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On the pointwise convergence of the Chebyshev best approximation on Jacobi nodes

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Abstract. This paper is devoted to obtain estimates and to point out convergence-type results and the superdense unbounded divergence for some pointwise approximation formulas, related to the Chebyshev best approximation on Jacobi node matrix.

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1. Introduction

Denote by C the Banach space of all continuous functions $f : [-1, 1] \to \mathbb{R}$, endowed with the uniform norm and let C^r, $r \geq 1$, be the subspace of C which contains the functions f whose derivatives up to the order r belong to C; we admit $C^0 = C$.

Let us consider, also, a strictly increasing sequence of positive integers m_n, with $m_n \geq n + 1$, $\forall\, n \geq 1$, and the node matrix

$$\mathcal{M} = \{x_{m_n}^k :\ n \geq 1,\ 1 \leq k \leq m_n\}, \tag{1.1}$$

where $-1 \leq x_{m_n}^1 < x_{m_n}^2 < x_{m_n}^3 < \ldots < x_{m_n}^{m_n} \leq 1$.

Define the operators $U_n : C \to \mathcal{P}_n$, $n \geq 1$, as follows: for each f in C, let $U_n f$ be the unique polynomial of $\mathcal{P}_n$ for which the infimum of the set

$$\{\max\{|f(x_{m_n}^k) - P(x_{m_n}^k)| :\ 1 \leq k \leq m_n\} :\ P \in \mathcal{P}_n\} \tag{1.2}$$

is attained, [1], [4]; in this paper, $\mathcal{P}_n$ is the usual notation for the set of all algebraic polynomials of degree at most $n \in \mathbb{N}$.

The polynomial $U_n f = U_n(f; \mathcal{M}) \in \mathcal{P}_n$, that provides the best approximation of f in the Chebyshev sense, with respect to the finite point set

$$J_n = \{x_{m_n}^k :\ 1 \leq k \leq m_n\}, \quad n \geq 1, \tag{1.3}$$

is said to be *the $\mathcal{M}$-projection* of f on the space $\mathcal{P}_n$.

We associate to each row J_n, $n \geq 1$, and $f \in C$, the Lagrange polynomial $L_{m_n} f$ which interpolates f at the nodes of J_n, namely

$$(L_{m_n} f)(x) = \sum_{k=1}^{m_n} f(x_{m_n}^k) l_{m_n}^k(x), \quad x \in [-1,1], \tag{1.4}$$

and the Lebesgue function $\Lambda_{m_n} : [-1,1] \to [0,\infty)$,

$$\Lambda_{m_n}(x) = \sum_{k=1}^{m_n} |l_{m_n}^k(x)|, \quad x \in [-1,1], \tag{1.5}$$

where

$$l_{m_n}^k(x) = \frac{u_{m_n}(x)}{(x - x_{m_n}^k) u'_{m_n}(x_{m_n}^k)},\ 1 \leq k \leq m_n;\ u_{m_n}(x) = \prod_{k=1}^{m_n} (x - x_{m_n}^k),\ n \geq 1. \tag{1.6}$$

Clearly, if $m_n = n+1$, $n \geq 1$, then the operators U_n coincide with the classical Lagrange projection operators, $f \mapsto L_{m_n} f$.

On the other hand, assuming that each row J_n of $\mathcal{M}$ contains exactly $n+2$ points, i.e. $m_n = n+2$, $\forall\ n \geq 1$, Ph. C. Curtis Jr., [4], has proved that the corresponding $\mathcal{M}$-projection operators U_n, $n \geq 1$, are linear and continuous operators and there exists a function $g \in C$ for which the sequence $(U_n g)_{n \geq 1}$ fails to converge uniformly on $[-1,1]$. As we proved in [5], the set of all functions $f \in C$ with the property that $\limsup_{n\to\infty} \|U_n f\| = \infty$ is, in fact, a superdense set in the Banach space $(C, \|\cdot\|)$; this superdense unbounded divergence remains valid if $m_n = n+3$ and the nodes of J_n are symmetric with respect to the origin, $\forall\ n \geq 1$, [6]. We recall that a subset $\mathcal{S}$ of a topological space $\mathcal{T}$ is said to be *superdense* in $\mathcal{T}$ if it is residual (i.e. its complement is of first Baire category), uncountable and dense in $\mathcal{T}$. These results of divergence type contrast with the well-known theorem concerning the uniform convergence of the best approximation polynomials in supremum norm, which states that the operators $Q_n : C \to \mathcal{P}_n$, defined by $\|f - Q_n f\| = \inf\{\|f - P\| :\ P \in \mathcal{P}_n\}$, $f \in C$, are continuous nonlinear projections and the sequence $(Q_n f)_{n \geq 1}$ is uniformly convergent to f, for each $f \in C$.

In the next sections, we consider the case $m_n = n+2$, $n \geq 1$. Our aim is to point out estimates, results of convergence type and the phenomenon of condensation of singularities for some pointwise approximation formulas associated to the Chebyshev best approximation on the Jacobi node matrix.

The paper is organized as follows. In the second section, we introduce the point-functionals that define the pointwise approximation formulas for an arbitrary node matrix $\mathcal{M}$ in (1.1) and we derive an estimate of the corresponding approximation error. In the third and fourth sections we establish results of convergence type and we prove the superdense unbounded divergence, respectively, for the pointwise approximation formulas corresponding to the Jacobi matrix. To this goal, we use the following *principle of condensation of singularities* from Functional Analysis.

Theorem 1.1. [2], [3]. *If X is a Banach space, Y is a normed space and $(A_n)_{n\geq 1}$ is a sequence of continuous linear operators from X into Y so that the set of norms $\{\|A_n\| : \ n \geq 1\}$ is unbounded, then the set of singularities of the family $\{A_n : \ n \geq 1\}$, namely*

$$\mathcal{S} = \left\{ x \in X : \ \limsup_{n\to\infty} \|A_n x\| = \infty \right\},$$

is superdense in X.

In this paper, the notations m, M, M_k, $k \geq 1$, stand for some generic positive constants, which do not depend on n. If (a_n) and (b_n) are sequences of real numbers with $b_n \neq 0$, we write $a_n \sim b_n$ if $0 < m \leq |a_n/b_n| \leq M$, for all $n \geq 1$. Also, $\omega(f;\cdot)$ denotes the modulus of continuity of a function $f \in C$.

2. Estimates for pointwise approximation formulas

Firstly, let us derive, according to [4], the formula of computing $U_n f$, for a given $n \geq 1$. Let $\sigma_{n+2} \in C$ be a function satisfying the conditions $\sigma_{n+2}(x_{n+2}^k) = (-1)^k$, $1 \leq k \leq n+2$. By means of Theorem of Charles de la Vallée-Poussin, [1], [8], and taking into account (1.2), we get:

$$U_n f = L_{n+2} f - \frac{a_{n+1}(f)}{a_{n+1}(\sigma_{n+2})} L_{n+2}\sigma_{n+2}; \quad f \in C, \ n \geq 1, \tag{2.1}$$

where $a_{n+1}(f)$ is the leading-coefficient of $L_{n+2} f$.

Further, by introducing the notation

$$\tau_{n+2}^k = (u_{n+2}'(x_{n+2}^k))^{-1}, \quad 1 \leq k \leq n+2, \tag{2.2}$$

and remarking that $\mathrm{sign}\tau_{n+2}^k = (-1)^{n-k}$, $1 \leq k \leq n+2$, we have:

$$a_{n+1}(f) = \sum_{k=1}^{n+2} \tau_{n+2}^k f(x_{n+2}^k) \tag{2.3}$$

and

$$a_{n+1}(\sigma_{n+2}) = (-1)^{n+2} \sum_{k=1}^{n+2} |\tau_{n+2}^k|. \tag{2.4}$$

The relations (2.1), (1.4) and the definition of σ_{n+2} lead to:

$$(U_n f)(x) = \sum_{k=1}^{n+2} d_{n+2}^k(f) l_{n+2}^k(x); \quad f \in C, \ |x| \leq 1, \ n \geq 1, \tag{2.5}$$

where the linear functionals $d_{n+2}^k : C \to \mathbb{R}$, are given by:

$$d_{n+2}^k(f) = f(x_{n+2}^k) + (-1)^{k+1} \frac{a_{n+1}(f)}{a_{n+1}(\sigma_{n+2})}, \quad 1 \leq k \leq n+2. \tag{2.6}$$

The relations (2.3) and (2.4) give $|a_{n+1}(f)| \leq |a_{n+1}(\sigma_{n+2})| \cdot \|f\|$ which, combined with (2.5) and (2.6), yield $|d_{n+2}^k(f)| \leq 2\|f\|$, so:

$$|(U_n f)(x)| \leq 2\Lambda_{n+2}(x) \cdot \|f\|; \quad f \in C,\ |x| \leq 1,\ n \geq 1. \tag{2.7}$$

Now, for a given point $t \in [-1,1]$, let us define the point-functionals $T_n^t : C \to \mathbb{R}$ by

$$T_n^t(f) = (U_n f)(t) = \sum_{k=1}^{n+2} d_{n+2}^k(f) \cdot l_{n+2}^k(t); \quad f \in C,\ n \geq 1 \tag{2.8}$$

and let us consider the approximation-errors $R_n^t f$, of the pointwise approximation formulas

$$f(t) = T_n^t(f) + R_n^t(f); \quad f \in C,\ n \geq 1, \tag{2.9}$$

associated to the Chebyshev discrete best approximation on the nodes (1.3) of J_n.

By using the relation $U_n P = P$, $\forall\ P \in \mathcal{P}_n$, that follows from (2.1), we obtain, taking into account (2.9):

$$|R_n^t f| = |R_n^t(f-P)| \leq |f(t) - P(t)| + |T_n^t(f-P)|, \quad f \in \mathcal{P}_n.$$

The last inequality, combined with (2.7), leads to:

$$|R_n^t f| \leq (1 + 2\Lambda_{n+2}(t)) \cdot \|f - P\|; \quad f \in C,\ P \in \mathcal{P}_n. \tag{2.10}$$

Further, let $f \in C^r$, $r \geq 0$. It follows from the inequality of Gopengauz, [9], the existence of a polynomial $\widetilde{P} \in \mathcal{P}_n$ so that:

$$\|f - \widetilde{P}\| \leq M_1 n^{-r} \omega\left(f^{(r)}; \frac{1}{n}\right); \quad n \geq 1, \tag{2.11}$$

where $M_1 = M_1(r)$.

We derive from (2.10) and (2.11):

$$|R_n^t f| \leq M_1 n^{-r}(1 + 2\Lambda_{n+2}(t))\omega\left(f^{(r)}; \frac{1}{n}\right); \quad f \in C^r,\ n \geq 1. \tag{2.12}$$

Finally, (2.12) leads to the following statement.

Theorem 2.1. *The pointwise approximation formulas (2.8) and (2.9), with respect to an arbitrary point $t \in [-1,1]$, are convergent on C^r, $r \geq 0$, i.e.*

$$\lim_{n \to \infty} T_n^t(f) = f(t),\ \forall\ f \in C^r,$$

if the corresponding Lebesgue functions satisfy the condition

$$\Lambda_{n+2}(t) = O(n^r).$$

3. Results of convergence-type for pointwise Jacobi approximation formulas

In this section and the next section, we take as node matrix $\mathcal{M}$ the Jacobi ultraspherical matrix $\mathcal{M}^{(\alpha)}$, $\alpha > -1$, whose n-th row contains the roots of the Jacobi ultraspherical polynomial $P_{n+2}^{(\alpha)}$, $n \geq 1$. In this framework, the formulas (2.8) and (2.9) will be refereed to as *pointwise Jacobi approximation formulas*, associated to the Chebyshev discrete best approximation.

The following estimate is valid, [7]:

$$\Lambda_n(t) - 1 \sim |P_n^{(\alpha)}(t)|\sqrt{n} \cdot k_n(\alpha); \quad n \geq 2,\ t \in [-1,1], \tag{3.1}$$

with

$$k_n(\alpha) = \begin{cases} 1 + (1-t)^{\alpha/2+1/4} \ln n, & \text{if} \quad \alpha > -1/2 \\ \ln n, & \text{if} \quad \alpha = -1/2 \\ \dfrac{\ln(2 + n\sqrt{1-t})}{(1-t)^{-\alpha/2-1/4} + n^{\alpha+1/2}}, & \text{if} \quad \alpha < -1/2. \end{cases} \tag{3.2}$$

It follows from (2.12), (3.1) and (3.2):

$$|R_n^t f| \leq M_2 n^{-r}(1 + |P_{n+2}^{(\alpha)}(t)|\sqrt{n+2}\ k_{n+2}(\alpha))\omega\left(f^{(r)}; \frac{1}{n}\right); \tag{3.3}$$

$f \in C^r,\ n \geq 2,\ t \in [-1,1]$.

3.1. First case

Suppose that $t \in (-1,1)$. The following statement holds.

Theorem 3.1. *Let consider the Jacobi pointwise approximation formulas with respect to an arbitrary point $t \in (-1,1)$.*

$1°$. *If $r > \alpha + 1/2 > 0$* **or** *$\alpha \leq -\frac{1}{2}$ and $r \geq 1$, then these formulas are convergent on the space C^r, namely*

$$\lim_{n\to\infty} T_n^t(f) = f(t),\ \forall\ f \in C^r.$$

$2°$. *If $\alpha + \frac{1}{2} \in \mathbb{N}^*$ and $r = \alpha + \frac{1}{2}$* **or** *$\alpha \leq -\frac{1}{2}$ and $r = 0$, then these formulas are convergent on the subset of all $f \in C^r$, whose r-th derivatives satisfy the Dini-Lipschitz condition*

$$\lim_{\delta \searrow 0} \omega(f^{(r)}; \delta) \ln \delta = 0.$$

Proof. The estimate $\|P_n^{(\alpha)}\| \sim n^q$, with $q = \max\{\alpha, -1/2\}$, [10], together with (3.3), yields:

$$\begin{cases} |R_n^t f| \leq M_3 n^{\alpha-r+1/2}(\ln n)\omega\left(f^{(r)}; \frac{1}{n}\right), & \text{if} \quad \alpha > -1/2 \\ |R_n^t f| \leq M_3 n^{-r}(\ln n)\omega\left(f^{(r)}; \frac{1}{n}\right), & \text{if} \quad \alpha \leq -1/2, \end{cases} \tag{3.4}$$

for each $f \in C^r$, $n \geq 2$ and $t \in (-1,1) \setminus J_n$.

If $t \in J_n$, then $P_{n+2}^{(\alpha)}(t) = 0$ and (2.12) provides:

$$|R_n^t f| \leq 3M_1 n^{-r} \omega\left(f^{(r)}; \frac{1}{n}\right),$$

so the formulas (3.4) are valid for $t \in (-1,1)$.

The estimates (3.4) and the properties of ω imply the validity of the assertions 1° and 2° of this theorem, which completes the proof. □

3.2. Second case

Let us examine the remaining cases $t = \pm 1$.

Theorem 3.2. *Let consider the Jacobi pointwise approximation formulas with respect to the end points $t = -1$ and $t = 1$.*

1°. If $r \geq \alpha + 1/2 > 0$ **or** *$\alpha = -1/2$ and $r \geq 1$* **or** *$\alpha < -1/2$ and $r \geq 0$, then these formulas are convergent on the space C^r.*

2°. If $\alpha = -1/2$ and $r = 0$, then these formulas are convergent on the subset of all functions f in C satisfying the Dini-Lipschitz condition

$$\lim_{\delta \searrow 0} \omega(f;\delta) \ln \delta = 0.$$

Proof. Using the estimate $|P_n^{(\alpha)}(\pm 1)| \sim n^\alpha$, $\alpha > -1$, [10], we derive from (3.1) and (3.2)

$$\Lambda_n(1) \sim \begin{cases} n^{\alpha+1/2}, & \text{if} \quad \alpha > -1/2 \\ \ln n, & \text{if} \quad \alpha = -1/2 \\ 1, & \text{if} \quad \alpha < -1/2. \end{cases} \tag{3.5}$$

The relations (2.12) and (3.5) yield:

$$|R_n^{\pm 1} f| \leq M_4 n^{\alpha - r + 1/2} \omega\left(f^{(r)}; \frac{1}{n}\right), \quad \text{if} \quad \alpha > -1/2$$

$$|R_n^{\pm 1} f| \leq M_5 n^{-r} \ln n \omega\left(f^{(r)}; \frac{1}{n}\right), \quad \text{if} \quad \alpha = -1/2$$

$$|R_n^{\pm 1} f| \leq M_6 n^{-r} \omega\left(f^{(r)}; \frac{1}{n}\right), \quad \text{if} \quad \alpha < -1/2,$$

which proves the assertions 1° and 2° of this theorem. □

4. Superdense unbounded divergence for a class of Jacobi pointwise approximation formulas

In this section, we emphasize the phenomenon of condensation of singularities for the family of the pointwise approximating functionals $\{T_n^0 : \ n \geq 1\}$.

Theorem 4.1. *The set of all functions $f \in C$ for which the Jacobi pointwise approximation formulas (2.8) and (2.9) with respect to the origin are unboundedly divergent, i.e.* $\limsup_{n\to\infty} |T_n^0 f| = \infty$, *is superdense in the Banach space* $(C, \|\cdot\|)$.

Proof. Define $f_{n+2} \in C$ by

$$f_{n+2}(x) = \begin{cases} \operatorname{sign} l_{n+2}^k(0), & \text{if } x \in J_n \\ 1, & \text{if } x \in \{-1, 1\} \\ \text{linear}, & \text{otherwise.} \end{cases}$$

We obtain from (2.5), (2.6) and (2.8):

$$T_{4n-2}^0(f_{4n}) = \sum_{k=1}^{4n} \left[1 + (-1)^{k+1} \frac{a_{4n-1}(f)}{a_{4n-1}(\sigma_{4n})} \operatorname{sign} l_{4n}^k(0)\right] |l_{4n}^k(0)|. \tag{4.1}$$

On the other hand, the relations $\operatorname{sign} l_{4n}^k(0) = (-1)^k$, $1 \le k \le 2n$ and $\operatorname{sigm} l_{4n}^k(0) = (-1)^{k+1}$, $2n+1 \le k \le 4n$, show that f_{4n} is an even function, so we derive from (2.2) and (2.3):

$$a_{4n-1}(f_{4n}) = \sum_{k=1}^{4n} \tau_{4n}^k f_{4n}(x_{4n}^k) = \sum_{k=1}^{4n} \tau_{4n}^{4n-k+1} f_{4n}(x_{4n}^{4n-k+1})$$

$$= \sum_{k=1}^{4n} \frac{f_{4n}(-x_{4n}^k)}{u_{4n}'(-x_{4n}^k)} = -\sum_{k=1}^{4n} \frac{f_{4n}(x_{4n}^k)}{u_{4n}'(x_{4n}^k)} = -a_{4n-1}(f_{4n}) \tag{4.2}$$

because the nodes of J_n in $\mathcal{M}^{(\alpha)}$ are symmetric with respect to the origin and u_{4n} is an even function. So, we obtain from (4.2):

$$a_{4n-1}(f_{4n}) = 0, \quad n \ge 1. \tag{4.3}$$

The equalities (4.1) and (4.3) leads to:

$$T_{4n-2}^0(f_{4n}) = \Lambda_{4n}(0), \quad n \ge 1. \tag{4.4}$$

Using the estimates (3.1), (3.2) and taking into account that

$$|P_{2n}^{(\alpha)}(0)| \sim 1/\sqrt{n},$$

[10], we infer:

$$\Lambda_{4n}(0) - 1 \sim \ln n, \ \forall\, \alpha > -1. \tag{4.5}$$

The relations (4.4) and (4.5) give:

$$|T_{4n-2}^0(f_{4n})| \sim \ln n, \quad n \ge 2, \ \alpha > -1. \tag{4.6}$$

Finally, apply Theorem 1.1, with $X = C$, $Y = \mathbb{R}$, $A_n = T_n^0$ and remark that:

$$\sup\{\|A_n\| : \ n \ge 1\} \ge \sup\{\|T_{4n-2}^0\| : \ n \ge 1\} \ge \sup\{|T_{4n-2}^0(f_{4n})| : \ n \ge 1\},$$

which together with (4.6), proves the unboundedness of the set of norms $\{\|A_n\| : \ n \ge 1\}$. This completes the proof. □

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