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On a quaternion valued Gaussian random variables

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Abstract. In the present note we show that Polya's type characterization theorem of Gaussian distributions does not hold. This happens because in the linear form, constituted by the independent copies of quaternion random variables, a part of the quaternion coefficients is written on the right hand side and another part on the left side. This gives a negative answer to the question posed in [1].

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The present note is a natural extension of paper [1] where the formulation and proof of Polya's theorem on the characterization of Gaussian random variables with values in quaternion algebra is considered. We mean the following well-known theorem of Polya:

Theorem 1.1. *Let $\xi_1, \xi_2, ..., \xi_n$, $n \geq 2$ be i.i.d. random variables and $(a_1, a_2, ..., a_n)$ be nonzero reals that satisfy the condition $\sum_{h=1}^{n} a_h^2 = 1$. If the sum $\sum_{h=1}^{n} a_h \xi_h$ has the same distribution as ξ_1, then ξ_1 is a Gaussian random variable.*

If the random variable takes values in the quaternion algebra then three types of Gaussian random variables are considered: real, complex and quaternion Gaussian random variables. Let us recall the definition of complex and quaternion Gaussian random variables. The usual motivation for these definitions comes from the form of characteristic function of a centered Gaussian random variable, see e.g. [2]. For the real case this is given as

$$\exp\{-\frac{1}{2}t^2 E\xi^2\}, \ \forall t \in R.$$

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For the complex (quaternion) case we would analogously expect the characteristic function to be

$$\exp\{-c|q|^2 E|\xi|^2|\}, \forall q \in C, (\forall q \in Q), c > 0. \tag{1.1}$$

The characteristic function of a complex (quaternion) random variables ξ is defined as

$$\chi_\xi(q) = E\exp(i\mathrm{Re}(\xi\overline{q}))$$

and if we want the characteristic function of centered complex (quaternion) Gaussian random variable to have the form (1.1), then the covariance matrix of real two dimensional vector $(\xi^{'}, \xi^{''})$ (four dimensional vector $(\xi^{'}, \xi^{''}, \xi^{'''}, \xi^{IV})$) should be proportional to the identity matrix. Thus the covariance matrices of complex (quaternion) Gaussian random variables have a quite specific form: they are proportional to unit matrices in R^2 (in R^4). Therefore the coordinates of corresponding two dimension (four dimension) random vector $(\xi^{'}, \xi^{''})$, $((\xi^{'}, \xi^{''}, \xi^{'''}, \xi^{IV}))$ are mutually independent and have the same variances.

In [3] there is formulated Polya's theorem for the case of complex random variables.

Theorem 1.2. *Let ξ be a complex random variable, $\xi_1, \xi_2, ..., \xi_n$, $n \geq 2$ be independent copies of ξ and $(a_1, a_2, ..., a_n)$ be nonzero complex numbers such that $\sum_{h=1}^n |a_h|^2 = 1$ and at least one of them is not a real number. If $\sum_{h=1}^n a_h\xi_h$ has the same distribution as ξ, then ξ is a complex Gaussian random variable.*

As we see in the complex case there is an additional condition on the complex coefficients $(a_1, a_2, ..., a_n)$, for the Theorem 1.2 to be true, namely one of these coefficients should be essentially complex number. In [1] there is shown that in the quaternion case, such additional condition on the quaternions $(a_1, a_2, ..., a_n)$, plays condition which we call jointly quaternion system, i.e. the following theorem is true.

Theorem 1.3. *Let ξ be a quaternion random variable, $\xi_1, \xi_2, ..., \xi_n$, $n \geq 2$, be independent copies of ξ, and $(a_1, a_2, ..., a_n)$ be nonzero quaternions that form jointly quaternion system and satisfy the condition $\sum_{h=1}^n |a_h|^2 = 1$. Then, if the sum $\eta = \sum_{h=1}^n a_h\xi_h$ has the same distribution as ξ, ξ is quaternion Gaussian random variable.*

Now let us recall the definition of jointly quaternion system.

Definition 1.4. *We say that a collection of n quaternions $(a_1, a_2, ..., a_n)$, $n \geq 2$, constitutes a jointly quaternion system (JQS) if there does not exist imaginary number $\tilde{i} = \alpha i + \beta j + \gamma k$, with real α, β, γ, such that the following expressions holds: $a_1 = a_1^{'} + a_1^{''}\tilde{i}$, $a_2 = a_2^{'} + a_2^{''}\tilde{i}$,..., $a_n = a_n^{'} + a_n^{''}\tilde{i}$, $a_i^{'}, a_i^{''} \in R$, $1 \leq i \leq n$.*

This definition has also another interpretation: let $A \equiv (a_1, a_2, \ldots, a_n)$, $n \geq 2$, be the collection of quaternions not necessarily different to each other. Denote by $A'' \equiv (a_1'', a_2'', \ldots, a_n'')$, $A''' \equiv (a_1''', a_2''', \ldots, a_n''')$, $A^{IV} \equiv (a_1^{IV}, a_2^{IV}, \ldots, a_n^{IV})$. We say that the collection A is JQS if at least one of the three pairs (A'', A'''), (A'', A^{IV}) and (A''', A^{IV}) is a pair of non-collinear vectors in R^n or, in other words, if the vectors A'', A''' and A^{IV} do not belong to an one dimensional subspace of R^n. This name is motivated by the following observation: Any (one) quaternion $a = a' + ia'' + ja''' + ka^{IV}$ can be written as a complex number with respect to some imaginary unit $\tilde{i}$, defined by the following equality

$$\tilde{i} = \frac{ia'' + ja''' + ka^{IV}}{(a''^2 + a'''^2 + a^{IV^2})^{1/2}}.$$

Indeed, we have, $\tilde{i}^2 = -1$ and $a = a' + \tilde{i}\,\widetilde{a''}$, where $\widetilde{a''} = (a''^2 + a'''^2 + a^{IV^2})^{1/2}$. However, the collection of quaternions $A \equiv (a_1, a_2, \ldots, a_n)$, $n \geq 2$, not always can be expressed as complex numbers with the common imaginary unit. This can be done if and only if A is not a JQS.

Since the multiplication of quaternions is not commutative, the following natural question was posed at the end of [1]: is the Theorem 1.3 true if in the linear form $\eta = \sum_{h=1}^{n} a_h \xi_h$, a part of the coefficients $a_h, 1 \leq h \leq n$, are written on the left side of $\xi_h, 1 \leq h \leq n$ and other part on the right? The following example shows that the answer of this question is negative, i.e. it may happen that $a_1\xi_1 + \xi_2 a_2$ has the same distribution as ξ, (a_1, a_2) form the jointly quaternion system, but ξ is not a quaternion Gaussian random variable.

Example 1.5. Let $\xi = \xi^{'} + i\xi^{''} + j\xi^{''} - k\xi^{'}$, where $\xi^{'}$ and $\xi^{''}$ are independent standard Gaussian random variables. It is clear that the covariance matrix of the random vector $(\xi^{'}, \xi^{''}, \xi^{''}, -\xi^{'})$ has the form

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

Hence, ξ is not a quaternion Gaussian random variable, however using technique of characteristic functions it is not hard to show that ξ and $\eta = \frac{i}{\sqrt{2}}\xi_1 + \xi_2 \frac{j}{\sqrt{2}}$ are equally distributed, and $(\frac{i}{\sqrt{2}}, \frac{j}{\sqrt{2}})$ is the jointly quaternion system.

References

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