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Note on q-Bernstein-Schurer operators

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Abstract. In this paper, we introduce a generalization of the Bernstein-Schurer operators based on q-integers and get a Bohman-Korovkin type approximation theorem of these operators. We also compute the rate of convergence by using the first modulus of smoothness.

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1. Preliminaries

Lupaş [18] introduced in 1987 a q-type of the Bernstein operators and in 1997 another generalization of these operators based on q-integers was introduced by Phillips [20]. He obtained the rate of convergence and a Voronovskaja type asymptotic formula for the new Bernstein operators. After this, many authors studied new classes of q-generalized operators. To show the extend of this research direction, we mention in the following some achievements in this field. In [5] Bărbosu introduced a Stancu type generalization of two dimensional Bernstein operators based on q-integers. In [1] O. Agratini introduced a new class of q-Bernstein-type operators which fix certain polynomials and studied the limit of iterates of Lupaş q-analogue of the Bernstein operators. In [4] Aral and Doğru obtained the uniform approximation of q-Bleimann-Butzer-Hahn (BBH) operators and in [9] O. Doğru and V. Gupta studied the monotonicity properties and the Voronovskaja type asymptotic estimate of these operators. See also the recent paper [2].

T. Trif [21] investigated Meyer-König and Zeller (MKZ) operators based on q-integers. Some approximation properties of q-MKZ operators were investigated by W. Heping in [16]. O. Doğru and O. Duman introduced also a new generalization of Meyer-König and Zeller operators and studied some statistical approximation properties in [7]. O. Doğru and Gupta [8] constructed a q-type generalization of Meyer-König and Zeller operators in bivariate case. A new q-generalization of Meyer-König and Zeller type operators was constructed by Doğru and Muraru for improve the rate of convergence, see [10].

O. Doğru and M. Orkcu proved in [11] that a new modification of q-MKZ operators provides a better estimation on the $[\alpha_n, 1] \subset [1/2, 1)$ by means of the modulus of continuity.

An extension in q-Calculus of Szász-Mirakyan operators was constructed by Aral [3] who formulated also a Voronovskaya theorem related to q-derivatives for these operators.

Durrmeyer type generalization of the operators based on q-integers was studied by Derriennic in [6]. Gupta and Heping introduced a q-analoque of Bernstein-Durrmeyer operators in [13] and in 2009 Gupta and Finta [14] studied some local and global approximation properties for q-Durrmeyer operators. See also [12]. In [15] Gupta and Radu constructed a q-analoque of Baskakov-Kantorovich operators and investigated their weighted statistical approximation properties. Also, N. Mahmudov introduced in [19] new classes of q-Baskakov and q-Baskakov-Kantorovich operators.

First of all, we recall elements of q-Calculus, see, e.g., [17]. For any fixed real number $q > 0$, the q-integer $[k]_q$, for $k \in \mathbb{N}$ is defined as

$$[k]_q = \begin{cases} (1-q^k)/(1-q), & q \neq 1, \\ k, & q = 1. \end{cases}$$

Set $[0]_q = 0$. The q-factorial $[k]_q!$ and q-binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$ are defined as follows

$$[k]_q! = \begin{cases} [k]_q[k-1]_q \dots [1]_q, & k = 1, 2, \dots, \\ 1, & k = 0, \end{cases}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} \quad (0 \le k \le n).$$

The q-analogue of $(x-a)^n$ is the polynomial

$$(x-a)_q^n = \begin{cases} 1 & \text{if} \quad n = 0, \\ (x-a)(x-qa)\dots(x-q^{n-1}a) & \text{if} \quad n \geq 1. \end{cases}$$

$C([a,b])$ represents the space of all real valued continuous functions defined on $[a,b]$. The space is endowed with usual norm $\|\cdot\|$ given by

$$\|f\| = \sup_{x \in [a,b]} |f(x)|.$$

Let $p \in \mathbb{N}$ be fixed. In 1962 Schurer [22] introduced and studied the operators $\widetilde{B}_{m,p} : C([0,p+1]) \to C([0,1])$ defined for any $m \in \mathbb{N}$ and any function $f \in C([0,p+1])$ as follows

$$\widetilde{B}_{m,p}(f;x) = \sum_{k=0}^{m+p} \binom{m+p}{k} x^k (1-x)^{m+p-k} f\left(\frac{k}{m}\right), \quad x \in [0,1].$$

Our aim is to introduce a q-analogue of the above operators. We investigate the approximation properties of this class and we estimate the rate of convergence by using modulus of continuity.

2. Construction of generalized q-Bernstein-Schurer and approximation properties

Throughout the paper we consider $q \in (0,1)$.
For any $m \in \mathbb{N}$ and $f \in C([0,p+1])$, p is fixed, we construct the class of generalized q-Bernstein-Schurer operators as follows

$$\widetilde{B}_{m,p}(f;q;x) = \sum_{k=0}^{m+p} \begin{bmatrix} m+p \\ k \end{bmatrix}_q x^k \prod_{s=0}^{m+p-k-1} (1-q^s x) f\left(\frac{[k]_q}{[m]_q}\right), \; x \in [0,1]. \tag{2.1}$$

From here on, an empty product is taken to be equal 1. Clearly, the operator defined by (2.1) is linear and positive.

Lemma 2.1. *Let $B_{m,p}(\cdot;q;\cdot)$ be given by (2.1). The following identities*

1° $\widetilde{B}_{m,p}(e_0;q;x) = 1,$

2° $\widetilde{B}_{m,p}(e_1;q;x) = \dfrac{x[m+p]_q}{[m]_q},$

3° $\widetilde{B}_{m,p}(e_2;q;x) = \dfrac{[m+p]_q}{[m]_q^2}([m+p]_q x^2 + x(1-x))$

hold, where $e_j(x) = x^j$, $j = 0,1,2$.

Proof. 1° We use the known identity

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k (1-x)_q^{n-k} = 1,$$

which can be proved by induction with respect to n. Actually, the left hand side represents $(B_{n,q}e_0)(x)$ where $B_{n,q}$ is the q-analogue of Bernstein operator introduced by G. M. Phillips [20]. Phillips proved $B_{n,q}e_0 = e_0$.

In the above we choose $n := m+p$.

Since

$$(1-x)_q^{m+p-k} = \prod_{s=0}^{m+p-k-1} (1-q^s x),$$

we get

$$\sum_{k=0}^{m+p} \begin{bmatrix} m+p \\ k \end{bmatrix}_q x^k \prod_{s=0}^{m+p-k-1} (1-q^s x) = 1.$$

Consequently, we obtain $\widetilde{B}_{m,p}(e_0;q;x) = 1$.

$$2^\circ \; \widetilde{B}_{m,p}(e_1;q;x) = \sum_{k=1}^{m+p} \begin{bmatrix} m+p \\ k \end{bmatrix}_q x^k \prod_{s=0}^{m+p-k-1} (1-q^s x) \frac{[k]_q}{[m]_q}$$

$$\overset{k \to k+1}{=} x \cdot \frac{[m+p]_q}{[m]_q} \sum_{k=0}^{m+p-1} \frac{[m+p-1]_q!}{[k]![m+p-k-1]_q!} x^k \prod_{s=0}^{m+p-k-2} (1-q^s x)$$

$$= x \cdot \frac{[m+p]_q}{[m]_q}.$$

$$3^\circ\ \widetilde{B}_{m,p}(e_2;q;x) = \sum_{k=1}^{m+p} \begin{bmatrix} m+p \\ k \end{bmatrix}_q x^k \prod_{s=0}^{m+p-k-1} (1-q^s x) \frac{[k]_q^2}{[m]_q^2}$$

$$= \sum_{k=1}^{m+p} \frac{[k]_q}{[m]_q} \cdot \frac{[k]_q}{[m]_q} \cdot \frac{[m+p]_q!}{[m+p-k]_q![k]_q!} \cdot x^k \prod_{s=0}^{m+p-k-1} (1-q^s x).$$

Taking into account that $[k]_q = q[k-1]_q + 1$, we obtain

$$\widetilde{B}_{m,p}(e_2;q;x) = \frac{[m+p]_q}{[m]_q^2} \sum_{k=2}^{m+p} \frac{q[k-1]_q[m+p-1]_q!}{[k-1]_q![m+p-k]_q!} x^k \prod_{s=0}^{m+p-k-1} (1-q^s x)$$

$$+ \frac{[m+p]_q}{[m]_q^2} \sum_{k=1}^{m+p} \frac{[m+p-1]_q!}{[k-1]_q![m+p-k]_q!} x^k \prod_{s=0}^{m+p-k-1} (1-q^s x).$$

Replacing $k \to k+2$ in first sum and $k \to k+1$ in the second, we have

$$\widetilde{B}_{m,p}(e_2;q;x) = \frac{[m+p-1]_q[m+p]_q}{[m]_q^2} q \sum_{k=0}^{m+p-2} \frac{[m+p-2]_q!}{[k]_q![m+p-k-2]_q}$$

$$\cdot x^{k+2} \prod_{s=0}^{m+p-k-3} (1-q^s x)$$

$$+ \frac{[m+p]_q}{[m]_q^2} \sum_{k=0}^{m+p-1} \frac{[m+p-1]_q!}{[k]_q![m+p-k-1]_q} x^{k+1} \prod_{s=0}^{m+p-k-2} (1-q^s x)$$

$$= \frac{[m+p-1]_q[m+p]_q}{[m]_q^2} q x^2 + \frac{[m+p]_q}{[m]_q^2} x.$$

Since $[m+p-1]_q q x^2 + x = [m+p]_q x^2 + x(1-x)$, the conclusion follows. □

We can give now the following result, a theorem of Korovkin type.

Theorem 2.2. *Let $q = q_m$ satisfy $0 < q_m < 1$, $\lim_{m\to\infty} q_m = 1$ and $\lim_{m\to\infty} q_m^m = a$, $a < 1$. Then, for any $f \in C([0,p+1])$, the following relation holds*

$$\lim_{m\to\infty} \widetilde{B}_{m,p}(f;q_m) = f \text{ uniformly on } [0,1].$$

Proof. The proof is based on the well known Korovkin theorem regarding the convergence of a sequence of linear and positive operators. So, it is enough to prove the conditions

$$\lim_{m\to\infty} \widetilde{B}_{m,p}(e_i;q_m;x) = x^i, \quad i = 0,1,2,$$

uniformly on $[0,1]$.

To prove the theorem we take into account the next relations obtained by simple calculations, where p is a fixed natural number.

$$\lim_{m\to\infty} \frac{[m+p]_{q_m}}{[m]_{q_m}} = 1, \quad \lim_{m\to\infty} \frac{[m+p]_{q_m}}{[m]_{q_m}^2} = 0. \tag{2.2}$$

Taking into account Lemma 2.1 and the relations (2.2), our statement is proved. □

3. On the rate of convergence

We will estimate the rate of convergence in terms of the modulus of continuity. Let $f \in C([0,b])$. The modulus of continuity of f denoted by $\omega_f(\delta)$ gives the maximum oscillation of f in any interval of length not exceeding $\delta > 0$ and it is given by relation

$$\omega_f(\delta) = \sup_{|y-x|\le\delta} |f(y)-f(x)|, \quad x, y \in [0,b].$$

It is known that $\lim\limits_{\delta\to 0^+} \omega_f(\delta) = 0$ for $f \in C([0,b])$, and for any $\delta > 0$ one has

$$|f(y)-f(x)| \le \omega_f(\delta)\left(\frac{|y-x|}{\delta}+1\right). \tag{3.1}$$

Our result will be read as follows.

Theorem 3.1. *If $f \in C([0,1+p])$, then*

$$|\widetilde{B}_{m,p}(f;q;x) - f(x)| \le 2\omega_f(\delta_m)$$

takes place, where

$$\delta_m = \frac{1}{\sqrt{[m]_q}}\left(p + \frac{1}{2\sqrt{1-q^m}}\right), \quad q \in (0,1). \tag{3.2}$$

Proof. Since $B_{m,p}e_0 = e_0$, we have

$$|\widetilde{B}_{m,p}(f;q;x) - f(x)|$$
$$\le \sum_{k=0}^{m+p}\left|f\left(\frac{[k]_q}{[m]_q}\right) - f(x)\right| \frac{[m+p]_q!}{[m+p-k]_q![k]_q!}\, x^k \prod_{s=0}^{m+p-k-1}(1-q^s x).$$

In view of (3.1) we get

$$|\widetilde{B}_{m,p}(f;q;x) - f(x)|$$
$$\le \omega_f(\delta)\Bigg\{\frac{1}{\delta}\sum_{k=0}^{m+p}\left|\frac{[k]_q}{[m]_q} - x\right| \frac{[m+p]_q!}{[m+p-k]_q![k]_q!}\, x^k \prod_{s=0}^{m+p-k}(1-q^s x)$$
$$+ \sum_{k=0}^{m+p}\frac{[m+p]_q!}{[m+p-k]_q![k]_q!}\, x^k \prod_{s=0}^{m+p-k}(1-q^s x)\Bigg\}$$
$$= \omega_f(\delta)\left\{\frac{1}{\delta}\sum_{k=0}^{m+p}\left|\frac{[k]_q}{[m]_q} - x\right| \frac{[m+p]_q!}{[m+p-k]_q![k]_q!} x^k \prod_{s=0}^{m+p-k}(1-q^s x) + (\widetilde{B}_{m,p,q}e_0)(x)\right\}.$$

Using Cauchy-Schwartz inequality and Lemma 2.1 we can write

$$|\widetilde{B}_{m,p}(f;q;x) - f(x)|$$
$$\le \omega_f(\delta)\left\{\frac{1}{\delta}\left(\sum_{k=0}^{m+p}\left(\frac{[k]_q}{[m]_q} - x\right)^2 \frac{[m+p]_q!}{[m+p-k]_q![k]_q!} x^k \prod_{s=0}^{m+p-k}(1-q^s x)\right)^{1/2} + 1\right\}$$
$$= \omega_f(\delta)\left\{\frac{1}{\delta}((\widetilde{B}_{m,p,q}e_2)(x) - 2x(\widetilde{B}_{m,p,q}e_1)(x) + x^2(\widetilde{B}_{m,p,q}e_0)(x))^{1/2} + 1\right\}$$

$$=\omega_f(\delta)\left\{\frac{1}{\delta}\left(\frac{[m+p]_q}{[m]_q^2}([m+p]_q x^2+x(1-x))-2x^2\frac{[m+p]_q}{[m]_q}+x^2\right)^{1/2}+1\right\}$$

$$=\omega_f(\delta)\left\{\frac{1}{\delta}\left(x^2\left(\frac{[m+p]_q}{[m]_q}-1\right)^2+x(1-x)\frac{[m+p]_q}{[m]_q^2}\right)^{1/2}+1\right\}.$$

On the basis of the relation $(a^2+b^2)^{1/2} \le |a|+|b|$, the above inequality implies

$$|\widetilde{B}_{m,p}(f;q;x)-f(x)|$$

$$\le \omega_f(\delta)\left\{\frac{1}{\delta}\left(x\left|\frac{[m+p]_q}{[m]_q}-1\right|+\sqrt{\frac{x(1-x)}{[m]_q}}\sqrt{\frac{[m+p]_q}{[m]_q}}\right)+1\right\}. \tag{3.3}$$

Since

$$x\left|\frac{[m+p]_q}{[m]_q}-1\right| \le \frac{p}{\sqrt{[m]_q}}, \qquad \sqrt{\frac{[m+p]_q}{[m]_q}} \le \frac{1}{\sqrt{1-q^m}}$$

and $\max_{x\in[0,1]} x(1-x)=1/4$, choosing $\delta=\delta_m$ as in (3.2), we obtain the desired result. □

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