Stud. Univ. Babeş-Bolyai Math. 56(2011), No. 2, 261–272

# Approximation of cosine functions and Rogosinski type operators

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**Abstract.** We study some quantitative estimates of the convergence of the iterates of some Rogosinski type operators to their associated cosine functions. We also consider a general cosine counterpart of the quantitative version of Trotter's theorem on the approximation of $C_0$-semigroups.

**Mathematics Subject Classification (2010):** 41A25, 42A10, 47D09.

**Keywords:** Convergence of iterates of trigonometric polynomials, Rogosinski operators, representation of cosine functions.

## 1. Introduction and preliminary results

The convergence of iterates of trigonometric polynomials to suitable cosine functions in the setting of spaces of continuous periodical functions has been considered in [7, 8] from a qualitative point of view; these results extend to cosine function the possibility of using iterates of positive operators in the approximation of $C_0$-semigroups (see [2, Chapter 6] for more details). Recently, some quantitative versions of the classical Trotter's theorem [17] on the approximation of $C_0$-semigroups have been obtained in [14, 15] and [9, 10]. Here, we consider the possibility of obtaining quantitative estimates of the convergence to suitable cosine functions. We study in particular the Rogosinski type operators introduced in [7, 8] and establish some quantitative estimates of the convergence of their iterates to a cosine function generated by the square of a first order differential operator. Our discussion is based on the following general quantitative cosine version of Trotter's approximation theorem [17, Theorem 5.3], which provides a quantitative estimate of the convergence and, besides the Rogosinski type operators, can be applied also to other sequences of operators, such as Fejér operators and the general sequences of averages of trigonometric interpolating operators considered in [6]. A partial result on the generation of cosine functions has been obtained in [8, Theorem 1.2] without quantitative estimates.

**Theorem 1.1.** *Let $E$ be a Banach space, let $(L_n)_{n\geq 1}$ and $(M_n)_{n\geq 1}$ be two sequences of bounded linear operators from $E$ into itself and assume that there exist $M \geq 1$ and $\omega \geq 0$ such that*

$$\|L_n^k\| \leq M\, e^{\omega\, k/n}\,, \qquad \|M_n^k\| \leq M e^{\omega\, k/n}\,, \quad n\,, k \geq 1. \tag{1.1}$$

*Suppose also that $D$ is a dense subspace of $E$ and $A : D \to E$ is a linear operator such that*

$$\lim_{n\to+\infty} n(L_n u - u) = Au\ , \qquad \lim_{n\to+\infty} n(M_n u - u) = -Au$$

*and $(\lambda - A)(D)$ is dense in $E$ for some $\lambda > \omega$.*

*Then the closure of $(A, D)$ generates a $C_0-$group $(G(t))_{t\in\mathbb{R}}$ in $E$ and then the square $A^2$ of the closure of $(A, D)$ generates a cosine function $(C(t))_{t\in\mathbb{R}}$ in $E$ and, for every $t \geq 0$,*

$$C(t) = \frac{1}{2}\ \lim_{n\to\infty}\left(L_n^{k(n)} + M_n^{k(n)}\right)\ , \tag{1.2}$$

*where $(k(n)_n)_{n\in\mathbb{N}}$ is a sequence of positive integers such that*

$$\lim_{n\to+\infty} k(n)/n = t$$

*(in particular, we can take $k(n) = [n\,t]$). Consequently, for every $t \in \mathbb{R}$, we have $\|C(t)\| \leq M\, e^{\omega\,|t|}$.*

*Moreover, for every $t \geq 0$ and for every increasing sequence $(k(n))_{n\geq 1}$ of positive integers and $u \in \{v \in D|\, G(s)v,\ G(-s)v \in D$ for every $0 \leq s \leq t\}$, we have*

$$\begin{aligned}
&\left\| C(t)u - \frac{1}{2}\left(L_n^{k(n)}u + M_n^{k(n)}u\right)\right\| \qquad\qquad (1.3)\\
&\quad\leq \frac{M}{2}\ \exp(\omega\, e^{\omega/n}\, t) \int_0^t \exp(-\omega e^{\omega/n} s)\, (\|(n(L_n - I) - A)G(s)u\| \\
&\qquad\qquad + \|(n(M_n - I) + A)G(-s)u\|)\, ds \\
&\qquad + \frac{M}{2}\left(\exp(\omega\, e^{\omega/n}\, t_n)\ |k(n) - nt| + \sqrt{\frac{2k(n)}{\pi}}\ e^{\omega\, k(n)/n}\right. \\
&\qquad\qquad \left. + \frac{\omega k(n)}{n}\ \exp\left(\omega\, e^{\omega/n}\ \frac{k(n)}{n}\right)\right)\ (\|L_n u - u\| + \|M_n u - u\|)
\end{aligned}$$

*where $t_n := \sup\{t, k(n)/n\}$.*

*Proof.* From the classical Trotter's theorem [17, Theorem 5.3] it follows that the closure of the operators $A$ and $-A$ generate a $C_0$-semigroup $(T_+(t))_{t\geq 0}$ and respectively $(T_-(t))_{t\geq 0}$ in $E$. Consequently, the closure of $A$ generates a $C_0$-group $(G(t))_{t\in\mathbb{R}}$ in $E$ and, for every $t \geq 0$,

$$G(t) = T_+(t)\ , \qquad G(-t) = T_-(t)\ .$$

Moreover, again from [17, Theorem 5.3], we obtain the representation of the group $(G(t))_{t\in\mathbb{R}}$ in terms of iterates of the operators $L_n$ and $M_n$; indeed, for every $t \geq 0$ and for every sequence $(k(n)_n)_{n\in\mathbb{N}}$ of positive integers such that $\lim_{n\to+\infty} k(n)/n = t$, we have

$$G(t) = \lim_{n\to+\infty} L_n^{k(n)}\,, \qquad G(-t) = \lim_{n\to+\infty} M_n^{k(n)}\,.$$

Consequently, it follows that the square of the closure of $(A, D)$ generates a cosine function $(C(t))_{t\in\mathbb{R}}$ in $E$ (see [4, Example 3.14.15, p. 217]) and, for every $t \in \mathbb{R}$, $C(t) = (G(t)+G(-t))/2$. Hence the representation of the cosine function is a consequence of the representation of $(G(t))_{t\in\mathbb{R}}$ and the estimate $\|C(|t|)\| \leq M\, e^{\omega\, t}$ follows from (1.1) and (1.2).

Finally, we prove (1.3).

Let $t \geq 0$, $(k(n))_{n\geq 1}$ an increasing sequence of positive integers and $u \in \{v \in D|\, G(s)v,\ G(-s)v \in D$ for every $0 \leq s \leq t\}$.

From [9, Theorem 1.2] and our assumptions we get

$$\begin{aligned}
&\left\|T_+(t)u - L_n^{k(n)}u\right\| \\
&\quad\leq M\, \exp(\omega\, e^{\omega/n}\, t) \int_0^t \exp(-\omega e^{\omega/n} s)\|(n(L_n - I) - A)T_+(s)u\| ds \\
&\qquad + M \left( \exp(\omega\, e^{\omega/n}\, t_n)\ |k(n) - nt| + \sqrt{\frac{2k(n)}{\pi}}\, e^{\omega\, k(n)/n} \right. \\
&\qquad \left. + \frac{\omega k(n)}{n}\, \exp\left(\omega\, e^{\omega/n}\, \frac{k(n)}{n}\right)\right)\ \|L_n u - u\|
\end{aligned}$$

and

$$\begin{aligned}
&\left\|T_-(t)u - M_n^{k(n)}u\right\| \\
&\quad\leq M\, \exp(\omega\, e^{\omega/n}\, t) \int_0^t \exp(-\omega e^{\omega/n} s)\|(n(M_n - I) + A)T_-(s)u\| ds \\
&\qquad + M \left( \exp(\omega\, e^{\omega/n}\, t_n)\ |k(n) - nt| + \sqrt{\frac{2k(n)}{\pi}}\, e^{\omega\, k(n)/n} \right. \\
&\qquad \left. + \frac{\omega k(n)}{n}\, \exp\left(\omega\, e^{\omega/n}\, \frac{k(n)}{n}\right)\right)\ \|M_n u - u\|\,.
\end{aligned}$$

Taking into account that

$$\begin{aligned}
&\left\|C(t)u - \frac{1}{2}\left(L_n^{k(n)}u + M_n^{k(n)}u\right)\right\| \\
&\qquad\qquad = \frac{1}{2}\left\|T_+(t)u + T_-(t)u - L_n^{k(n)}u - M_n^{k(n)}u\right\| \\
&\qquad\qquad \leq \frac{1}{2}\left(\left\|T_+(t)u - L_n^{k(n)}u\right\| + \left\|T_-(t)u - M_n^{k(n)}u\right\|\right)
\end{aligned}$$

the proof follows from the preceding inequalities. □

**Remark 1.2.** In many applications it is natural to consider the sequence $k(n) = [nt]$ for which $t_n = t$ and $|[nt]/n - t| = nt/n - [nt]/n \leq 1/n$. Hence estimate (1.3) yields

$$\begin{aligned}
&\left\|C(t)u - \frac{1}{2}\left(L_n^{[nt]}u + M_n^{[nt]}u\right)\right\| \qquad (1.4)\\
&\quad \le \frac{M}{2}\exp(\omega\, e^{\omega/n}\, t)\int_0^t \exp(-\omega e^{\omega/n} s)\times\\
&\quad\ \times (\|(n(L_n - I) - A)G(s)u\| + \|(n(M_n - I) + A)G(-s)u\|)ds\\
&\quad\ + \frac{M}{2}\left(\exp(\omega\, e^{\omega/n}\, t) + \sqrt{\frac{2nt}{\pi}}\, e^{\omega\, t} + \omega\, t\, \exp\left(\omega\, e^{\omega/n}\, t\right)\right)\times\\
&\qquad \times (\|L_n u - u\| + \|M_n u - u\|)\ .
\end{aligned}$$

From the classical theory of the cosine functions (see [16] and [13, Chapter II] for more details) we have that the unique solution of the following second-order Cauchy problem

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} u(t,x) = A^2 u(t,x)\ , & t \in \mathbb{R}\ ;\\ u(0,x) = u_0(x)\ , & x \in \mathbb{R}\ ;\\ \dfrac{\partial}{\partial t} u(t,x)|_{t=0} = u_1(x)\ , & x \in \mathbb{R}\ , \end{cases} \qquad (1.5)$$

with $u_0,\ u_1 \in D$, is given by

$$\begin{aligned}
u(t,x) &= C(t)u_0(x) + \int_0^t C(v)u_1(x)\, dv \qquad (1.6)\\
&= \frac{1}{2}\lim_{n\to\infty}\left(L_n^{[n\,t]}u_0 + M_n^{[n\,t]}u_0 + \int_0^t \left(L_n^{[n\,v]}u_1 + M_n^{[n\,v]}u_1\right)\, dv\right)\ ,
\end{aligned}$$

for every $t \in \mathbb{R}$ and $x \in \mathbb{R}$. We explicitly observe that the sequences $(L_n^{[n\,v]}u_1)_{n\ge1}$ and $(M_n^{[n\,v]}u_1)_{n\ge1}$ are equibounded with respect to $v \in [0,t]$ and this allows us to apply the Lebesgue dominated convergence theorem.

## 2. Rogosinski type operators

Denote by $C_{2\pi}$ the space of all $2\pi$-periodic continuous real functions on $\mathbb{R}$ and put $\Pi := \{\pi + 2k\pi \mid k \in \mathbb{Z}\}$. Moreover, let $a \in C_{2\pi} \cap C^1(\mathbb{R}\setminus\Pi)$ be such that $a \neq 0$ in $]-\pi,\pi[$ and consider the first-order differential operator $(A, D(A))$ defined by

$$Au := au', \qquad u \in D(A) := \left\{u \in C_{2\pi} \cap C^1(\mathbb{R}\setminus\Pi) \mid Au \in C_{2\pi}\right\}\ .$$

In order to consider the generation of cosine functions, we also consider the operator $A^2$ on the following domain

$$D(A^2) := \left\{u \in C_{2\pi} \cap C^2(]-\pi,\pi[) \mid a(au')' \in C_{2\pi}\right\}\ .$$

It is well-known (see e.g. [8, Theorem 1.1]) that $(A^2, D(A^2))$ generates a cosine functions $(C(t))_{t\in\mathbb{R}}$ in $C_{2\pi}$ if and only if

$$\frac{1}{a} \in L^1(-\pi,0)\ ,\ \frac{1}{a} \in L^1(0,\pi)\ . \qquad (2.1)$$

Now, we consider the Rogosinski kernel defined by setting, for every $n \in \mathbb{N}$ and $x \in \mathbb{R}$,

$$r_n(x) := 1 + 2\sum_{k=1}^{n} \cos\left(\frac{k\pi}{2n+1}\right)\cos(kx) \ ,$$

and the corresponding $n$-th *Rogosinski operator* $R_n : C_{2\pi} \to C_{2\pi}$ given by

$$R_n f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-v) r_n(v)\, dv \ , \qquad f \in C_{2\pi} \ , \ x \in \mathbb{R} \ .$$

The $n$-th generalized Rogosinski operator $R_{a,n} : C_{2\pi} \to C_{2\pi}$ introduced in [8] is defined by putting

$$R_{a,n} f(x) = R_n f\left(x + \frac{2\pi}{2n+1}\, a(x)\right) \ , \qquad f \in C_{2\pi} \ , \quad x \in \mathbb{R} \ .$$

From [8, Theorem 2.1] the sequence $(\|R_{a,n}\|)_{n\in\mathbb{N}}$ is equibounded and moreover $\|R^k_{a,n}\| \le 2\pi$ for every $n, k \ge 1$. Further, there exists a positive constant $C > 0$ such that

$$\|R_{a,n} f - f\| \le C\ \omega\left(f; \frac{1}{n}\right) \ , \qquad f \in C_{2\pi} \ . \tag{2.2}$$

In order to apply Theorem 1.1, our next aim is to establish a quantitative estimate of the Voronovskaja-type formula associated with these operators.

**Lemma 2.1.** *Let $0 < \alpha \le 1$. Then, for every $f \in C^{1,\alpha}_{2\pi}$,*

$$\left\| \frac{2n+1}{2\pi}\left(R_{a,n} f - f\right) - Af \right\|_\infty \le 49(\|a\|_\infty + 1)\left(\frac{2\pi}{2n+1}\right)^{\alpha} L_{f'} \ ,$$

*where $L_{f'}$ is the constant of $\alpha$-hölderianity of $f'$.*

*Proof.* For every $f \in C^{1,\alpha}_{2\pi}$ we have

$$\begin{aligned}\left\| \frac{2n+1}{2\pi}\left(R_{a,n} f - f\right) - Af \right\|_\infty \le{} & \left\| \frac{2n+1}{2\pi}\left(R_{a,n} f - R_n f\right) - Af \right\|_\infty \\ & + \left\| \frac{2n+1}{2\pi}\left(R_n f - f\right) \right\|_\infty \ . \end{aligned} \tag{2.3}$$

As regards to the first term at the right-hand side of (2.3), from Lagrange's theorem we can write

$$f(y+t) - f(y) = f'(y)t + (f'(\xi) - f'(y))\, t \ , \qquad y, t \in \mathbb{R}$$

where $\xi$ lies between $y$ and $y+t$. For every $x \in \mathbb{R}$ and $n \in \mathbb{N}$ we have

$$\begin{aligned}
&\frac{2n+1}{2\pi}(R_{a,n}f(x) - R_n f(x)) - a(x)f'(x) \\
&\quad = \frac{2n+1}{2\pi}\,\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(f\left(x - v + \frac{2\pi}{2n+1}\,a(x)\right) - f(x-v)\right)\,r_n(v)\,dv \\
&\qquad - a(x)f'(x) \\
&\quad = \frac{2n+1}{2\pi}\,\frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x-v)\frac{2\pi}{2n+1}\,a(x)\,r_n(v)dv - a(x)f'(x) \\
&\qquad + \frac{2n+1}{2\pi}\,\frac{1}{2\pi}\int_{-\pi}^{\pi}(f'(\xi) - f'(x-v))\,\frac{2\pi}{2n+1}a(x)\,r_n(v)\,dv \\
&\quad = a(x)(R_n f'(x) - f'(x)) + a(x)\frac{1}{2\pi}\int_{-\pi}^{\pi}(f'(\xi) - f'(x-v))\,r_n(v)\,dv\ ,
\end{aligned}$$

where $\xi \in ]x-v, x-v+2\pi a(x)/(2n+1)[$.

We recall that (see e.g. [5, Theorem 2.4.8, p. 106 ])

$$\|R_n g - g\|_\infty \le (2\pi+1)E_n(g) + 4\,\omega\left(g; \frac{1}{n}\right)\ , \qquad g \in C_{2\pi}\ ,$$

where $E_n(g)$ is the best approximation of the function $g$ by trigonometric polynomials of degree $n$ and hence, from the classical Jackson's theorem,

$$\|R_n g - g\|_\infty \le 6(2\pi+1)\,\omega\left(g; \frac{1}{n}\right) + 4\,\omega\left(g; \frac{1}{n}\right) \le (12\pi+10)\omega\left(g; \frac{1}{n}\right)\ .$$

Applying the above inequality to $f'$ and $f$ we get

$$\begin{aligned}
&\left|\frac{2n+1}{2\pi}(R_{a,n}f(x) - R_n f(x)) - a(x)f'(x)\right| \\
&\qquad \le \|a\|_\infty\left((12\pi+10)\omega\left(f'; \frac{1}{n}\right) + \omega\left(f'; \frac{2\pi}{2n+1}\right)\right) \\
&\qquad \le \|a\|_\infty(12\pi+11)\,\omega\left(f'; \frac{2\pi}{2n+1}\right)\ ,
\end{aligned}$$

and consequently

$$\begin{aligned}
\left\|\frac{2n+1}{2\pi}\,(R_{a,n}f - f) - Af\right\|_\infty \le{}& \|a\|_\infty(12\pi+11)\omega\left(f'; \frac{2\pi}{2n+1}\right) \\
&+ \frac{2n+1}{2\pi}(12\pi+10)\omega\left(f; \frac{1}{n}\right)\ .
\end{aligned}$$

Since $f \in C^{1,\alpha}_{2\pi}$ we have $\omega(f,\delta) \le \frac{L_{f'}}{2}\delta^{\alpha+1}$ and $\omega(f',\delta) \le L_{f'}\delta^{\alpha}$. Thus we conclude that

$$\begin{aligned}&\left\| \frac{2n+1}{2\pi}\left(R_{a,n}f - f\right) - Af \right\|_\infty \\ &\qquad\le \|a\|_\infty (12\pi+11)\left(\frac{2\pi}{2n+1}\right)^\alpha L_{f'} + (12\pi+10)\frac{2n+1}{4n\pi}\frac{1}{n^\alpha}L_{f'} \\ &\qquad\le (12\pi+11)(\|a\|_\infty+1)\left(\frac{2\pi}{2n+1}\right)^\alpha L_{f'}.\end{aligned}$$

□

From [8, Theorem 1.1] we already know that if $a \in C^{0,1}_{2\pi} \cap C^1(\mathbb{R}\setminus \Pi)$ satisfies condition (2.1), then the operator $(A, D(A))$ generates a $C_0$-semigroup $(T(t))_{t\ge 0}$ of positive contractions on $C_{2\pi}$. In the next lemma we state a more precise quantitative estimate of Voronovskaja's formula.

**Lemma 2.2.** *Let $a \in C^3_{2\pi}$ satisfy condition* (2.1) *and let $(T(t))_{t\ge0}$ be the $C_0$-semigroup on $C_{2\pi}$ generated by $(A, D(A))$.*

*Then, for every $t \ge 0$ and $f \in C^{1,\alpha}_{2\pi}$,*

$$\begin{aligned}&\left\| \left(\frac{2n+1}{2\pi}\left(R_{a,n} - I\right) - A\right) T(t) f \right\|_\infty \qquad (2.4)\\ &\qquad\le 49(\|a\|_\infty+1)\left(\frac{2\pi}{2n+1}\right)^\alpha (C_1+t)\|a''\|_\infty e^{C_2\|a\|_{C^2} t}\|f\|_{C^{1,\alpha}}\,,\end{aligned}$$

*where $K = C_2\|a\|_{C^2}$ and the constants $C$, $C_1$ and $C_2$ are independent of $t$ and $n$.*

*Proof.* Let us consider the flow $\phi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ as the unique solution of this problem

$$\begin{cases} \dfrac{\partial \phi(t,x)}{\partial t} = a(\phi(t,x))\,, & \text{for every } t,\ x \in \mathbb{R} \\ \phi(0,x) = x\,, & \text{for every } x \in \mathbb{R}\,. \end{cases}$$

Now consider the $C_0$-semigroup $(T(t))_{t\ge0}$ defined by

$$T(t)f(x) := f(\phi(t,x)) \qquad \text{for all } t \ge 0,\ x\in\mathbb{R}\,,\ f \in C_{2\pi}\,. \tag{2.5}$$

Notice that the operator $(A, D(A))$ is the generator of the semigroup defined in (2.5).

Since $a \in C^3_{2\pi}$, then $\phi \in C^3_{2\pi}$ (see [3, Theorem 10.3]), and hence for all $f \in C^m_{2\pi}$, we have that $T(t)f = f(\phi(t,\cdot)) \in C^m_{2\pi}$, $m = 0,1,2,3$.

Let us consider the following Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = Au & \text{in } (0,\infty)\times\mathbb{R}\,, \\ u(0,x) = f(x) & \text{in } \mathbb{R}\,, \end{cases} \tag{2.6}$$

with $f \in C^3_{2\pi}$. Let $u(t,x) = T(t)f(x)$ be the solution of the previous problem, we have $\|u(t,\cdot)\|_\infty = \|T(t)f\|_\infty \le \|f\|_\infty$ for every $t \ge 0$; moreover $u \in C^3_{2\pi}$ since $f \in C^3_{2\pi}$.

Consider also the problem

$$\begin{cases} \dfrac{\partial v}{\partial t} = Av + Bv & \text{in } (0,\infty)\times\mathbb{R}\,, \\ v(0,\cdot) = f' & \text{in } \mathbb{R}\,, \end{cases} \tag{2.7}$$

where $Bv := a'v$ for every $v \in D(A)$. Since $\|B\| = \|a'\|_\infty$, then the operator $A+B$ on $D(A)$ is a bounded perturbation of the operator $(A, D(A))$ and it generates a semigroup $(S(t))_{t\geq 0}$ on $C_{2\pi}$ such that

$$\|S(t)\| \leq e^{\|a'\|_\infty t}\,, \qquad t \geq 0$$

see [12, Chapter 3].

Let us notice that $u'$ solves (2.7) on $[0,\infty)\times\mathbb{R}$, indeed

$$(Au)' = a'u' + au'' = Bu' + Au'\;;$$

then $u'(t,x) = S(t)f'(x)$ for every $t \geq 0$, $x \in \mathbb{R}$ and

$$\|T(t)\|_{\mathcal{L}(C^1;C^1)} \leq 1 + e^{\|a'\|_\infty t}\,, \qquad t \geq 0\,.$$

Now let us consider the problem

$$\begin{cases} \dfrac{\partial w(t,x)}{\partial t} = Aw + Cw + a''u'(t,0) & \text{in } (0,\infty)\times\mathbb{R}\,, \\ w(0,\cdot) = f'' & \text{in } \mathbb{R}\,, \end{cases} \tag{2.8}$$

where $Cw := a'' \int_0^x \left(w(y) - \frac{1}{2\pi}\int_0^{2\pi} w\right) dy + 2a'w$ for every $w \in D(A)$. Then $A + C$ is a bounded perturbation of $(A, D(A))$ and hence $(A + C, D(A))$ generates the $C_0$-semigroup $\left(\tilde{S}(t)\right)_{t\geq 0}$ on $C_{2\pi}$. Since $\|C\| \leq 2\pi\|a''\|_\infty + 2\|a'\|_\infty \leq 2\pi\|a\|_{C^2}$, we have (see [12, Chapter 3])

$$\|\tilde{S}(t)\| \leq e^{2\pi\|a\|_{C^2}\, t}\,, \qquad t \geq 0\,.$$

Therefore

$$w(t,x) = \tilde{S}(t)f'' + \int_0^t \tilde{S}(t-s)\left(a''u'(t,0)\right) ds$$

is a mild solution of (2.8) in $C_{2\pi}$; moreover

$$\begin{aligned} \|w(t,\cdot)\|_\infty &\leq \|\tilde{S}(t)f''\|_\infty + \int_0^t \|\tilde{S}(t-s)\left(a''u'(s,0)\right)\| ds \\ &\leq e^{2\pi\|a\|_{C^2}\, t}\|f''\|_\infty + \int_0^t \|\tilde{S}(t-s)\left(a''(T(s)f)'(0)\right)\| ds \\ &\leq e^{2\pi\|a\|_{C^2}\, t}\|f''\|_\infty + \int_0^t e^{2\pi\|a\|_{C^2}\,(t-s)}\|a''(T(s)f)'(0)\| ds \\ &\leq e^{2\pi\|a\|_{C^2}\, t}\|f''\|_\infty + \int_0^t e^{2\pi\|a\|_{C^2}\,(t-s)}\|a''\|_\infty\|f'\|_\infty e^{\|a'\|_\infty s} ds \\ &\leq e^{2\pi\|a\|_{C^2}\, t}\|f''\|_\infty + t\|a''\|_\infty\|f'\|_\infty e^{2\pi\|a\|_{C^2}\, t}\,. \end{aligned}$$

Since $(Au)'' = a''u' + 2a'u'' + Au''$, we have

$$\begin{aligned}\frac{\partial}{\partial t}u''(t,x) &= Au''(t,x) + a''(x)u''(t,x) + 2a'(x)u''(x)\\ &= Au''(t,x) + a''(x)\left(\int_0^x u'(t,y)dy + u'(t,0)\right) + 2a'(x)u''(x)\\ &= Au''(t,x) + Cu''(t,x) + a''(x)u'(t,0)\ ,\end{aligned}$$

then $u''(t,x)$ is a solution of (2.8) and $u(t,\cdot) = w(t,\cdot)$. So we can apply the previous estimate to $u''(t,\cdot)$ and we get, for every $t \geq 0$,

$$\|(T(t)f)''\|_\infty = \|u''(t,\cdot)\|_\infty \leq e^{2\pi\|a\|_{C^2}t}\|f''\|_\infty + t\|a''\|_\infty\|f'\|_\infty e^{2\pi\|a\|_{C^2}t}\ .$$

Therefore

$$\|T(t)\|_{\mathcal{L}(C^2;C^2)} \leq 1 + e^{\|a'\|_\infty t} + e^{2\pi\|a\|_{C^2}t} + t\|a''\|_\infty e^{2\pi\|a\|_{C^2}t}\ , \qquad t \geq 0\ .$$

Finally, since $C^{1,\alpha}_{2\pi}$ is an intermediate space between $C^1_{2\pi}$ and $C^2_{2\pi}$, then we get

$$\begin{aligned}\|T(t)\|_{\mathcal{L}(C^{1,\alpha};C^{1,\alpha})} &\leq C\|T(t)\|^{1-\alpha}_{\mathcal{L}(C^1;C^1)}\|T(t)\|^{\alpha}_{\mathcal{L}(C^2;C^2)}\\ &\leq (C_1+t)\|a''\|_\infty e^{C_2\|a\|_{C^2}\,t} \qquad \text{for all } t\geq 0\ ,\end{aligned}\tag{2.9}$$

where $C_1$ and $C_2$ are positive constant independent of $t$.

Finally from Lemma 2.1 and taking into account (2.9), we get

$$\begin{aligned}&\left\|\left(\frac{2n+1}{2\pi}\left(R_{a,n}-I\right)-A\right)T(t)f\right\|_\infty \\ &\quad\leq 49(\|a\|_\infty+1)\left(\frac{2\pi}{2n+1}\right)^\alpha L_{(T(t)f)'}\\ &\quad\leq 49(\|a\|_\infty+1)\left(\frac{2\pi}{2n+1}\right)^\alpha \|T(t)f\|_{C^{1,\alpha}}\\ &\quad\leq 49(\|a\|_\infty+1)\left(\frac{2\pi}{2n+1}\right)^\alpha \|T(t)\|_{\mathcal{L}(C^{1,\alpha};C^{1,\alpha})}\|f\|_{C^{1,\alpha}}\\ &\quad\leq 49(\|a\|_\infty+1)\left(\frac{2\pi}{2n+1}\right)^\alpha (C_1+t)\|a''\|_\infty e^{C_2\|a\|_{C^2}t}\|f\|_{C^{1,\alpha}}\ ,\end{aligned}\tag{2.10}$$

for all $t \geq 0$. □

In [8, Theorem 2.7] Campiti and Ruggeri established that besides the generation of the cosine function $(C(t))_{t\in\mathbb{R}}$, condition (2.1) also ensures that $C^1_{2\pi}\cap D(A^2)$ is a core for $(A^2, D(A^2))$ and further, for every $t > 0$,

$$C(t) = \frac{1}{2}\lim_{n\to\infty}\left(R^{k(n)}_{a,n}u + R^{k(n)}_{-a,n}u\right)\ ,\tag{2.11}$$

where $(k(n))_{n\geq 1}$ is a sequence of positive integers such that

$$\lim_{n\to+\infty}\frac{2\pi k(n)}{2n+1} = t.$$

From (2.2) it follows that there exits a constant $C > 0$ such that

$$\|R_{a,n}f - f\|_\infty \le \frac{C}{n^{\alpha+1}} L_{f'}$$

for all $f \in C^{1,\alpha}_{2\pi}$ and the same estimate also holds for $R_{-a,n}$. We obtain the following quantitative version of (2.11).

**Theorem 2.3.** *Let $a \in C^3_{2\pi}$ satisfy* (2.1). *Then for every $t \ge 0$ and $u \in C^{1,\alpha} \cap D(A^2)$*

$$\left\| C(t)u - \frac{1}{2}\left(R^{k(n)}_{a,n}u + R^{k(n)}_{-a,n}u\right)\right\|_\infty \tag{2.12}$$

$$\le 2\pi\, C \left(\frac{2\pi}{2n+1}\right)^\alpha \left( \|a''\|_\infty \left(\|a\|_\infty + 1\right) \left[\frac{e^{Kt}-1}{K}\left(C_1 + \frac{1}{K}\right) + t\frac{e^{Kt}}{K}\right] \right. \tag{2.13}$$

$$\left. + \left|\frac{2\pi k(n)}{2n+1} - t\right| + \sqrt{\frac{2}{n}}\sqrt{\frac{2k(n)}{2n+1}}\right) \|u\|_{C^{1,\alpha}} \,,$$

*where $K = C_2\|a\|_{C^2}$, $(k(n))_{n\ge 1}$ is a sequence of positive integers such that*

$$\lim_{n\to+\infty} \frac{2\pi k(n)}{2n+1} = t$$

*and $C$, $C_1$ and $C_2$ are positive constants independent of $n \in \mathbb{N}$ and $t \ge 0$.*

*Proof.* Consider $u \in C^{1,\alpha} \cap D(A^2)$, taking into account (2.4) we have

$$\int_0^t \left\| \left(\frac{2n+1}{2\pi}\left(R_{a,n} - I\right) - A\right) T(s)u \right\|_\infty ds \tag{2.14}$$

$$\le 49\|a''\|_\infty(\|a\|_\infty + 1)\left(\frac{2\pi}{2n+1}\right)^\alpha \|u\|_{C^{1,\alpha}} \left[\frac{e^{Kt}-1}{K}\left(C_1 + \frac{1}{K}\right) + t\frac{e^{Kt}}{K}\right] \,,$$

where $K = C_2\|a\|_{C^2}$. The same estimate also holds for the sequences of operators $(R_{-a,n})_{n\ge1}$ and the differential operator $-A$. Then from (1.3) we have

$$\left\| C(t)u - \frac{1}{2}\left(R^{k(n)}_{a,n}u + R^{k(n)}_{-a,n}u\right)\right\|_\infty \tag{2.15}$$

$$\le 2\pi C\|a''\|_\infty \left(\|a\|_\infty + 1\right)\left(\frac{2\pi}{2n+1}\right)^\alpha \left[\frac{e^{Kt}-1}{K}\left(C_1 + \frac{1}{K}\right) + t\frac{e^{Kt}}{K}\right] \|u\|_{C^{1,\alpha}}$$

$$+ 2\pi\left(\left|k(n) - \frac{2n+1}{2\pi}t\right| + \sqrt{\frac{2k(n)}{2\pi}}\right) C\left(\frac{1}{n}\right)^{\alpha+1} \|u\|_{C^{1,\alpha}} \,.$$

□

Finally, we observe that arguing as in [11] we can also establish a quantitative estimate of the resolvent operators.

Exactly the same procedure can be also applied to other sequences of trigonometric polynomials such as Fejér operators and more general averages of trigonometric interpolating operator considered in [8, 6]. Since in these cases the cosine function is the same, we limit ourselves to observe that (2.12) remains still valid when considering these other sequences of trigonometric interpolating operators too.

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