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On Grüss-type inequalities for positive linear operators

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Abstract. The classical form of Grüss' inequality gives an estimate of the difference between the integral of the product and the product of the integrals of two functions in $C[a, b]$. It was first published by G. Grüss in [7]. The aim of this article is to discuss Grüss-type inequalities in $C(X)$, the set of continuous functions defined on a compact metric space X. We consider a functional $L(f) := H(f; x)$, where $H : C(X) \to C(X)$ is a positive linear operator and $x \in X$ is fixed. Generalizing a result of Acu et al. [1], a quantitative Grüss-type inequality is obtained in terms of the least concave majorant of the classical modulus of continuity. The interest is in the degree of non-multiplicativity of the functional L. Moreover, for the case $X = [a, b]$ we improve the inequality and apply it to various known operators, in particular those of Bernstein-, convolution- and Shepard-type.

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1. Introduction

The classical form of Grüss' inequality gives an estimate of the difference between the integral of the product and the product of the integrals of two functions in $C[a, b]$. It was first published by G. Grüss in [7]. The aim of this article is to discuss Grüss-type inequalities in $C(X)$, the set of continuous functions defined on a compact metric space X. We consider a functional $L(f) := H(f; x)$, where $H : C(X) \to C(X)$ is a positive linear operator and $x \in X$ is fixed. Generalizing a result of Acu et al. [1], a quantitative Grüss-type inequality is obtained in terms of the least concave majorant of the classical modulus of continuity. The interest is in the degree of non-multiplicativity of the functional L. Moreover, for the case $X = [a, b]$ we improve the inequality and apply it to various known operators, in particular those of Bernstein-, convolution- and Shepard-type.

2. Auxiliary results

Before giving our main results, we need some introductory notions that will be used in the sequel. Let $C(X) = C_{\mathbb{R}}((X,d))$ represent the Banach lattice of real-valued continuous functions defined on the compact metric space (X,d). Then we have the following definition:

Definition 2.1. *Let $f \in C(X)$. If, for $t \in [0,\infty)$, the quantity*

$$\omega_d(f;t) := \sup\{|f(x)-f(y)|\,,\ d(x,y) \le t\}$$

is the usual modulus of continuity, then its least concave majorant is given by

$$\widetilde{\omega_d}(f,t) = \begin{cases} \sup\limits_{0\le x\le t\le y\le d(X),x\ne y} \frac{(t-x)\omega_d(f,y)+(y-t)\omega_d(f,x)}{y-x} & \text{for } 0 \le t \le d(X)\ , \\ \omega_d(f,d(X)) & \text{if } t > d(X)\ , \end{cases}$$

and $d(X) < \infty$ is the diameter of the compact space X.

For $0 < r \le 1$, let Lip_r be the set of all functions $g \in C(X)$ with the property that

$$|g|_{Lip_r} := \sup_{d(x,y)>0} |g(x)-g(y)|\,/d^r(x,y) < \infty.$$

Lip_r is a dense subspace of $C(X)$ equipped with the supremum norm $\|\cdot\|_\infty$, and $|\cdot|_{Lip_r}$ is a seminorm on Lip_r.

We also need to define the $K-$functional with respect to $(Lip_r, |\cdot|_{Lip_r})$, which is given by

$$K\left(t,f;C(X),Lip_r\right) := \inf_{g\in Lip_r}\left\{\|f-g\|_\infty + t\cdot|g|_{Lip_r}\right\},$$

for $f \in C(X)$ and $t \ge 0$.

Another tool for some proofs that follow is a lemma of Brudnyĭ (see [10]) that gives the relationship between the K-functional and the least concave majorant of the modulus of continuity.

Lemma 2.2. *Every continuous function f on X satisfies*

$$K\left(\frac{t}{2},f;C(X),Lip_1\right) = \frac{1}{2}\cdot\widetilde{\omega_d}(f,t),\ 0 \le t \le d(X).$$

In the case $X = [a,b]$, we also have

$$\begin{aligned} K\left(\frac{t}{2},f;C[a,b],C^1[a,b]\right) &:= \inf_{g\in C^1[a,b]}\left\{\|f-g\|_\infty + \frac{t}{2}\cdot\|g'\|_\infty\right\} \\ &= \frac{1}{2}\cdot\widetilde{\omega}(f;t),\ t \ge 0. \end{aligned}$$

3. Grüss-type inequalities in a compact metric space

What we do here is generalize Theorem 4 in [1] in the case of a compact metric space.

We consider (X, d) a compact metric space, $x \in X$ fixed, with diameter $d(X) > 0$. Now let $H : C(X) \to C(X)$ be a positive linear operator reproducing constant functions. We define the positive linear functional $H(\cdot; x)$ and consider the positive bilinear functional

$$D(f, g) := H(f \cdot g; x) - H(f; x) \cdot H(g; x).$$

It was remarked after Theorem 4 in [1] that the assertion given there can be generalized by replacing $([a, b], |\cdot|)$ by a compact metric space (X, d), the second moment $H((e_1 - x)^2; x)$ by $H(d^2(\cdot, x); x)$, and the K-functional $K(\cdot, f; C[a, b], C^1[a, b])$ by $K(\cdot, f; C(X), Lip_1)$.

We then obtain the following result:

Theorem 3.1. *If $f, g \in C(X)$, (X, d) a compact metric space and $x \in X$ fixed, then the inequality*

$$|D(f, g)| \leq \frac{1}{4}\widetilde{\omega_d}\left(f; 4\sqrt{H(d^2(\cdot, x); x)}\right) \cdot \widetilde{\omega_d}\left(g; 4\sqrt{H(d^2(\cdot, x); x)}\right) \tag{3.1}$$

holds.

Proof. Let $f, g \in C[a, b]$ and $r, s \in Lip_1$. We use the Cauchy-Schwarz inequality for positive linear functionals:

$$|H(f; x)| \leq H(|f|; x) \leq \sqrt{H(f^2; x) \cdot H(1; x)} = \sqrt{H(f^2; x)},$$

so we have

$$D(f, f) = H(f^2; x) - H(f; x)^2 \geq 0.$$

Hence D is a positive bilinear form on $C(X)$. Using Cauchy-Schwarz for D gives us

$$|D(f, g)| \leq \sqrt{D(f, f)D(g, g)} \leq \|f\|_\infty \cdot \|g\|_\infty .$$

Because $H : C(X) \to C(X)$ is a positive linear operator reproducing constant functions, $H(f; x)$, with fixed $x \in X$, is a positive linear functional that we can represent as follows

$$H(f; x) := \int_X f(t)d\mu_x(t),$$

where μ_x is a Borel probability measure on X, i.e., $\int_X d\mu_x(t) = 1$. For r as above, we have

$$\begin{aligned} D(r, r) = H(r^2; x) - H(r; x)^2 &= \int_X r^2(t)d\mu_x(t) - \left(\int_X r(u)d\mu_x(u)\right)^2 \\ &= \int_X \left(r(t) - \int_X r(u)d\mu_x(u)\right)^2 d\mu_x(t) \\ &= \int_X \left(\int_X (r(t) - r(u))\, d\mu_x(u)\right)^2 d\mu_x(t) \end{aligned}$$

$$\le \int_X \left(\int_X (r(t)-r(u))^2 \, d\mu_x(u) \right) d\mu_x(t)$$

$$\le |r|_{Lip_1}^2 \int_X \left(\int_X d^2(t,u) d\mu_x(u) \right) d\mu_x(t)$$

$$\le |r|_{Lip_1}^2 \int_X \left(\int_X [d(t,x)+d(x,u)]^2 \, d\mu_x(u) \right) d\mu_x(t)$$

$$= |r|_{Lip_1}^2 \int_X \int_X \left\{ d^2(t,x) + 2 \cdot d(t,x) \cdot d(x,u) + d^2(x,u) \right\} d\mu_x(u) d\mu_x(t)$$

$$= |r|_{Lip_1}^2 \left[\int_X d^2(t,x) d\mu_x(t) + 2 \int_X \int_X d(t,x) d(x,u) d\mu_x(u) d\mu_x(t) + \int_X d^2(x,u) d\mu_x(u) \right]$$

$$= |r|_{Lip_1}^2 \left[H(d^2(\cdot,x);x) + 2 \left(\int_X d(t,x) d\mu_x(t) \right) \left(\int_X d(u,x) d\mu_x(u) \right) + H(d^2(\cdot,x);x) \right]$$

$$= |r|_{Lip_1}^2 \left[H(d^2(\cdot,x);x) + 2H(d(\cdot,x);x) \cdot H(d(\cdot,x);x) + H(d^2(\cdot,x);x) \right]$$

$$= |r|_{Lip_1}^2 \left[2H(d^2(\cdot,x);x) + 2H(d(\cdot,x);x)^2 \right]$$

$$\le |r|_{Lip_1}^2 \left[2H(d^2(\cdot,x);x) + 2H(d^2(\cdot,x);x) \right]$$

$$= 4 \, |r|_{Lip_1}^2 \cdot H(d^2(\cdot,x);x).$$

For r, s as above, we have the estimate

$$|D(r,s)| \le \sqrt{D(r,r)D(s,s)} \le 4 \, |r|_{Lip_1} \cdot |s|_{Lip_1} \cdot H(d^2(\cdot,x);x).$$

Moreover, for $f \in C(X)$ and $s \in Lip_1$, the inequality

$$|D(f,s)| \le \sqrt{D(f,f)D(s,s)} \le 2 \, \|f\|_\infty \cdot |s|_{Lip_1} \cdot \sqrt{H(d^2(\cdot,x);x)}$$

holds. Similarly, if $r \in Lip_1$ and $g \in C(X)$, we have

$$|D(r,g)| \le \sqrt{D(r,r)D(g,g)} \le 2 \, \|g\|_\infty \cdot |r|_{Lip_1} \cdot \sqrt{H(d^2(\cdot,x);x)}.$$

Now let $f, g \in C(X)$ be fixed and $r, s \in Lip_1$ arbitrary. Then

$$\begin{aligned}
&|D(f,g)| \\
&= |D(f-r+r, g-s+s)| \\
&\le |D(f-r, g-s)| + |D(f-r, s)| + |D(r, g-s)| + |D(r,s)| \\
&\le \|f-r\|_\infty \cdot \|g-s\|_\infty + 2 \, \|f-r\|_\infty \cdot |s|_{Lip_1} \cdot \sqrt{H(d^2(\cdot,x);x)} \\
&+ 2 \, \|g-s\|_\infty \cdot |r|_{Lip_1} \cdot \sqrt{H(d^2(\cdot,x);x)} + 4 \, |r|_{Lip_1} \cdot |s|_{Lip_1} \cdot H(d^2(\cdot,x);x) \\
&= \|f-r\|_\infty \cdot \{ \|g-s\|_\infty + 2 \, |s|_{Lip_1} \cdot \sqrt{H(d^2(\cdot,x);x)} \} \\
&+ 2 \, |r|_{Lip_1} \cdot \sqrt{H(d^2(\cdot,x);x)} \cdot \{ \|g-s\|_\infty + 2 \, |s|_{Lip_1} \cdot \sqrt{H(d^2(\cdot,x);x)} \} \\
&= \{ \|f-r\|_\infty + 2 \, |r|_{Lip_1} \sqrt{H(d^2(\cdot,x);x)} \} \\
&\cdot \{ \|g-s\|_\infty + 2 \, |s|_{Lip_1} \sqrt{H(d^2(\cdot,x);x)} \}.
\end{aligned}$$

We now pass to the infimum over r and s, respectively, which leads us to

$$\begin{aligned}
&|D(f,g)| \\
&\leq K\left(\sqrt{4H(d^2(\cdot,x);x)}, f; C(X), Lip_1\right) K\left(\sqrt{4H(d^2(\cdot,x);x)}, g; C(X), Lip_1\right) \\
&= \frac{1}{2}\widetilde{\omega}\left(f; 2\cdot\sqrt{4H(d^2(\cdot,x);x)}\right)\cdot\frac{1}{2}\widetilde{\omega}\left(g; 2\cdot\sqrt{4H(d^2(\cdot,x);x)}\right) \\
&= \frac{1}{4}\widetilde{\omega}\left(f; 4\sqrt{H(d^2(\cdot,x);x)}\right)\cdot\widetilde{\omega}\left(g; 4\sqrt{H(d^2(\cdot,x);x)}\right).
\end{aligned}$$

This ends our proof. □

3.1. Shepard-type operators

The latter result from Theorem 3.1 can be applied to Shepard-type operators defined in the general setting. An example of such Shepard-type operators goes back to the work of I.K. Crain and B.K. Bhattacharyya [3] and D. Shepard [11] and was first investigated by W.J. Gordon and J.A. Wixom [6]. Other important references are e.g. the Habilitationsschrift [4] and the paper [5], both by H. Gonska.

In both of the latter references, we have the following:

Definition 3.2. *Let (X,d) be a metric space and let $x_1,\ldots,x_n$ be a finite collection of distinct points in X. We further suppose that for each n-tuple $(x_1,\ldots,x_n)$ we have a finite given sequence $(\mu_1,\ldots,\mu_n)$ of real numbers $\mu_i > 0$. Then the Crain-Bhattacharyya-Shepard (CBS) operator is given by*

$$S_n(f;x) := S^{\mu_1,\ldots,\mu_n}_{x_1,\ldots,x_n}(f,x)$$

$$:= \begin{cases} \sum_{i=1}^n f(x_i)\cdot\frac{d(x,x_i)^{-\mu_i}}{\sum_{l=1}^n d(x,x_l)^{-\mu_l}} & , x\notin\{x_1,\ldots,x_n\} \\ f(x_i) & , \textit{otherwise.}\end{cases}$$

Here $x\in X$ and f is a real-valued function defined on X.

Remark 3.3. From the above definition, we can state that S_n is a positive linear operator on $C(X)$ that satisfies $S_n(1_X,x) = 1$ for all $x\in X$. Also it holds that $S_n(f,x_i) = x_i$, for all $x_i, 1\leq i\leq n$.

We now restrict ourselves to the simpler case $1\leq\mu=\mu_1=\ldots=\mu_n$ and denote the corresponding operator by S_n^μ. Now let $H := S_n^\mu$. Then we have the following main result:

Theorem 3.4. *Let $f,g\in C(X)$ be two given functions. Then the inequality*

$$|D(f,g)|\leq\frac{1}{4}\widetilde{\omega_d}\left(f;4\sqrt{\sum_{i=1}^n\frac{d(x,x_i)^{2-\mu}}{\sum_{l=1}^n d(x,x_l)^{-\mu}}}\right)\widetilde{\omega_d}\left(g;4\sqrt{\sum_{i=1}^n\frac{d(x,x_i)^{2-\mu}}{\sum_{l=1}^n d(x,x_l)^{-\mu}}}\right)$$

holds, for $x\notin\{x_1,\ldots,x_n\}$. For $x=x_i$, $|D(f,g)|=0$.

Proof. If we substitute the CBS operator S_n^μ in the result of Theorem 3.1, the following inequality

$$\begin{aligned}|D(f,g)| &= |S_n^\mu(f\cdot g;x) - S_n^\mu(f;x)\cdot S_n^\mu(g;x)| \\ &\le \frac{1}{4}\widetilde{\omega_d}\left(f;4\sqrt{S_n^\mu(d^2(\cdot,x);x)}\right)\cdot\widetilde{\omega_d}\left(g;4\sqrt{S_n^\mu(d^2(\cdot,x);x)}\right)\end{aligned}$$

holds. The second moment of the CBS-operator can be written as

$$S_n^\mu(d^2(\cdot,x);x) = \begin{cases}\sum_{i=1}^n \frac{d(x,x_i)^{2-\mu}}{\sum_{l=1}^n d(x,x_l)^{-\mu}} & , x\notin\{x_1,\dots,x_n\},\\ 0 & , \text{otherwise.}\end{cases} \tag{3.2}$$

Using (3.2) in the previous estimate, we get the claimed result and this ends our proof. □

Remark 3.5. We can also apply the Grüss-type inequality for the CBS operator defined on $X=[a,b]$, but we are not doing this here. What will be done in the sequel is improve the inequality from Theorem 4 in [1] and then apply it to different types of operators.

4. Grüss-type inequalities in $C[a,b]$

In a recent paper [1], Grüss-type inequalities in $C[a,b]$ were treated. The degree of non-multiplicativity of a positive linear operator $H: C[a,b]\to C[a,b]$ that reproduces constant functions was examined. For fixed $x\in[a,b]$ and two functions $f,g\in C[a,b]$, the positive linear functional $H(\cdot;x)$ was defined and the positive bilinear functional

$$D(f,g) := H(f\cdot g;x) - H(f;x)\cdot H(g;x)$$

was considered. We improve a result from the above stated article (see Theorem 4) by removing the constant $\sqrt{2}$ in the arguments of the least concave majorants. The idea of the proof was given by two of the authors of the article, namely H. Gonska and I. Raşa.

We state and prove the following:

Theorem 4.1. *If $f,g\in C[a,b]$ and $x\in[a,b]$ is fixed, then the inequality*

$$|D(f,g)| \le \frac{1}{4}\widetilde{\omega}\left(f;2\sqrt{H((e_1-x)^2;x)}\right)\cdot\widetilde{\omega}\left(g;2\sqrt{H((e_1-x)^2;x)}\right)$$

holds, where e_1 denotes the first monomial given by $e_1(t)=t$, $t\in[a,b]$.

Proof. Let $f,g\in C[a,b]$ and $r,s\in C^1[a,b]$. Just like in the proof of Theorem 4 in [1], we use the Cauchy-Schwarz inequality for positive linear functionals:

$$|H(f;x)| \le H(|f|;x) \le \sqrt{H(f^2;x)\cdot H(1;x)} = \sqrt{H(f^2;x)},$$

so we have

$$D(f,f) = H(f^2;x) - H(f;x)^2 \ge 0.$$

Then we can say that D is a positive bilinear form on $C[a,b]$. Using Cauchy-Schwarz for D, we obtain

$$|D(f,g)| \le \sqrt{D(f,f)D(g,g)} \le \|f\|_\infty\cdot\|g\|_\infty\,.$$

As stated before, $H : C[a,b] \to C[a,b]$ is a positive linear operator that reproduces constant functions, so that $H(\cdot;x)$, with fixed $x \in [a,b]$, is a positive linear functional that can be represented as

$$H(f;x) = \int_a^b f(t)d\mu_x(t),$$

where μ_x is a probability measure on $[a,b]$, i.e., $\int_a^b d\mu_x(t) = 1$. The interest is in finding an upper bound for the following:

$$\begin{aligned}|D(f,g)| &= |D(f-r+r, g-s+s)| \\ &\le |D(f-r,g-s)| + |D(f-r,s)| + |D(r,g-s)| + |D(r,s)|\,.\end{aligned}$$

What is different from Theorem 4 in [1] is that we replace a part of the proof with the following results. We first consider Theorem 12 from the same paper [1]. Let the function h in this theorem be equal to e_1. Then we can write

$$|D(r,s)| \le \|r'\|_\infty \cdot \|s'\|_\infty \cdot |D(e_1,e_1)|$$

and we know that

$$0 \le |D(e_1,e_1)| = H(e_2;x) - H(e_1;x)^2 \le H((e_1-x)^2;x).$$

This last inequality is true, because

$$\begin{aligned}H((e_1-x)^2;x) &= H(e_2 - 2\cdot e_1 \cdot x + x^2;x) \\ &= H(e_2;x) - 2\cdot x \cdot H(e_1;x) + x^2 \cdot H(e_0;x) \\ &\ge H(e_2;x) - H(e_1;x)^2\end{aligned}$$

is equivalent to

$$x^2 - 2\cdot x\cdot H(e_1;x) + H(e_1;x)^2 = (x - H(e_1;x))^2 \ge 0.$$

We then get

$$|D(r,s)| \le \|r'\|_\infty \cdot \|s'\|_\infty \cdot H((e_1-x)^2;x).$$

For $f-r \in C[a,b]$ and $g-s \in C[a,b]$ we have

$$|D(f-r,g-s)| \le \|f-r\|_\infty \cdot \|g-s\|_\infty\,.$$

Moreover, if $f-r \in C[a,b]$ and $s \in C^1[a,b]$, then

$$\begin{aligned}|D(f-r,s)| &\le \sqrt{D(f-r,f-r)\cdot D(s,s)} \\ &\le \|f-r\|_\infty \cdot \|s'\|_\infty \cdot \sqrt{H((e_1-x)^2;x)}\end{aligned}$$

and similarly, for $r \in C^1[a,b]$, $g-s \in C[a,b]$, we obtain

$$|D(r,g-s)| \le \|r'\|_\infty \cdot \|g-s\|_\infty \cdot \sqrt{H((e_1-x)^2;x)}.$$

If we combine all these inequalities, we have

$$\begin{aligned}
&|D(f,g)| \le \|f-r\|_\infty \cdot \|g-s\|_\infty + \|f-r\|_\infty \cdot \|s'\|_\infty \cdot \sqrt{H((e_1-x)^2;x)} \\
&+ \|r'\|_\infty \cdot \|g-s\|_\infty \cdot \sqrt{H((e_1-x)^2;x)} + \|r'\|_\infty \cdot \|s'\|_\infty \cdot H((e_1-x)^2;x) \\
&= \|f-r\|_\infty \cdot \left\{\|g-s\|_\infty + \|s'\|_\infty \cdot \sqrt{H((e_1-x)^2;x)}\right\} \\
&+ \|r'\|_\infty \cdot \sqrt{H((e_1-x)^2;x)} \cdot \left\{\|g-s\|_\infty + \|s'\|_\infty \cdot \sqrt{H((e_1-x)^2;x)}\right\} \\
&= \left\{\|f-r\|_\infty + \|r'\|_\infty \cdot \sqrt{H((e_1-x)^2;x)}\right\} \\
&\cdot \left\{\|g-s\|_\infty + \|s'\|_\infty \cdot \sqrt{H((e_1-x)^2;x)}\right\}.
\end{aligned}$$

We now pass to the infimum with respect to each of r, s and we obtain the wanted result:

$$\begin{aligned}
&|D(f,g)| \\
&\le K\left(\sqrt{H((e_1-x)^2;x)}, f; C^0, C^1\right) \cdot K\left(\sqrt{H((e_1-x)^2;x)}, g; C^0, C^1\right) \\
&= \frac{1}{2}\widetilde{\omega}\left(f; 2\sqrt{H((e_1-x)^2;x)}\right) \cdot \frac{1}{2}\widetilde{\omega}\left(g; 2\sqrt{H((e_1-x)^2;x)}\right) \\
&= \frac{1}{4}\widetilde{\omega}\left(f; 2\sqrt{H((e_1-x)^2;x)}\right) \cdot \widetilde{\omega}\left(g; 2\sqrt{H((e_1-x)^2;x)}\right).
\end{aligned}$$

This ends our proof. □

At present it is an open problem if the improved inequality in Theorem 4.1 can be generalized to $C(X)$ with (X, d) a compact metric space.

5. Applications

We can now apply the above improved result for different kinds of operators, like Bernstein- , convolution- and a special kind of Shepard-type operators.

5.1. Bernstein operator

As a first example, we have the following remark:

Remark 5.1. We consider $H := B_n$, the Bernstein operator defined by

$$B_n(f;x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \cdot \binom{n}{k} \cdot x^k (1-x)^{n-k},$$

where $f \in C[0,1]$ and $x \in [0,1]$, $n = 1, 2, \ldots$. It is well known that the second moment of the Bernstein polynomial is equal to

$$B_n((e_1-x)^2;x) = \frac{x(1-x)}{n}.$$

Using Theorem 4.1, we get the Grüss-type inequality for the Bernstein operator as follows:

$$\begin{aligned}
&|B_n(fg;x) - B_n(f;x)B_n(g;x)| \\
&\leq \frac{1}{4}\widetilde{\omega}\left(f; 2\sqrt{B_n((e_1-x)^2;x)}\right) \cdot \widetilde{\omega}\left(g; 2\sqrt{B_n((e_1-x)^2;x)}\right) \\
&= \frac{1}{4}\widetilde{\omega}\left(f; 2\sqrt{\frac{x(1-x)}{n}}\right) \cdot \widetilde{\omega}\left(g; 2\sqrt{\frac{x(1-x)}{n}}\right) \\
&\leq \frac{1}{4}\cdot\widetilde{\omega}\left(f; \frac{1}{\sqrt{n}}\right) \cdot \widetilde{\omega}\left(g; \frac{1}{\sqrt{n}}\right),
\end{aligned}$$

for two functions $f, g \in C[0,1]$.

5.2. Convolution-type operators

These types of operators were treated by many authors, like J.-D. Cao, H. Gonska and H.-J. Wenz (see [2]). One of the first authors to give the following definition was H.G. Lehnhoff in [8]:

Definition 5.2. *For the case $X = [-1,1]$, given a function $f \in C(X)$ and any natural number n, the convolution operator $G_{m(n)}$ is given by*

$$G_{m(n)}(f,x) := \frac{1}{\pi}\cdot\int_{-\pi}^{\pi} f(\cos(\arccos(x)+v))\cdot K_{m(n)}(v)dv,$$

where the kernel $K_{m(n)}$ is a positive and even trigonometric polynomial of degree $m(n)$ satisfying

$$\int_{-\pi}^{\pi} K_{m(n)}(v)dv = \pi,$$

meaning that $G_{m(n)}(1,x) = 1$ for $x \in X$.

It is clear that $G_{m(n)}(f,\cdot)$ is an algebraic polynomial of degree $m(n)$ and the kernel $K_{m(n)}$ has the following form:

$$K_{m(n)}(v) = \frac{1}{2} + \sum_{k=1}^{m(n)} \rho_{k,m(n)}\cdot\cos(kv),$$

for $v \in [-\pi,\pi]$.

We also need another result that goes back to H.G. Lehnhoff [8]:

Lemma 5.3. *For $x \in X$ the inequality*

$$\begin{aligned}
&G_{m(n)}((e_1-x)^2,x) \\
&= x^2\left\{\frac{3}{2} - 2\cdot\rho_{1,m(n)} + \frac{1}{2}\cdot\rho_{2,m(n)}\right\} + (1-x^2)\cdot\left\{\frac{1}{2} - \frac{1}{2}\cdot\rho_{2,m(n)}\right\}
\end{aligned}$$

holds. Here e_1 denotes the first monomial given by $e_1(t) = t$ for $|t| \leq 1$.

This lemma gives the second moment of the convolution-type operator, which we will need in the sequel.

Furthermore, we take into account different degrees $m(n)$, different convolution operators and Grüss-type inequalities, respectively.

5.2.1. Convolution-type operator with Fejér-Korovkin kernel. If we consider degree $m(n) = n - 1$, for $n \in \mathbb{N}$, the Fejér-Korovkin kernel is given by

$$K_{n-1}(v) = \frac{1}{n+1} \left(\frac{\sin\left(\frac{\pi}{n+1}\right) \cdot \cos\left((n+1)\frac{v}{2}\right)}{\cos(v) - \cos\left(\frac{\pi}{n+1}\right)} \right)^2$$

with

$$\rho_{1,n-1} = \cos\left(\frac{\pi}{n+1}\right), \ \rho_{2,n-1} = \frac{n}{n+1}\cos\left(\frac{2\pi}{n+1}\right) + \frac{1}{n+1}.$$

Using the latter relations, we get

$$\begin{aligned} G_{n-1}\left((e_1 - x)^2; x\right) &\leq \left|\frac{3}{2} - 2 \cdot \rho_{1,n-1} + \frac{1}{2}\rho_{2,n-1}\right| + \frac{1}{2}\left|1 - \rho_{2,n-1}\right| \\ &\leq \left|\frac{3}{2} - 2\cos\left(\frac{\pi}{n+1}\right) + \frac{1}{2(n+1)} + \frac{n}{2(n+1)}\cos\left(\frac{2\pi}{n+1}\right)\right| \\ &+ \frac{1}{2} \cdot \left|1 - \frac{1}{n+1} - \frac{n}{n+1} \cdot \cos\left(\frac{2\pi}{n+1}\right)\right| \\ &\leq 3 \cdot \left(\frac{\pi}{n+1}\right)^2 + \left(\frac{\pi}{n+1}\right)^2 \\ &= 4 \cdot \left(\frac{\pi}{n+1}\right)^2. \end{aligned}$$

Having this preamble, we can now state the following result.

Theorem 5.4. *If we consider $f, g \in C(X)$ and the convolution-type operator of degree $n-1$ with the Fejér-Korovkin kernel, we have*

$$\begin{aligned} |D(f,g)| &= |G_{n-1}(f \cdot g; x) - G_{n-1}(f; x) \cdot G_{n-1}(g; x)| \\ &\leq \frac{1}{4}\widetilde{\omega}\left(f; \frac{4\pi}{n+1}\right) \cdot \widetilde{\omega}\left(g; \frac{4\pi}{n+1}\right) \\ &= \mathcal{O}\left(\widetilde{\omega}\left(f; \frac{1}{n}\right) \cdot \widetilde{\omega}\left(g; \frac{1}{n}\right)\right). \end{aligned}$$

5.2.2. Convolution-type operator with de La Vallée Poussin kernel. We now have degree $m(n) = n \in \mathbb{N}_0$ and we define the de La Vallée Poussin kernel by

$$V_n(v) = \frac{(n!)^2}{(2n)!} \cdot \left(2\cos\left(\frac{v}{2}\right)\right)^{2n}$$

with

$$\rho_{1,n} = \frac{n}{n+1}, \ \rho_{2,n} = \frac{(n-1)n}{(n+1)(n+2)}.$$

Using the two relations, we have the second moment:

$$\begin{aligned}
G_n\left((e_1-x)^2;x\right) &\le \left|\frac{3}{2}-\frac{2n}{n+1}+\frac{1}{2}\cdot\frac{n(n-1)}{(n+1)(n+2)}\right| \\
&+\frac{1}{2}\left|1-\frac{n(n-1)}{(n+1)(n+2)}\right| \\
&\le \left|\frac{3}{(n+1)(n+2)}\right|+\left|\frac{2n+1}{(n+1)(n+2)}\right| \\
&\le \frac{2}{n+1}.
\end{aligned}$$

Taking this into account, we give the following theorem:

Theorem 5.5. *If we consider the convolution-type operator with the de La Vallée Poussin kernel we have*

$$\begin{aligned}
|D(f,g)| &= |G_n(f\cdot g;x)-G_n(f;x)\cdot G_n(g;x)| \\
&\le \frac{1}{4}\widetilde{\omega}\left(f;\frac{2\sqrt{2}}{\sqrt{n+1}}\right)\cdot\widetilde{\omega}\left(g;\frac{2\sqrt{2}}{\sqrt{n+1}}\right) \\
&= \mathcal{O}\left(\widetilde{\omega}\left(f;\frac{1}{\sqrt{n}}\right)\cdot\widetilde{\omega}\left(g;\frac{1}{\sqrt{n}}\right)\right).
\end{aligned}$$

5.2.3. Convolution-type operator with Jackson kernel. Finally, the last operator we consider is of degree $m(n)=2n-2$, with $n\in\mathbb{N}$. For this, the Jackson kernel has the form

$$J_{2n-2}(v)=\frac{3}{2n(2n^2+1)}\cdot\left(\frac{\sin\left(n\frac{v}{2}\right)}{\sin\left(\frac{v}{2}\right)}\right)^4$$

with

$$\rho_{1,2n-2}=\frac{2n^2-2}{2n^2+1},\ \rho_{2,2n-2}=\frac{2n^3-11n+9}{n(2n^2+1)}$$

and the second moment

$$\begin{aligned}
G_{2n-2}\left((e_1-x)^2;x\right) &\le \left|\frac{3}{2}-\frac{4n^2-4}{2n^2+1}+\frac{1}{2}\cdot\frac{2n^3-11n+9}{n(2n^2+1)}\right| \\
&+\frac{1}{2}\cdot\left|1-\frac{2n^3-11n+9}{n(2n^2+1)}\right| \\
&\le \left|\frac{9}{2n(2n^2+1)}\right|+\left|\frac{12n-9}{2n(2n^2+1)}\right| \\
&\le \frac{6}{2n^2+1}\le\frac{3}{n^2}.
\end{aligned}$$

The result is as follows:

Theorem 5.6. *If we consider the convolution-type operator with the Jackson kernel we have*

$$\begin{aligned}|D(f,g)| &= |G_{2n-2}(f\cdot g;x) - G_{2n-2}(f;x)\cdot G_{2n-2}(g;x)|\\ &\le \frac{1}{4}\widetilde{\omega}\left(f;\frac{2\sqrt{3}}{n}\right)\cdot\widetilde{\omega}\left(g;\frac{2\sqrt{3}}{n}\right)\\ &= \mathcal{O}\left(\widetilde{\omega}\left(f;\frac{1}{n}\right)\cdot\widetilde{\omega}\left(g;\frac{1}{n}\right)\right).\end{aligned}$$

As we can see, the best degrees of approximation are obtained when dealing with the Grüss-type inequality for convolution operators in the cases of Fejér-Korovkin and Jackson kernels.

Remark 5.7. Another possibility is to apply the above obtained Grüss inequality for the Shepard-type operator defined on $C[0,1]$. But this result, just like in the case of the Hermite-Fejér operator, is disappointing (see Remark 7 in [1]).

6. A pre-Grüss-type inequality for the CBS operator

We now try to find a pre-Grüss inequality for the CBS-operator. Just like in the case of the pre-Grüss-type inequality for the Hermite-Fejér operator, obtained in [1] (see Theorem 8), the idea is to find a different approach. We consider the special case $X=[0,1]$, $d(x,y)=|x-y|$. Then, taking $H:=S_{n+1}^{\mu}$ the CBS operator based on $n+1$ equidistant points $x_i=\frac{i}{n}$, for $0\le i\le n$ and $1\le\mu\le 2$, we get:

Theorem 6.1. *Let $f,g\in C[0,1]$. Then the inequality*

$$\begin{aligned}&|D(f,g)|\\ &\le\frac{1}{2}\min\left\{\|f\|_{\infty}\,\widetilde{\omega_d}\left(g;4S_{n+1}^{\mu}\left(|e_1-x|\,;x\right)\right);\|g\|_{\infty}\,\widetilde{\omega_d}\left(f;4S_{n+1}^{\mu}\left(|e_1-x|\,;x\right)\right)\right\}\end{aligned}$$

holds.

Proof. We want to estimate

$$|D(f,g)| = \left|S_{n+1}^{\mu}(f\cdot g;x) - S_{n+1}^{\mu}(f;x)\cdot S_{n+1}^{\mu}(g;x)\right|.$$

For two fixed functions $f,g\in C[0,1]$ and an arbitrary $s\in C^1[0,1]$, we have

$$|D(f,g)| = |D(f,g-s+s)| \le |D(f,g-s)| + |D(f,s)|. \qquad (6.1)$$

First, if we have $f \in C[0,1]$ and $s \in C^1[0,1]$, we continue with

$$\begin{aligned}
|D(f,s)| &= \left|S^{\mu}_{n+1}(f\cdot s;x) - S^{\mu}_{n+1}(f;x)\cdot S^{\mu}_{n+1}(s;x)\right| \\
&= \left|S^{\mu}_{n+1}(f(s - S^{\mu}_{n+1}(s;x));x)\right| \\
&= \left|S^{\mu}_{n+1,t}(f(t)(s(t)-s(x)+s(x)-S^{\mu}_{n+1}(s;x));x)\right| \\
&\le \|f\|_\infty \cdot S^{\mu}_{n+1,t}(|s(t)-s(x)| + \left|s(x)-S^{\mu}_{n+1}(s;x)\right|;x) \\
&\le \|f\|_\infty \cdot S^{\mu}_{n+1}(\|s'\|_\infty \cdot |e_1 - x| + \|s'\|_\infty \cdot S^{\mu}_{n+1}(|e_1-x|;x);x) \\
&= 2\cdot \|f\|_\infty \cdot \|s'\|_\infty \cdot S^{\mu}_{n+1}(|e_1-x|;x).
\end{aligned}$$

If we now use this result in (6.1), we get

$$\begin{aligned}
|D(f,g)| &\le \|f\|_\infty \cdot \|g-s\|_\infty + 2\cdot \|f\|_\infty \cdot \|s'\|_\infty \cdot S^{\mu}_{n+1}(|e_1-x|;x) \\
&= \|f\|_\infty \left\{\|g-s\|_\infty + 2\cdot \|s'\|_\infty \cdot S^{\mu}_{n+1}(|e_1-x|;x)\right\}.
\end{aligned}$$

Passing to the infimum over $s \in C^1[0,1]$, it follows

$$\begin{aligned}
|D(f,g)| &\le \|f\|_\infty \cdot K(2\cdot S^{\mu}_{n+1}(|e_1-x|;x), g; C[0,1], C^1[0,1]) \\
&= \frac{1}{2}\cdot \|f\|_\infty \cdot \widetilde{\omega}\left(g, 4\cdot S^{\mu}_{n+1}(|e_1-x|;x)\right).
\end{aligned}$$

The same estimate holds if we interchange f and g. Putting both inequalities together, we get the result we were looking for. □

In the above result, the first absolute moment of the CBS operator appears, which can be represented by

$$S^{\mu}_{n+1}(|e_1-x|;x) = \begin{cases} \sum_{i=0}^{n} \frac{\left|x-\frac{i}{n}\right|^{1-\mu}}{\sum_{l=0}^{n}\left|x-\frac{l}{n}\right|^{-\mu}} & , x \notin \{x_0,\dots,x_n\} \\ 0 & , \text{otherwise.} \end{cases}$$

The idea is to further estimate this quantity. For that, we use an idea from [5] (see proof of Theorem 4.3).

We distinguish three important cases for different values of μ.

The first case is $\mu = 1$. The first absolute moment of the CBS operator becomes

$$\begin{aligned}
S^{1}_{n+1}(|e_1-x|;x) &= \begin{cases} \sum_{i=0}^{n} \frac{1}{\sum_{l=0}^{n}\left|x-\frac{l}{n}\right|^{-1}} & , x \notin \{x_0,\dots,x_n\} \\ 0 & , \text{otherwise} \end{cases} \\
&= \begin{cases} (n+1)\left(\sum_{l=0}^{n} \frac{1}{\left|x-\frac{l}{n}\right|}\right)^{-1} & , x \notin \{x_0,\dots,x_n\} \\ 0 & , \text{otherwise.} \end{cases}
\end{aligned}$$

Let now l_0 be defined by $\frac{l_0}{n} < x < \frac{l_0+1}{n}$. Then we have

$$\begin{aligned}\frac{1}{n+1}\cdot\left(\sum_{l=0}^{n}\frac{1}{\left|x-\frac{l}{n}\right|}\right) &\geq \frac{n}{n+1}\cdot\left\{\sum_{l=0}^{l_0}\frac{1}{l_0+1-l}+\sum_{l=l_0+1}^{n}\frac{1}{l-l_0}\right\}\\ &\geq \frac{n}{n+1}\left\{\int_1^{l_0+2}\frac{1}{x}dx+\int_1^{n-l_0+1}\frac{1}{x}dx\right\}\\ &= \frac{n}{n+1}\ln((l_0+2)\cdot(n-l_0+1))\\ &\geq \frac{n}{n+1}\cdot\ln(2n+2),\end{aligned}$$

and the second absolute moment is then

$$S^1_{n+1}(|e_1-x|\,;x) \leq \frac{n+1}{n\cdot\ln(2n+2)},$$

for $x \notin \{x_0,\ldots,x_n\}$. In the end we get

$$\begin{aligned}&|D(f,g)|\\ &\leq \frac{1}{2}\min\left\{\|f\|_\infty\cdot\widetilde{\omega_d}\left(g;\frac{4(n+1)}{n\cdot\ln(2n+2)}\right),\|g\|_\infty\cdot\widetilde{\omega_d}\left(f;\frac{4(n+1)}{n\cdot\ln(2n+2)}\right)\right\}.\end{aligned}$$

For the other two cases we will consider, first let l_0 defined by

$$\left|x-\frac{l_0}{n}\right| = \min\left\{\left|x-\frac{l}{n}\right| : 0\leq l\leq n\right\}.$$

Then for the case $x \notin \{x_0,\ldots,x_n\}$, we have

$$\begin{aligned}S^\mu_{n+1}(|e_1-x|\,;x) &\leq |x-x_{l_0}|^\mu\cdot\sum_{i=0}^{n}|x-x_i|^{1-\mu}\\ &\leq \frac{1}{n}+\left(\frac{1}{n}\right)\cdot\left\{\sum_{i<l_0}|x-x_i|^{1-\mu}+\sum_{i>l_0}|x-x_i|^{1-\mu}\right\}\\ &\leq \frac{1}{n}+\left(\frac{1}{n}\right)\cdot\left\{\sum_{k=0}^{l_0-1}\left(\frac{1}{2}+k\right)^{1-\mu}+\sum_{k=0}^{n-l_0-1}\left(\frac{1}{2}+k\right)^{1-\mu}\right\},\end{aligned}$$

with $0 \leq l_0 \leq n$. Either of the two last sums may be empty. Estimating the result in the accolades from above, we get

$$S^\mu_{n+1}(|e_1-x|\,;x) \leq \begin{cases}\frac{1}{n}+\frac{1}{n}\cdot\left[2^\mu+\frac{2}{2-\mu}\cdot\left(\frac{n+1}{2}\right)^{2-\mu}\right] & \text{, for } 1<\mu<2\\ \frac{1}{n}+\frac{1}{n}\cdot[4+2\cdot\ln(n+1)] & \text{, for } \mu=2\end{cases}. \tag{6.2}$$

For $1 < \mu < 2$, we obtain

$$\begin{aligned}&|D(f,g)|\\ &\leq \frac{1}{2}\min\left\{\|f\|_\infty\,\widetilde{\omega_d}\left(g;4S^\mu_{n+1}(|e_1-x|\,;x)\right),\|g\|_\infty\,\widetilde{\omega_d}\left(f;4S^\mu_{n+1}(|e_1-x|\,;x)\right)\right\}\end{aligned}$$

where the first absolute moment can be estimated from above as in (6.2). For $\mu = 2$ we obtain

$$|D(f,g)| \leq \frac{1}{2}\min\left\{\|f\|_\infty \widetilde{\omega_d}\left(g; \frac{20+8\cdot\ln(n+1)}{n}\right), \|g\|_\infty \widetilde{\omega_d}\left(f; \frac{20+8\cdot\ln(n+1)}{n}\right)\right\}.$$

One can also obtain results for $\mu > 2$. This was done by G. Somorjai [12](see also J. Szabados [13] for $\mu > 4$), but we are not treating other cases in this article.

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