Approximation by max-product type nonlinear operators

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Abstract. The purpose of this survey is to present some approximation and shape preserving properties of the so-called nonlinear (more exactly sublinear) and positive, max-product operators, constructed by starting from any discrete linear approximation operators, obtained in a series of recent papers jointly written with B. Bede and L. Coroianu. We will present the main results for the max-product operators of: Bernstein-type, Favard-Szász-Mirakjan-type, truncated Favard-Szász-Mirakjan-type, Baskakov-type, truncated Baskakov-type, Meyer-König and Zeller-type, Bleimann-Butzer-Hahn-type, Hermite-Fejér interpolation-type on Chebyshev nodes of first kind, Lagrange interpolation-type on Chebyshev knots of second kind, Lagrange interpolation-type on arbitrary knots, generalized sampling-type, sampling sinc-type, Cardaliaguet-Euvrard neural network-type.

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1. Introduction

The idea of construction of these operators goes back to a paper of Bede, B., Nobuhara, H., Fodor, J. and Hirota K. [11], where it is applied to the rational approximation operators of Shepard. How could be applied to any linear and discrete Bernstein-type operator I have shown in my book Gal [18], pp. 324-326, Open Problem 5.5.4, where also a general form for the estimate in terms of the modulus of continuity is obtained.

The construction is based on a simple idea, exemplified for the case of Bernstein polynomials, as follows.

Let $B_n(f)(x) = \sum_{k=0}^{n} p_{n,k}(x)f(k/n)$ be with $p_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}$ and $f : [0,1] \to \mathbb{R}$. If in the obvious formula

$$B_n(f)(x) = \frac{\sum_{k=0}^{n} p_{n,k}(x)f(k/n)}{\sum_{k=0}^{n} p_{n,k}(x)}, x \in [0,1],$$

we replace the $\sum$ operator with the max operator denoted by $\bigvee$, then we obtain the so-called max-product Bernstein nonlinear (sublinear), piecewise rational operator by (Gal [18], p. 325)

$$B_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} p_{n,k}(x)f(k/n)}{\bigvee_{k=0}^{n} p_{n,k}(x)}, x \in [0,1],$$

where recall

$$\bigvee_{k=0}^{n} p_{n,k}(x)f(k/n) := \max_{0 \le k \le n} \{p_{n,k}(x)f(k/n)\}.$$

The same idea of construction can be applied to any discrete linear Bernstein-type operator or to any discrete linear interpolation operator, obtaining thus the corresponding nonlinear max-product operators (well-defined because the denominators of these new operators always are strictly positive).

Surprisingly, the max-product operators do not lose the approximation properties of the corresponding linear operators to which they are attached. Moreover, for large classes of functions, they improve the order of approximation to the Jackson-type order. The most important improvement is in the case of interpolation (on any arbitrary system of nodes), when for the whole class of continuous functions the Jackson order $\omega_1(f; 1/n)$ is achieved. Also, the max-product Bernstein-type operators preserve the monotonicity and the quasi-convexity of the functions.

In this survey we will present the main results for the max-product operators of: Bernstein-type, Favard-Szász-Mirakjan-type, truncated Favard-Szász-Mirakjan-type, Baskakov-type, truncated Baskakov-type, Meyer-König and Zeller-type, Bleimann-Butzer-Hahn-type, Hermite-Fejér interpolation-type on Chebyshev nodes of first kind, Lagrange interpolation-type on Chebyshev knots of second kind, Lagrange interpolation-type on arbitrary knots, generalized sampling-type, sampling sinc-type, Cardaliaguet-Euvrard neural network-type.

2. Approximation by max-product operators of Bernstein-type

Denote

$$C_+[0,1] = \{f : [0,1] \to \mathbb{R}_+; f \text{ is continuous on } [0,1]\}.$$

This section contains the approximation and shape preserving properties for a series of important max-product Bernstein-type operators.

Theorem 2.1. *For $f \in C_+[0,1]$, define the max-product Bernstein operator by (Gal [18], p. 325)*

$$B_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} p_{n,k}(x) f(k/n)}{\bigvee_{k=0}^{n} p_{n,k}(x)}, x \in [0,1].$$

(i) *(Bede-Coroianu-Gal [4]) For any $j \in \{0, 1, ..., n\}$ and $x \in [\frac{j}{n+1}, \frac{j+1}{n+1}]$ we have*

$$B_n^{(M)}(f)(x) = \bigvee_{k=0}^{n} f_{k,n,j}(x).$$

where $f_{k,n,j}(x) = \frac{\binom{n}{k}}{\binom{n}{j}} \left(\frac{x}{1-x}\right)^{k-j} f\left(\frac{k}{n}\right)$. This form suggested the denomination of "max-product" operator for $B_n^{(M)}$ (that is the maximum of the product of the values of f on nodes with some rational functions).

(ii) *(Bede-Coroianu-Gal [4]) $B_n^{(M)}(f)(x)$ is a continuous, piecewise convex and piecewise rational function on $[0,1]$.*

(iii) *(Bede-Coroianu-Gal [4]) For all $x \in [0,1]$, $n \in \mathbb{N}$ we have*

$$|B_n^{(M)}(f)(x) - f(x)| \leq 12\omega_1\left(f; \frac{1}{\sqrt{n+1}}\right),$$

where

$$\omega_1(f; \delta) = \sup\{|f(x) - f(y)|; x, y \in [0,1], |x - y| \leq \delta\}.$$

(iv) *(Coroianu-Gal [15]) There exists $f \in C_+[0,1]$ such that the order in (iii) is exactly $1/\sqrt{n+1}$, that is on the whole class $C_+[0,1]$, the order in (iii) cannot be improved.*

(v) *(Coroianu-Gal [15]) If $f \in C_+[0,1]$ is strictly positive on $[0,1]$ then*

$$\|B_n^{(M)}(f) - f\| \leq C_f \left\{ n \left[\omega_1\left(f; \frac{1}{n}\right)\right]^2 + \omega_1\left(f; \frac{1}{n}\right)\right\}.$$

(vi) *(Coroianu-Gal [15]) If $f \in Lip1$ then by (v)*

$$\|B_n^{(M)}(f) - f\| \leq \frac{C_f}{n}, n \in \mathbb{N}.$$

(vii) *(Coroianu-Gal [15]) If $f \in Lip\,\alpha$, then (v) gives the approximation order $1/n^{2\alpha-1}$, which for $\alpha \in (2/3, 1]$ is essentially better than the general approximation order $O[\omega_1(f; 1/\sqrt{n})] = O[1/n^{\alpha/2}]$ given by (iii).*

(viii) *(Bede-Coroianu-Gal [4]) If $f : [0,1] \to \mathbb{R}_+$ is a concave function then we have the Jackson-type estimate*

$$\left\|B_n^{(M)}(f)(x) - f(x)\right\| \leq 2\omega_1\left(f; \frac{1}{n}\right), n \in \mathbb{N}.$$

(ix) *(Coroianu-Gal [15]) If $f \in C_+[0,1]$ is strictly positive then the pointwise estimate holds*

$$|B_n^{(M)}(f)(x) - f(x)| \leq 24\omega_1\left(f, \sqrt{\frac{x(1-x)}{n}}\right),$$

for all $x \in [0, 1/(n+1)] \cup [n/(n+1), 1]$, *and*

$$\left| B_n^{(M)}(f)(x) - f(x) \right| \leq \left(\frac{n\omega_1(f, \frac{1}{n})}{m_f} + 4 \right) \omega_1(f, \frac{1}{n}),$$

for all $x \in [1/(n+1), n/(n+1)]$.

   (x) (Bede-Coroianu-Gal [4]) $f : [0,1] \to \mathbb{R}$ *is called quasi-convex (quasi-concave) on* $[0,1]$ *if it satisfies the inequality (for all* $x, y, \lambda \in [0,1]$)

$$f(\lambda x + (1 - \lambda)y) \leq (\geq) \max\{f(x), f(y)\}.$$

$B_n^{(M)}(f)$, $n \in \mathbb{N}$, *preserve the quasi-convexity, quasi-concavity and monotonicity of* f.

Remarks. 1) Comparing with the approximation by the Bernstein polynomials, clearly for large classes of functions, $B_n^{(M)}$ gives essentially better estimates.

   2) The problem of finding the saturation class for $B_n^{(M)}$ is still open. Clearly it is different from the saturation class of the Bernstein polynomials.

   For $f \in C_+[0, \infty)$ we define the Bleimann-Butzer-Hahn max-product operators by (Gal [18], p. 326)

$$H_n^{(M)}(f)(x) = \frac{\bigvee\limits_{k=0}^{n} \binom{n}{k} x^k f\left(\frac{k}{n+1-k}\right)}{\bigvee\limits_{k=0}^{n} \binom{n}{k} x^k}.$$

Theorem 2.2. (Bede-Coroianu-Gal [8]) *(i) If* $f : [0, \infty) \to \mathbb{R}_+$ *is continuous, then for any* $n + 1 \geq \max\{1 + 2x, 16x(1 + x)\}$ *we have*

$$|H_n^{(M)}(f)(x) - f(x)| \leq 5\omega_1\left(f, \frac{(1+x)^{\frac{3}{2}}\sqrt{x}}{\sqrt{n+1}} \right), x \in [0, \infty),$$

where

$$\omega_1(f, \delta) = \sup\{|f(x) - f(y)|; x, y \in [0, \infty), |x - y| \leq \delta\}.$$

   (ii) If $f : [0, \infty) \to \mathbb{R}_+$ *is a nondecreasing concave function, then for* $x \in [0, \infty)$, $n \geq 2x$,

$$\left| H_n^{(M)}(f)(x) - f(x) \right| \leq 2\omega_1\left(f; \frac{(1+x)^2}{n} \right).$$

   (iii) $H_n^{(M)}(f)$, $n \in \mathbb{N}$, *preserve the monotonicity and the quasi-convexity of* f.

   For $f \in C_+[0, 1)$ we define the Meyer-König and Zeller max-product operators by (Gal [18], p. 326)

$$Z_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{\infty} \binom{n+k}{k} x^k f(k/(n+k))}{\bigvee_{k=0}^{\infty} \binom{n+k}{k} x^k}, x \in [0, 1), n \in \mathbb{N}.$$

Theorem 2.3. (Bede-Coroianu-Gal [5]) *(i) If $f : [0,1] \to \mathbb{R}_+$ is continuous on $[0,1]$, then for $n \geq 4$ we have*

$$|Z_n^{(M)}(f)(x) - f(x)| \leq 18\omega_1\left(f, \frac{(1-x)\sqrt{x}}{\sqrt{n}}\right), x \in [0,1],$$

where

$$\omega_1(f,\delta) = \sup\{|f(x) - f(y)|; x, y \in [0,1], |x - y| \leq \delta\}.$$

(ii) If $f : [0,1] \to \mathbb{R}_+$ is a continuous, nondecreasing concave function, then

$$\left|Z_n^{(M)}(f)(x) - f(x)\right| \leq \omega_1\left(f; \frac{1}{n}\right), x \in [0,1], n \in \mathbb{N}.$$

(iii) $Z_n^{(M)}(f)$, $n \in \mathbb{N}$, preserve the monotonicity and the quasi-convexity of f.

For $f \in C_+[0,\infty)$ and $f \in C_+[0,1]$, we define the Favard-Szász-Mirakjan max-product (Gal [18], p. 326) and the truncated Favard-Szász-Mirakjan max-product operators (Bede-Coroianu-Gal [7]) by

$$F_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right)}{\bigvee_{k=0}^{\infty} \frac{(nx)^k}{k!}}, x \in [0,\infty), n \in \mathbb{N}$$

and

$$T_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right)}{\bigvee_{k=0}^{n} \frac{(nx)^k}{k!}}, x \in [0,1], n \in \mathbb{N},$$

respectively.

Theorem 2.4. (Bede-Coroianu-Gal [10], [7]) *(i)*

$$|F_n^{(M)}(f)(x) - f(x)| \leq 8\omega_1\left(f, \frac{\sqrt{x}}{\sqrt{n}}\right), n \in \mathbb{N}, x \in [0,\infty),$$

where

$$\omega_1(f,\delta) = \sup\{|f(x) - f(y)|; x, y \in [0,\infty), |x - y| \leq \delta\},$$

and

$$|T_n^{(M)}(f)(x) - f(x)| \leq 6\omega_1\left(f, \frac{1}{\sqrt{n}}\right), n \in \mathbb{N}, x \in [0,1].$$

(ii) If $f : [0,\infty) \to \mathbb{R}_+$ is a nondecreasing concave function on $[0,\infty)$, then

$$\left|F_n^{(M)}(f)(x) - f(x)\right| \leq \omega_1\left(f; \frac{1}{n}\right), x \in [0,\infty), n \in \mathbb{N}.$$

(iii) If $f : [0,1] \to \mathbb{R}_+$ is a nondecreasing concave function on $[0,1]$, then

$$|T_n^{(M)}(f)(x) - f(x)| \leq 6\omega_1\left(f, \frac{1}{n}\right), n \in \mathbb{N}, x \in [0,1].$$

(iv) $F_n^{(M)}(f)$ *and* $T_n^{(M)}(f)$, $n \in \mathbb{N}$, *preserve the monotonicity and the quasi-convexity of* f *on the corresponding intervals.*

For $f \in C_+[0, \infty)$ and $f \in C_+[0, 1]$, we define Baskakov max-product (Gal [18], p. 326) and the truncated Baskakov max-product operators (Bede-Coroianu-Gal [9]) by, respectively

$$V_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{\infty} b_{n,k}(x) f\left(\frac{k}{n}\right)}{\bigvee_{k=0}^{\infty} b_{n,k}(x)},$$

and

$$U_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} b_{n,k}(x) f\left(\frac{k}{n}\right)}{\bigvee_{k=0}^{n} b_{n,k}(x)}, x \in [0,1], n \in \mathbb{N}, n \geq 1,$$

where $b_{n,k}(x) = \binom{n+k-1}{k} x^k / (1+x)^{n+k}$.

Theorem 2.5. (Bede-Coroianu-Gal [6], [9]) *(i) For $n \geq 3$ and $x \in [0, \infty)$ we have*

$$|V_n^{(M)}(f)(x) - f(x)| \leq 12\omega_1\left(f, \sqrt{\frac{x(x+1)}{n-1}}\right),$$

where

$$\omega_1(f, \delta) = \sup\{|f(x) - f(y)|; x, y \in [0, \infty), |x - y| \leq \delta\}.$$

Also, for $n \in \mathbb{N}, n \geq 2, x \in [0, 1]$ we have

$$|U_n^{(M)}(f)(x) - f(x)| \leq 24\omega_1\left(f, \frac{1}{\sqrt{n+1}}\right),$$

where

$$\omega_1(f, \delta) = \sup\{|f(x) - f(y)|; x, y \in [0, 1], |x - y| \leq \delta\}.$$

(ii) If $f : [0, \infty) \to [0, \infty)$ is a nondecreasing concave function on $[0, \infty)$, then for $n \geq 3$, $x \in [0, \infty)$,

$$\left|V_n^{(M)}(f)(x) - f(x)\right| \leq 2\omega_1\left(f; \frac{x+1}{n-1}\right).$$

(iii) If $f : [0, 1] \to [0, \infty)$ is a nondecreasing concave function on $[0, 1]$, then

$$\left|U_n^{(M)}(f)(x) - f(x)\right| \leq 2\omega_1\left(f; \frac{1}{n}\right), x \in [0, 1], n \in \mathbb{N}.$$

(iv) $V_n^{(M)}(f)$ and $U_n^{(M)}(f)$, $n \in \mathbb{N}$, preserve the monotonicity and the quasi-convexity of f on the corresponding intervals.

Remark. The estimates in Theorems 2.1, (iii), and Theorems 2.2-2.5, (i), were obtained by using the following general result:

Theorem 2.6. (Gal [18], p. 326, Bede-Gal [3]) *Let $I \subset \mathbb{R}$ be a bounded or unbounded interval,*

$$CB_+(I) = \{f : I \to \mathbb{R}_+; f \text{ continuous and bounded on } I\},$$

and $L_n : CB_+(I) \rightarrow CB_+(I)$, $n \in \mathbb{N}$ be a sequence of **positive homogenous** operators, satisfying in addition the following properties:

(i) (**Monotonicity**) if $f, g \in CB_+(I)$ satisfy $f \leq g$ then $L_n(f) \leq L_n(g)$ for all $n \in N$;

(ii) (**Sublinearity**) $L_n(f + g) \leq L_n(f) + L_n(g)$ for all $f, g \in CB_+(I)$.

Then for all $f \in CB_+(I)$, $n \in N$ and $x \in I$ we have

$$|f(x) - L_n(f)(x)| \leq$$

$$\left[\frac{1}{\delta} L_n(\varphi_x)(x) + L_n(e_0)(x)\right] \omega_1(f; \delta)_I + f(x) \cdot |L_n(e_0)(x) - 1|,$$

where $\delta > 0$, $e_0(t) = 1$ for all $t \in I$, $\varphi_x(t) = |t - x|$.

Remarks. 1) The above Theorem 2.6 is a generalization of the classical one for Positive Linear Operators, because the **Positivity + Linearity** imply the

$$\textbf{Positivity + Sublinearity + Positive homogeneity}$$

$$\textbf{+Monotonicity},$$

but the converse implication does not hold, taking into account that the max product operators are counterexamples.

2) The Jackson-type estimates (for subclasses of functions) in Theorems 2.1-2.5, were obtained by direct reasonings.

3) The saturation results for the above max-product Bernstein-type operators are interesting open questions.

3. Approximation by interpolation max-product operators

In this section we present the approximation properties of a series of max-product interpolation operators.

Consider the Hermite-Fejér interpolation polynomial of degree $\leq 2n+1$ attached to $f : [-1, 1] \rightarrow \mathbb{R}$ and to the Chebyshev knots of first kind, $x_{n,k} = \cos\left(\frac{2(n-k)+1}{2(n+1)}\pi\right)$,

$$H_{2n+1}(f)(x) = \sum_{k=0}^{n} h_{n,k}(x) f(x_{n,k}),$$

with

$$h_{n,k}(x) = (1 - x x_{n,k}) \cdot \left(\frac{T_{n+1}(x)}{(n+1)(x - x_{n,k})}\right)^2,$$

$T_{n+1}(x) = \cos[(n+1)arccos(x)]$-Chebyshev polynomials. Because

$$H_{2n+1}(f)(x) = \frac{\sum_{k=0}^{n} h_{n,k}(x) f(x_{n,k})}{\sum_{k=0}^{n} h_{n,k}(x)},$$

by the max-product method the corresponding max-product Hermite-Fejér interpolation operator is

$$H_{2n+1}^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} h_{n,k}(x)f(x_{n,k})}{\bigvee_{k=0}^{n} h_{n,k}(x)}.$$

Remark. We have $H_{2n+1}^{(M)}(f)(x_{n,j}) = f(x_{n,j})$, for all $j \in \{0, ..., n\}$.

Theorem 3.1. (Coroianu-Gal [14]) *If $f : [-1, 1] \to \mathbb{R}_+$ is continuous on $[-1, 1]$ then for all $x \in [-1, 1]$ and $n \in \mathbb{N}$*

$$\|H_{2n+1}^{(M)}(f) - f\| \leq 14\omega_1\left(f, \frac{1}{n+1}\right).$$

Remark. For $f \in Lip_1[-1, 1]$, we have $\|H_{2n+1}^{(M)}(f) - f\| \leq \frac{c}{n+1}$, while it is well-known that $\|H_{2n+1}(f) - f\| \sim \frac{ln(n+1)}{n+1}$.

Let now $x_{n,k} \in [-1, 1]$, $k \in \{1, ..., n\}$, be arbitrary and consider the Lagrange interpolation polynomial of degree $\leq n - 1$ attached to f and to the nodes $(x_{n,k})_k$,

$$L_n(f)(x) = \sum_{k=1}^{n} l_{n,k}(x)f(x_{n,k}),$$

with

$$l_{n,k}(x) =$$
$$\frac{(x - x_{n,1})...(x - x_{n,k-1})(x - x_{n,k+1})...(x - x_{n,n})}{(x_{n,k} - x_{n,1})...(x_{n,k} - x_{n,k-1})(x_{n,k} - x_{n,k+1})...(x_{n,k} - x_{n,n})}.$$

Because $\sum_{k=1}^{n} l_{n,k}(x) = 1$, for all $x \in \mathbb{R}$, we can write

$$L_n(f)(x) = \frac{\sum_{k=1}^{n} l_{n,k}(x)f(x_{n,k})}{\sum_{k=1}^{n} l_{n,k}(x)}, \text{ for all } x \in I.$$

Therefore, its corresponding max-product interpolation operator will be given by

$$L_n^{(M)}(f)(x) = \frac{\bigvee_{k=1}^{n} l_{n,k}(x)f(x_{n,k})}{\bigvee_{k=1}^{n} l_{n,k}(x)}, x \in I.$$

Remark. We have $L_n^{(M)}(f)(x_{n,k}) = f(x_{n,k})$, $k = 1, ..., n$.

Theorem 3.2. (Coroianu-Gal [12]) *If $x_{n,k} = \cos\left(\frac{n-k}{n-1}\pi\right)$, $k = 1, ..., n$ and $f : [-1, 1] \to \mathbb{R}_+$ then*

$$\|L_n^{(M)}(f) - f\| \leq 28\omega_1\left(f, \frac{1}{n-1}\right), n \geq 3.$$

Remarks. 1) For the linear Lagrange polynomials we have the worst estimate

$$\|L_n(f) - f\| \le C\omega_1\left(f; \frac{1}{n}\right) ln(n), n \in \mathbb{N}.$$

2) The case of other kind of nodes (e.g. equidistant, or roots of orthogonal polynomials, etc) can be found in the joint paper [17] with L. Coroianu published in this proceedings.

Now, consider the truncated Whittaker (sinc) series defined by

$$W_n(f)(x) = \sum_{k=0}^{n} \frac{\sin(nx - k\pi)}{nx - k\pi} \cdot f\left(\frac{k\pi}{n}\right), x \in [0, \pi],$$

and the truncated max-product Whittaker operator given by

$$W_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^{n} \frac{\sin(nx-k\pi)}{nx-k\pi} \cdot f\left(\frac{k\pi}{n}\right)}{\bigvee_{k=0}^{n} \frac{\sin(nx-k\pi)}{nx-k\pi}}, x \in [0, \pi]$$

Remark. Clearly, $W_n^{(M)}(f)(j\pi/n) = f(j\pi/n)$, for all $j \in \{0, ..., n\}$.

Theorem 3.3. (Coroianu-Gal [16]) *If* $f : [0, \pi] \to \mathbb{R}_+$ *is continuous then*

$$|W_n^{(M)}(f)(x) - f(x)| \le 4\omega_1\left(f; \frac{1}{n}\right)_{[0,\pi]}, n \in \mathbb{N}, x \in [0, \pi].$$

Remark. If $\lim_{n\to\infty} \omega_1(f; 1/n) \ln(n) = 0$ then $W_n(f)(x) \to f(x)$ uniformly inside of $(0, \pi)$ and pointwise in $[0, \pi]$, while it is known that $\|W_n(1) - 1\| \ge \frac{1}{3\pi}$, for all $n \ge 2$.

4. Approximation by sampling and neural networks max-prod operators

This section contains approximation results for some max-product sampling operators and for some max-product neural networks operators.

Definition 4.1. (Bardaro-Butzer-Stens-Vinti [2]) A function $\varphi \in C(\mathbb{R})$ is called a time-limited kernel (for a sampling operator), if:

(i) There exist $T_0, T_1 \in \mathbb{R}, T_0 < T_1$, such that $\varphi(t) = 0$ for all $t \notin [T_0, T_1]$;

(ii) $\sum_{k=-\infty}^{\infty} \varphi(u - k) = 1$, for all $u \in \mathbb{R}$.

If φ is a time-limited kernel and $W > 0$, then

$$S_{W,\varphi}(f)(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k}{W}\right) \varphi(Wt - k), t \in \mathbb{R},$$

will be called a generalized sampling operator.

Taking into account Definition 4.1, (ii), we can write

$$S_{W,\varphi}(f)(t) = \frac{\sum_{k=-\infty}^{\infty} f\left(\frac{k}{W}\right) \varphi(Wt - k)}{\sum_{k=-\infty}^{\infty} \varphi(Wt - k)}, t \in \mathbb{R}.$$

Remark. If e.g. $\varphi(t) = sinc(t) = \frac{sin(\pi t)}{\pi t}$, then $S_{W,\varphi}(f)(t)$ becomes the Whittaker cardinal (sinc) series.

Therefore, applying the max-product method, the corresponding max-product Whittaker operator will be given by

$$S_{W,\varphi}^{(M)}(f)(t) = \frac{\bigvee_{k=-\infty}^{\infty} \varphi(Wt-k)f\left(\frac{k}{W}\right)}{\bigvee_{k=-\infty}^{\infty} \varphi(Wt-k)}, t \in \mathbb{R}.$$

Theorem 4.2. (Coroianu-Gal [13]) *If $\varphi(t) = sinc(t) = \frac{\sin(\pi t)}{\pi t}$ and $f : \mathbb{R} \to \mathbb{R}_+$ is bounded and continuous on $\mathbb{R}$, then*

$$|S_{W,\varphi}^{(M)}(f)(t) - f(t)| \le 2\omega_1\left(f; \frac{1}{W}\right)_{\mathbb{R}}, \text{ for all } t \in \mathbb{R},$$

where $\omega_1(f; \delta)_{\mathbb{R}} = \sup\{|f(u) - f(v)|; u, v \in \mathbb{R}, |u-v| \le \delta\}$.

Remarks. 1) If $f \in Lip\alpha$, $\alpha \in (0,1]$, then in Theorem 4.2 we get $\|S_{W,\varphi}^{(M)}(f) - f\| = O\left(\frac{1}{W^\alpha}\right)$, while it is well-known that for the usual Whitaker cardinal series, we have the worst estimate

$$\|S_{W,\varphi}(f) - f\| = O\left(\frac{\log(W)}{W^\alpha}\right).$$

2) We get similar results for other kernels $\varphi(t)$ too.

The Cardaliaguet-Euvrard neural network is defined by

$$C_{n,\alpha}(f)(x) = \sum_{k=-n^2}^{n^2} \frac{f(k/n)}{I \cdot n^{1-\alpha}} \cdot b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right),$$

where $0 < \alpha < 1$, $n \in \mathbb{N}$ and $f : \mathbb{R} \to \mathbb{R}$ is continuous and bounded or uniformly continuous on $\mathbb{R}$.

The corresponding max-product Cardaliaguet-Euvrard network operator is formally given by

$$C_{n,\alpha}^{(M)}(f)(x) = \frac{\bigvee_{k=-n^2}^{n^2} b\left[n^{1-\alpha}\left(x - \frac{k}{n}\right)\right] f\left(\frac{k}{n}\right)}{\bigvee_{k=-n^2}^{n^2} b\left[n^{1-\alpha}\left(x - \frac{k}{n}\right)\right]}, x \in \mathbb{R}.$$

Theorem 4.3. (Anastassiou-Coroianu-Gal [1]) *Let $b(x)$ be a centered bell-shaped function, continuous and with compact support $[-T,T]$, $T > 0$ and $0 < \alpha < 1$. In addition, suppose that the following requirements are fulfilled:*

(i) There exist $0 < m_1 \le M_1 < \infty$ such that $m_1(T-x) \le b(x) \le M_1(T-x)$ for all $x \in [0,T]$;

(ii) There exist $0 < m_2 \le M_2 < \infty$ such that $m_2(x+T) \le b(x) \le M_2(x+T)$ for all $x \in [-T,0]$.

Then for all $f \in CB_+(R)$, $x \in \mathbb{R}$ and for all $n \in \mathbb{N}$ satisfying $n > \max\{T + |x|, (2/T)^{1/\alpha}\}$, we have the estimate

$$|f(x) - C_{n,\alpha}^{(M)}(f)(x)| \leq c\omega_1\left(f; n^{\alpha-1}\right)_\mathbb{R},$$

where

$$c = 2\left(\max\left\{\frac{TM_2}{2m_2}, \frac{TM_1}{2m_1}\right\} + 1\right).$$

Remark. Let $f \in Lip\alpha$. For $\frac{1}{2} \leq \alpha < 1$, we get the same order of approximation $O\left(\frac{1}{n^{1-\alpha}}\right)$ for both operators $C_{n,\alpha}(f)(x)$ and $C_{n,\alpha}^{(M)}(f)(x)$, while for $0 < \alpha < \frac{1}{2}$, the approximation order obtained by the max-product operator $C_{n,\alpha}^{(M)}(f)(x)$ is essentially better than that obtained by the linear operator $C_{n,\alpha}(f)(x)$.

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