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Rigid body time-stepping schemes in a quasi-static setting

Bogdan Gavrea

Abstract. We discuss how linear complementary problems (LCPs) can be used to simulate rigid-body systems in a quasi-static setting. LCP-based time-stepping schemes were successfully used in [1] in order to plan and control meso-scale manipulation tasks.

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1. Introduction

In [1] we considered the canonical problem of assembling a peg into a hole. Simulation of this quasi-static system was used in order to select the control parameters. The integration step in the simulator was formulated as a *mixed linear complementarity problem* (MLCP). MLCPs should be thought of as *linear complementarity problems*(LCPs) coupled with additional linear equality constraints. A brief description of the linear complementarity problem and results concerning LCPs with copositive matrices are given in the following subsections. For a detailed analysis of these problems we refer the reader to the excellent manuscript [2].

1.1. Linear complementarity problems

In this section we present the definitions for the linear complementarity problem (LCP) and the mixed linear complementarity problem(MLCP).

Definition 1.1. *The problem of finding $z \in \mathbb{R}^n$ such that*

$$z \geq 0, \quad Mz + b \geq 0, \quad \text{and} \quad z^T(Mz + b) = 0, \tag{1.1}$$

where $b \in \mathbb{R}^n$ and $M \in \mathbb{R}^{n \times n}$ is called a linear complementarity problem.

In the above definition the inequality $z \geq 0,\ z \in \mathbb{R}^n$ is to be understood componentwise, i.e., $z_i \geq 0,\ i = \overline{1,n}$. The non-negativity and complementarity conditions (1.1) can be also written in the more compact form:

$$0 \leq z \perp w := Mz + b \geq 0.$$

We denote the problem (1.1) by $LCP(b, M)$. If in addition to the complementarity constraints we add some equality constraints we obtain a *mixed linear complementarity problem (MLCP)*. To be more precise, we follow the definition in [2] and consider the matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times m}$, $C \in \mathbb{R}^{n\times m}$ and $D \in \mathbb{R}^{m\times n}$. Let $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ be given.

Definition 1.2. *The mixed linear complementarity problem is the problem of finding vectors $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$ such that*

$$\begin{array}{lcl} a + Au + Cv & = & 0 \\ b + Du + Bv & \geq & 0 \\ v & \geq & 0 \\ v^T(b + Du + Bv) & = & 0 \end{array} \tag{1.2}$$

We note that if the matrix A in (1.2) is invertible we can write u in terms of v and use this form to reduce the problem to a standard LCP formulation.

1.2. LCPs with copositive matrices

The matrix of the underlying LCP used in the time-stepping schemes such as the one used in [1] is a copositive matrix.

Definition 1.3. *A matrix $M \in \mathbb{R}^{n\times n}$ is said to be copositive if*

$$x^T Mx \geq 0 \ \text{ for all } \ x \in \mathbb{R}^n,\ \ x \geq 0.$$

In general a linear complementarity problem with a copositive matrix is not guaranteed to possess a solution. Solvability of such LCPs is discussed in the following Theorem.

Theorem 1.4 ([2], Th. 3.8.6). *Let $M \in \mathbb{R}^{n\times n}$ be a copositive matrix and let $b \in \mathbb{R}^n$ be given. If the implication*

$$\left[v \geq 0,\ Mv \geq 0,\ v^T Mv = 0\right] \ \Rightarrow\ \left[v^T b \geq 0\right]$$

holds, then $LCP(b, M)$ has a solution. Lemke's algorithm with precautions taken against cycling will always find a solution of $LCP(b, M)$.

Lemke's algorithm is a pivoting method similar to the simplex method of linear programming. Cycling here refers to the possibility of using the same basis twice.

2. The quasi-static model

The continuous-time model under the rigid body assumption is given by the following *differential complementarity problem (DCP)*:

$$\dot{q}(t) = v(t), \tag{2.1}$$

$$Ev(t) - W_n(q,u,t)\lambda_n(t) - W_t(q,u,t)\lambda_t(t) = 0, \tag{2.2}$$

$$0 \le \Psi_n(q,u,t) \quad \perp \quad \lambda_n(t) \ge 0, \tag{2.3}$$

$$\dot{s}^+_{tk}(t) - \dot{s}^-_{tk}(t) = (W_{tk}(q,u,t))^T v(t) + \frac{\partial \Psi_{tk}}{\partial t}(q,u,t), \quad k = 1,...,n_c, \tag{2.4}$$

$$0 \le \dot{s}^+_{tk}(t) \quad \perp \quad \mu_k \lambda_{nk}(t) + \lambda_{tk}(t) \ge 0, \quad k = 1,...,n_c, \tag{2.5}$$

$$0 \le \dot{s}^-_{tk}(t) \quad \perp \quad \mu_k \lambda_{nk}(t) - \lambda_{tk}(t) \ge 0, \quad k = 1,...,n_c. \tag{2.6}$$

Here q denotes the generalized system position and v the generalized system velocity. The control parameters are encoded in the vector u. The quasi-static assumption is reflected by the equilibrium equation (2.2), where E is a damping matrix, assumed to be symmetric positive definite. The vectors $\lambda_n(t) \in \mathbb{R}^{n_c}$ and $\lambda_t(t) \in \mathbb{R}^{n_c}$ represent all normal and tangential forces, while $W_n(q,u,t)$ and $W_t(q,u,t)$ are the normal and tangential wrench matrices. More precisely, the k-th column of $W_n(q,u,t)$ $(W_t(q,u,t))$ is the normal (tangential) wrench vector $W_{nk}(q,u,t)$ $(W_{tk}(q,u,t))$ corresponding to contact k, $k = \overline{1,n_c}$, with n_c denoting the number of active contacts. The vector $\Psi_n(q,u,t)$ contains the normal displacements for configuration q, controls u and time t. More precisely, $\Psi_n(q,u,t) = [\Psi_{n1}(q,u,t),...,\Psi_{nn_c}(q,u,t)]^T$, where $\Psi_{nk}(q,u,t)$ represents the normal displacement function corresponding to contact k. In a similar way, one defines the vector of tangential displacements, $\Psi_t(q,u,t) = [\Psi_{t1}(q,u,t),...,\Psi_{tn_c}(q,u,t)]^T$. Equation (2.3) represents the contact and non-penetration constraints; that is whenever the normal separation at contact k is strictly positive $(\Psi_{nk}(q,u,t) > 0)$, the corresponding normal force is 0 $(\lambda_{nk} = 0)$, while whenever contact k is established $(\Psi_{nk}(q,u,t) = 0)$, the corresponding normal force is nonnegative $(\lambda_{nk} \ge 0)$.

Equation (2.4) defines the positive, $\dot{s}^+_{tk}(t)$, and negative, $\dot{s}^-_{tk}(t)$, sliding velocities at contact k. The right-hand side of (2.4) represents the (overall) sliding velocity $\dot{s}_{tk}(t) := \dot{\Psi}_{tk}(q,u,t) = (W_{tk}(q,u,t))^T v(t) + \frac{\partial \Psi_{tk}}{\partial t}(q,u,t)$ at contact k. The last two equations, namely (2.5) and (2.6), represent Coulomb's friction law at contact k, with $\mu_k \in [0,1]$ being the friction coefficients.

3. The time-stepping scheme

Let t_l denote the time at which one has a solution configuration q^l and let $t_{l+1} = t_l + h$ denote the time at which one would want an estimate of the solution. We approximate the new configuration q^{l+1} using a backward Euler formula, as follows

$$q^{l+1} = q^l + hv^{l+1},$$

where v^{l+1} is an estimate for the new velocity and will be found by solving a mixed linear complementarity problem. At each integration step the unknowns $\left(hv^{l+1}, h\lambda_n^{l+1}, h\lambda_f^{l+1}, h\sigma^{l+1}\right)$ may be obtained as the solution of the following MLCP:

$$\begin{pmatrix} 0 \\ \rho_n^{l+1} \\ \rho_f^{l+1} \\ s^{l+1} \end{pmatrix} = \begin{pmatrix} E & -W_n^l & -W_f^l & 0 \\ (W_n^l)^T & 0 & 0 & 0 \\ (W_f^l)^T & 0 & 0 & E_f \\ 0 & U_f & -E_f^T & 0 \end{pmatrix} \begin{pmatrix} hv^{l+1} \\ h\lambda_n^{l+1} \\ h\lambda_f^{l+1} \\ h\sigma^{l+1} \end{pmatrix} + \begin{pmatrix} 0 \\ \Psi_n^l + h\frac{\partial \Psi_n}{\partial t}^l \\ h\frac{\partial \Psi_f}{\partial t}^l \\ 0 \end{pmatrix} \tag{3.1}$$

with $0 \le \left[\rho_n^{l+1}, \rho_f^{l+1}, s^{l+1}\right] \quad \perp \quad \left[h\lambda_n^{l+1}, h\lambda_f^{l+1}, h\sigma^{l+1}\right] \ge 0$. Here $U_f \in \mathbb{R}^{n_c \times n_c}$, $E_f \in \mathbb{R}^{2n_c \times n_c}$ with U_f a diagonal matrix with elements on its diagonal equal to μ_k, $k = 1, ..., n_c$ and E_f a block diagonal matrix, with diagonal blocks given by the vector e (e is a two-dimensional vector of all ones). That is,

$$U_f = \begin{pmatrix} \mu_1 & ... & 0 \\ \vdots & & \vdots \\ 0 & ... & \mu_{n_c} \end{pmatrix}, \quad E_f = \begin{pmatrix} 1 & ... & 0 \\ 1 & ... & 0 \\ \vdots & & \vdots \\ 0 & ... & 1 \\ 0 & ... & 1 \end{pmatrix}.$$

The superscript l used in the MLCP (3.1) indicates that all the corresponding quantities are calculated with $q := q^l$ and $t := t_l$. For each contact k we define the 3×2 matrix $W_{fk}(q, u, t)$ by joining the column vectors $W_{tk}(q, u, t)$ and $-W_{tk}(q, u, t)$. That is,

$$W_{fk}(q, u, t) = \left[W_{tk}(q, u, t) \quad - W_{tk}(q, u, t)\right].$$

If we put all the active contacts together we obtain the "frictional" wrench matrix $W_f(q, u, t)$ appearing in formulation (3.1). In a similar way, we get the vector $\Psi_f(q, u, t)$.

Solvability and the Friction Cone. For an active contact k, we define the friction cone corresponding to that contact by

$$FC_k(q, u, t) = \left\{z = W_{nk}\lambda_{nk} + W_{fk}\lambda_{fk} \mid \lambda_{nk} \ge 0,\ \lambda_{fk} \ge 0,\ e^T\lambda_{fk} \le \mu_k\lambda_{nk}\right\}, \tag{3.2}$$

where $W_{nk} := W_{n,k}(q, u, t)$, $W_{fk} := W_{fk}(q, u, t)$ and $e = [1, 1]^T$. The total friction cone, $FC(q, u, t)$, which accounts for all active contacts is defined by

$$FC(q, u, t) = \sum_{k=1}^{n_c} FC_k(q, u, t).$$

Using the fact that the matrix E in the MLCP (3.1) is positive definite, we can eliminate the variables hv^{l+1} and reduce the MLCP to a standard LCP with a copositive matrix. It can be shown that the resulting LCP, is solvable whenever the total friction cone $FC(q^l, u, t_l)$ is pointed. We recall that a cone is pointed if it doesn't contain any proper subspace. The lack of pointedness for the friction cone results in jammed configurations (see [3]) and therefore this regularity assumption is very realistic and can be successfully used in devising randomized plans (see [1]).

4. Conclusions

We have discussed an LCP-based time-stepping scheme that can be used to simulate rigid body systems in a quasi-static setting. The scheme was introduced and successfully used for a particular case in [1]. Solvability of the integration step is guaranteed by the pointedness of the friction cone, an assumption that is common in dynamic settings as well (see [3] and [4] for example).

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References

[1] Cheng, P., Cappelleri, S., Gavrea, B., Kumar, V., *Planning and Control of Mesoscale Manipulation Tasks with Uncertainties*, Proceedings of Robotics, Science and Systems (3rd: 2007, Atlanta GA), MIT PRESS, 2008.

[2] Cottle, R.W., Pang, J.S., Stone, R.E., *The Linear Complementarity Problem*, Academic Press, Boston, MA, 1992.

[3] Anitescu, M., *Optimization-based simulation of nonsmooth dynamics*, Mathematical Programming, **105**(1990), no. 1, 113-143.

[4] Gavrea, B., Anitescu, M., Potra, F.A., *Convergence of a Class of Semi-Implicit Time-Stepping schemes for Nonsmooth Rigid Multibody Dynamics*, SIAM Journal on Optimization, **19**(2008), no. 2, 969-1001.

Bogdan Gavrea
Technical University of Cluj-Napoca
Faculty of Automation and Computer Science
28, Str. Memorandumului
400114 Cluj-Napoca
Romania
e-mail: Bogdan.Gavrea@math.utcluj.ro