epsilon/2,$$

then

$$|(B\varphi)(t) - (B\psi)(t)| \leq |\varphi(t+1) - \psi(t+1)| \leq (1/2)\|\varphi - \psi\|.$$

So, for all t we have

$$|(B\varphi)(t) - (B\psi)(t)| \leq \max\left\{1/2, 1 - \frac{\epsilon^2}{16}\right\} \|\varphi - \psi\|.$$

Hence, for each $\epsilon > 0$, if $\delta = \max\left\{1/2, 1 - \frac{\epsilon^2}{16}\right\} < 1$, then

$$\|B\varphi - B\psi\| \leq \delta \|\varphi - \psi\|.$$

□

3. Existence of periodic solutions

To apply Theorem 2.3, we need to define a Banach space $\mathbb{B}$, a bounded convex subset M of $\mathbb{B}$ and construct two mappings, one is a large contraction and the other is compact. So, we let $(\mathbb{B}, \|.\|) = (C_T, \|.\|)$ and $M = \{\varphi \in \mathbb{B} \mid \|\varphi\| \leq L\}$, where $L = \sqrt{3}/3$. We express equation (2.5) as

$$\varphi(t) = (B\varphi)(t) + (A\varphi)(t) := (H\varphi)(t),$$

where $A, B : M \to \mathbb{B}$ are defined by

$$\begin{aligned}(A\varphi)(t) = &\left(1 - \prod_{s=t-T}^{t-1} (1 + a(s))^{-1}\right)^{-1} \\ &\times \sum_{r=t-T}^{t-1} G\left(r, \varphi^3(r), \varphi^3(r - g(r))\right) \prod_{s=r}^{t-1} (1 + a(s))^{-1},\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}(B\varphi)(t) = &\left(1 - \prod_{s=t-T}^{t-1} (1 + a(s))^{-1}\right)^{-1} \\ &\times \sum_{r=t-T}^{t-1} a(r)\left(\varphi(r+1) - \varphi^3(r+1)\right) \prod_{s=r}^{t-1} (1 + a(s))^{-1}.\end{aligned} \tag{3.2}$$

We suppose an additional condition, there is $J \geq 3$ with

$$J\left(\left(k_1+k_2\right)L^3+\left|G\left(t,0,0\right)\right|\right) \leq La\left(t\right),\ \forall t \in \mathbb{Z}. \tag{3.3}$$

We shall prove that the mapping H has a fixed point which solves (1.1).

Lemma 3.1. *For A defined in (3.1), suppose that (2.1)–(2.4) and (3.3) hold. Then $A : M \to M$ is continuous in the maximum norm and maps M into a compact subset of M.*

Proof. We first show that $A : M \to M$.
Let $\varphi \in M$. Evaluate (3.1) at $t+T$.

$$\begin{aligned}\left(A\varphi\right)\left(t+T\right) &= \left(1-\prod_{s=t}^{t+T-1}\left(1+a\left(s\right)\right)^{-1}\right)^{-1} \\ &\quad\times \sum_{r=t}^{t+T-1} G\left(r,\varphi^3\left(r\right),\varphi^3\left(r-g\left(r\right)\right)\right)\prod_{s=r}^{t+T-1}\left(1+a\left(s\right)\right)^{-1}.\end{aligned}$$

Let $j = r - T$, then

$$\begin{aligned}\left(A\varphi\right)\left(t+T\right) &= \left(1-\prod_{s=t}^{t+T-1}\left(1+a\left(s\right)\right)^{-1}\right)^{-1} \\ &\quad\times \sum_{j=t-T}^{t-1} G\left(j+T,\varphi^3\left(j+T\right),\varphi^3\left(j+T-g\left(j+T\right)\right)\right) \\ &\quad\times \prod_{s=j+T}^{t+T-1}\left(1+a\left(s\right)\right)^{-1}.\end{aligned}$$

Now let $k = s - t$, then

$$\begin{aligned}\left(A\varphi\right)\left(t+T\right) &= \left(1-\prod_{k=t-T}^{t-1}\left(1+a\left(k\right)\right)^{-1}\right)^{-1} \\ &\quad\times \sum_{j=t-T}^{t-1} G\left(j,\varphi^3\left(j\right),\varphi^3\left(j-g\left(j\right)\right)\right)\prod_{k=j}^{t-1}\left(1+a\left(k\right)\right)^{-1} \\ &= \left(A\varphi\right)\left(t\right).\end{aligned}$$

That is, $A : C_T \to C_T$.
In view of (2.4) we arrive at

$$\begin{aligned}\left|G\left(t,x,y\right)\right| &= \left|G\left(t,x,y\right)-G\left(t,0,0\right)+G\left(t,0,0\right)\right| \\ &\leq \left|G\left(t,x,y\right)-G\left(t,0,0\right)\right|+\left|G\left(t,0,0\right)\right| \\ &\leq k_1\left\|x\right\|+k_2\left\|y\right\|+\left|G\left(t,0,0\right)\right|.\end{aligned}$$

Note that from (2.2), we have $1-\prod_{s=t-T}^{t-1}(1+a(s))^{-1}>0$. So, for any $\varphi\in M$, we have

$$\begin{aligned}|(A\varphi)(t)| &\le \left(1-\prod_{s=t-T}^{t-1}(1+a(s))^{-1}\right)^{-1}\\ &\quad\times\sum_{r=t-T}^{t-1}\left|G\left(r,\varphi^3(r),\varphi^3(r-g(r))\right)\right|\prod_{s=r}^{t-1}(1+a(s))^{-1}\\ &\le \left(1-\prod_{s=t-T}^{t-1}(1+a(s))^{-1}\right)^{-1}\\ &\quad\times\sum_{r=t-T}^{t-1}\left((k_1+k_2)L^3+|G(r,0,0)|\right)\prod_{s=r}^{t-1}(1+a(s))^{-1}\\ &\le \left(1-\prod_{s=t-T}^{t-1}(1+a(s))^{-1}\right)^{-1}\sum_{r=t-T}^{t-1}\frac{La(r)}{J}\prod_{s=r}^{t-1}(1+a(s))^{-1}\\ &= \frac{L}{J}<L.\end{aligned}$$

Thus $A\varphi\in M$.
Consequently, we have $A:M\to M$.

We show that A is continuous in the maximum norm. Let $\varphi,\psi\in M$, and let

$$\alpha=\left(1-\prod_{s=t-T}^{t-1}(1+a(s))^{-1}\right)^{-1}.$$

Note that from (2.2), we have $\max_{r\in[t-T,t-1]}\prod_{s=r}^{t-1}(1+a(s))^{-1}\le 1$. So,

$$\begin{aligned}|(A\varphi)(t)-(A\psi)(t)| &\le \left(1-\prod_{s=t-T}^{t-1}(1+a(s))^{-1}\right)^{-1}\\ &\quad\times\sum_{r=t-T}^{t-1}\left|G\left(r,\varphi^3(r),\varphi^3(r-g(r))\right)\right.\\ &\quad\left.-G\left(r,\varphi^3(r),\varphi^3(r-g(r))\right)\right|\prod_{s=r}^{t-1}(1+a(s))^{-1}\\ &\le (k_1+k_2)\left\|\varphi^3-\psi^3\right\|\\ &\quad\times\left(1-\prod_{s=t-T}^{t-1}(1+a(s))^{-1}\right)^{-1}\sum_{r=t-T}^{t-1}\prod_{s=r}^{t-1}(1+a(s))^{-1}\\ &\le 3(k_1+k_2)T\alpha L^2\|\varphi-\psi\|.\end{aligned}$$

Let $\epsilon > 0$ be arbitrary. Define $\eta = \epsilon / K$ with $K = 3\left(k_1 + k_2\right) T \alpha L^2$, where k_1 and k_2 are given by (2.4). Then, for $\|\varphi - \psi\| < \eta$ we obtain

$$\|A\varphi - A\psi\| \leq K \|\varphi - \psi\| < \epsilon.$$

This proves that A is continuous.

Next, we show that A maps bounded subsets into compact sets. As M is bounded and A is continuous, then AM is a subset of $\mathbb{R}^T$ which is bounded. Thus AM is contained in a compact subset of M. Therefore A is continuous in M and AM is contained in a compact subset of M. □

Lemma 3.2. *Let B be defined by (3.2) and suppose that (2.1)–(2.2) hold. Then $B : M \to M$ is a large contraction.*

Proof. We first show that $B : M \to M$.
Let $\varphi \in M$. Evaluate (3.2) at $t + T$.

$$\begin{aligned}(B\varphi)(t+T) = {} & \left(1 - \prod_{s=t}^{t+T-1} \left(1 + a(s)\right)^{-1}\right)^{-1} \\ & \times \sum_{r=t}^{t+T-1} a(r) \left(\varphi(r+1) - \varphi^3(r+1)\right) \prod_{s=r}^{t+T-1} \left(1 + a(s)\right)^{-1}.\end{aligned}$$

Let $j = r - T$, then

$$\begin{aligned}(B\varphi)(t+T) = {} & \left(1 - \prod_{s=t}^{t+T-1} \left(1 + a(s)\right)^{-1}\right)^{-1} \\ & \times \sum_{j=t-T}^{t-1} a(j+T) \left(\varphi(j+T+1) - \varphi^3(j+T+1)\right) \\ & \times \prod_{s=j+T}^{t+T-1} \left(1 + a(s)\right)^{-1}.\end{aligned}$$

Now let $k = s - t$, then

$$\begin{aligned}(B\varphi)(t+T) = {} & \left(1 - \prod_{k=t-T}^{t-1} \left(1 + a(k)\right)^{-1}\right)^{-1} \\ & \times \sum_{j=t-T}^{t-1} a(j) \left(\varphi(j+1) - \varphi^3(j+1)\right) \prod_{k=j}^{t-1} \left(1 + a(k)\right)^{-1} \\ & = (B\varphi)(t).\end{aligned}$$

That is, $B : C_T \to C_T$.

Note that from (2.2), we have $1-\prod_{s=t-T}^{t-1}(1+a(s))^{-1}>0$. So, for any $\varphi\in M$, we have

$$\begin{aligned}|(B\varphi)(t)| &\le \left(1-\prod_{s=t-T}^{t-1}(1+a(s))^{-1}\right)^{-1}\\ &\quad\times\sum_{r=t-T}^{t-1}a(r)\left|\varphi(r+1)-\varphi^3(r+1)\right|\prod_{s=r}^{t-1}(1+a(s))^{-1}\\ &\le \left(1-\prod_{s=t-T}^{t-1}(1+a(s))^{-1}\right)^{-1}\\ &\quad\times\sum_{r=t-T}^{t-1}a(r)\left\|\varphi-\varphi^3\right\|\prod_{s=r}^{t-1}(1+a(s))^{-1}\\ &=\left\|\varphi-\varphi^3\right\|.\end{aligned}$$

Since $\|\varphi\|\le L$, we have $\left\|\varphi-\varphi^3\right\|\le\left(2\sqrt{3}\right)/9<L$. So, for any $\varphi\in M$, we have

$$\|B\varphi\|<L.$$

Thus $B\varphi\in M$. Consequently, we have $B:M\to M$.

It remains to show that B is large contraction in the maximum norm. From the proof of Proposition 2.4 we have for $\varphi,\psi\in M$, with $\varphi\ne\psi$

$$\begin{aligned}|(B\varphi)(t)-(B\psi)(t)| &\le \left(1-\prod_{s=t-T}^{t-1}(1+a(s))^{-1}\right)^{-1}\\ &\quad\times\sum_{r=t-T}^{t-1}a(r)\|\varphi-\psi\|\prod_{s=r}^{t-1}(1+a(s))^{-1}\\ &=\|\varphi-\psi\|.\end{aligned}$$

Then $\|B\varphi-B\psi\|\le\|\varphi-\psi\|$. Thus B is a large pointwise contraction. But B is still a large contraction for the maximum norm. To show this, let $\epsilon\in(0,1)$ be given and let $\varphi,\psi\in M$ with $\|\varphi-\psi\|\ge\varepsilon$. From the proof of the Proposition 2.4 we have found $\delta<1$ such that

$$\begin{aligned}|(B\varphi)(t)-(B\psi)(t)| &\le \left(1-\prod_{s=t-T}^{t-1}(1+a(s))^{-1}\right)^{-1}\\ &\quad\times\sum_{r=t-T}^{t-1}a(r)\,\delta\,\|\varphi-\psi\|\prod_{s=r}^{t-1}(1+a(s))^{-1}\\ &=\delta\,\|\varphi-\psi\|.\end{aligned}$$

Then $\|B\varphi-B\psi\|\le\delta\,\|\varphi-\psi\|$. Consequently, B is a large contraction. $\square$

Theorem 3.3. *Let* $(C_T, \|.\|)$ *be the Banach space of* T*-periodic real valued functions and* $M = \{\varphi \in C_T \mid \|\varphi\| \leq L\}$, *where* $L = \sqrt{3}/3$. *Suppose* (2.1)*–*(2.4) *and* (3.3) *hold. Then equation* (1.1) *has a* T*-periodic solution* φ *in the subset* M.

Proof. By Lemma 3.1, $A : M \to M$ is continuous and AM is contained in a compact set. Also, from Lemma 3.2, the mapping $B : M \to M$ is a large contraction. Moreover, if $\varphi, \psi \in M$, we see that

$$\|A\varphi + B\psi\| \leq \|A\varphi\| + \|B\psi\| \leq L/J + \left(2\sqrt{3}\right)/9 \leq L.$$

Thus $A\varphi + B\psi \in M$.
Clearly, all the hypotheses of Krasnoselskii-Burton Theorem 2.3 are satisfied. Thus there exists a fixed point $\varphi \in M$ such that $\varphi = A\varphi + B\varphi$. Hence the equation (1.1) has a T-periodic solution which lies in M. □

Example 3.4. We consider the totally nonlinear difference equation with functional delay

$$\triangle x(t) = -8x^3(t+1) + \sin\left(x^3(t)\right) + \cos\left(x^3(t - g(t))\right), \ t \in \mathbb{Z}, \tag{3.4}$$

where

$$g(t+T) = g(t).$$

So, we have

$$a(t) = 8, \ G\left(t, x^3(t), x^3(t-g(t))\right) = \sin\left(x^3(t)\right) + \cos\left(x^3(t-g(t))\right).$$

Clearly, $G(t, x, y)$ is periodic in t Lipschitz continuous in x and y. That is

$$G(t+T, x, y) = G(t, x, y),$$

and

$$\begin{aligned} |G(t,x,y) - G(t,z,w)| &= |\sin(x) - \sin(z) + \cos(y) - \cos(w)| \\ &\leq |\sin(x) - \sin(z)| + |\cos(y) - \cos(w)| \\ &\leq |x - z| + |y - w|. \end{aligned}$$

Note that if $J = 3$ we have

$$\begin{aligned} J\left((k_1 + k_2) L^3 + |G(t,0,0)|\right) &= 3\left(2\left(\sqrt{3}/3\right)^3 + 1\right) \\ &\leq \left(\sqrt{3}/3\right) 8 \\ &= La(t), \ \forall t \in \mathbb{Z}. \end{aligned}$$

Define $M = \{\varphi \in C_T \mid \|\varphi\| \leq L\}$, where $L = \sqrt{3}/3$. Then the difference (3.4) has a T-periodic solution in M, by Theorem 2.

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