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Area preserving maps from rectangles to elliptic domains

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Abstract. We construct a bijection from $\mathbb{R}^2$ to $\mathbb{R}^2$, which maps, for each $\alpha \in (0, \infty)$, rectangles of arbitrary edges $2\alpha L_1, 2\alpha L_2$ onto ellipses with semi-axes $\alpha a, \alpha b$, with a, b satisfying $4L_1L_2 = \pi ab$. This bijection preserves area and thus allows us to construct uniform and refinable grids on elliptic domains starting from uniform and refinable grids on rectangles.

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1. Introduction

Uniform and refinable grids (UR) are useful in many applications, like construction of multiresolution analysis and wavelets, or for solving numerically partial differential equations. While on a rectangle or on other polygonal domains the construction of UR grids is trivial, it is not immediate on an elliptic domain or on a disc.

In this paper we construct an area preserving bijection from $\mathbb{R}^2$ to $\mathbb{R}^2$, which maps rectangles of arbitrary edges $2\alpha L_1, 2\alpha L_2$ onto ellipses with semi-axes $\alpha a, \alpha b$, with a, b satisfying $4L_1L_2 = \pi ab$. This allows us to transport a rectangular grid to an elliptic grid, preserving the area of the cells. In particular, any uniform[1] rectangular grid is mapped into a uniform elliptic grid. A refinement process is needed when a grid is not fine enough to solve a problem accurately. A uniform refinement consists in dividing a cell into a given number of smaller cells with the same area. With the procedure described here, any uniform refinement of a rectangular grid leads to a uniform refinement of the corresponding elliptic grid.

[1] A grid is uniform if its cells have the same area.

In the particular case of the disc, such a bijection was constructed in a previous paper [1] and helped us to construct uniform grids on the sphere.

The particular case when $2L_1 = \sqrt{\pi}a$ and $2L_2 = \sqrt{\pi}b$ (the semi-axes a, b of the ellipse are proportional to the edges $2L_1, 2L_2$ of the rectangle) was considered in [2]. Here we consider the general case when the edges of the rectangle are arbitrary and the semi-axes satisfy the condition $4L_1L_2 = \pi ab$, implied by the fact that the rectangle and the ellipse have the same area. Also, we use another method of construction than the one in [1, 2].

2. Construction of an area preserving bijection in $\mathbb{R}^2$

Consider the ellipse $\mathcal{E}_{a,b}$ of semi-axes a and b, $a, b > 0$, of equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and the rectangle $\mathcal{R}_{L_1,L_2}$ with edges of lengths $2L_1$ and $2L_2$, defined as

$$\mathcal{R}_{L_1,L_2} = \{(x, y) \in \mathbb{R}^2,\ |x| = L_1, |y| = L_2\}.$$

The domains enclosed by $\mathcal{E}_{a,b}$ and $\mathcal{R}_{L_1,L_2}$ will be denoted by $\overline{\mathcal{E}}_{a,b}$ and $\overline{\mathcal{R}}_{L_1,L_2}$, respectively. We will construct a bijection $T^{a,b}_{L_1,L_2} : \mathbb{R}^2 \to \mathbb{R}^2$ which maps each rectangle $\mathcal{R}_{\alpha L_1,\alpha L_2}$ onto the ellipse $\mathcal{E}_{\alpha a,\alpha b}$ and has the area preserving property

$$\mathcal{A}(D) = \mathcal{A}(T^{a,b}_{L_1,L_2}(D)),\ \text{for every domain } D \subseteq \mathbb{R}^2. \tag{2.1}$$

Here $\mathcal{A}(D)$ denotes the area of D. Thus, $\mathcal{A}(\overline{\mathcal{R}}_{L_1,L_2}) = \mathcal{A}(\overline{\mathcal{E}}_{a,b})$ implies

$$\pi ab = 4L_1L_2.$$

We focus for the moment on the first octant I of the plane,

$$I = I_{L_1,L_2} = \{(x, y) \in \mathbb{R}^2,\ 0 \le L_1 y \le L_2 x\}.$$

The map $T^{a,b}_{L_1,L_2}$ will be defined in such a way that each half-line $d_m \subset I$ of equation $y = mx$ $(0 \le m \le \frac{L_2}{L_1})$ is mapped onto the half-line $d_{\varphi(m)}$ of equation $Y = \varphi(m)X$, such that

$$0 \le \varphi(m) \le \frac{b}{a},\quad \text{for } 0 \le m \le \frac{L_2}{L_1}.$$

Let $Q = Q(L_1, mL_1)$ and let $Q' = Q'(L_1, 0)$ be its projection on Ox. The area of the triangle OQQ' is

$$\mathcal{A}_\triangle = \frac{mL_1^2}{2} = \frac{yL_1^2}{2x}.$$

We denote by $(X, \varphi(m)X)$ the coordinates of the point $P = T^{a,b}_{L_1,L_2}(Q) \in \mathcal{E}_{a,b}$. The area of the portion of the elliptic domain $\overline{\mathcal{E}}_{a,b}$ located between the axis OX and the line $Y = \varphi(m)X$ will be

$$\mathcal{A}_e = \frac{ab\theta}{2},$$

where

$$\theta = \arctan \frac{a\varphi(m)}{b}$$

is the angle between the axis OX and OP. Next, we impose the area preserving property $\mathcal{A}_{\Delta} = \mathcal{A}_e$, which yields

$$\theta = \frac{\pi L_1 y}{4 L_2 x},$$

and therefore

$$\varphi(m) = \frac{b}{a} \tan \frac{\pi L_1 y}{4 L_2 x}.$$

It is easy to see that φ has the following properties:

$$\varphi(0) = 0, \quad \varphi\left(\frac{L_2}{L_1}\right) = \frac{b}{a}, \quad \text{and}$$

$$0 \leq \varphi(m) \leq \frac{b}{a}, \quad \text{for } 0 \leq m \leq \frac{L_2}{L_1}.$$

Consider now $M = M(x, mx)$ and $N = T^{a,b}_{L_1,L_2}(M) = (X, \varphi(m)X)$, which belongs to an ellipse $\mathcal{E}_{\alpha a, \alpha b}$ for a certain α. The portion of the elliptic domain $\overline{\mathcal{E}}_{\alpha a, \alpha b}$, located between ON and OX, has the area

$$\mathcal{A}_{e,\alpha} = \frac{ab\theta\alpha^2}{2},$$

whereas the area of the triangle OMM', with $M' = M'(x, 0)$ is $mx^2/2$.

Again, the area preserving property implies this time

$$\alpha = x\sqrt{\frac{m}{ab\theta}} = 2x\sqrt{\frac{L_2}{L_1} \cdot \frac{1}{\pi ab}}.$$

Finally, from $N \in \mathcal{E}_{\alpha a, \alpha b}$ we obtain

$$\frac{X^2}{a^2} + \frac{X^2 \varphi^2(m)}{b^2} = \alpha^2,$$

and therefore

$$\begin{aligned} X &= \frac{ab\alpha}{\sqrt{b^2 + a^2\varphi^2(m)}} = \frac{a\alpha}{\sqrt{1 + \tan^2\theta}} = a\alpha\cos\theta = 2x\sqrt{\frac{aL_2}{bL_1\pi}} \cos\frac{\pi L_1 y}{4L_2 x}, \\ Y &= \varphi(m)X = 2x\sqrt{\frac{bL_2}{aL_1\pi}} \sin\frac{\pi L_1 y}{4L_2 x}. \end{aligned}$$

A simple calculation shows that the Jacobian of $T^{a,b}_{L_1,L_2}$ is 1 and therefore relation (2.1) is fulfilled for domains $D \subseteq I$.

By similar arguments for the other seven octants, we find that the function $T^{a,b}_{L_1,L_2} : \mathbb{R}^2 \to \mathbb{R}^2$ which maps rectangles onto ellipses and preserves areas is defined as follows:

- For $L_1|y| \leq L_2|x|$,

$$(x, y) \longmapsto (X, Y) = \left(2x\sqrt{\frac{aL_2}{bL_1\pi}} \cos\frac{\pi L_1 y}{4L_2 x},\ 2x\sqrt{\frac{bL_2}{aL_1\pi}} \sin\frac{\pi L_1 y}{4L_2 x}\right);$$

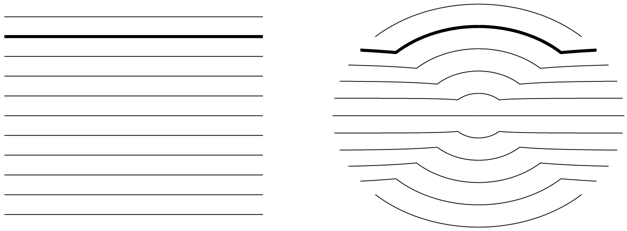

FIGURE 1. A horizontal grid and its image grid on the elliptic domain. The image of the bold line on the left is the bold curve on the right.

- For $L_2|x| \leq L_1|y|$,

$$(x,y) \longmapsto (X,Y) = \left(2y\sqrt{\frac{aL_1}{bL_2\pi}}\sin\frac{\pi L_2 x}{4L_1 y},\ 2y\sqrt{\frac{bL_1}{aL_2\pi}}\cos\frac{\pi L_2 x}{4L_1 y}\right).$$

For the origin we take $T^{a,b}_{L_1,L_2}(0,0) = (0,0)$. We can prove that $T^{a,b}_{L_1,L_2}$ is continuous and bijective and its inverse is given by the following formulas:

- For $a|Y| \leq b|X|$,

$$(X,Y) \longmapsto (x,y) = \operatorname{sign}(X)\sqrt{X^2 + \frac{a^2}{b^2}Y^2}\left(\frac{\sqrt{\pi}}{2},\ \frac{2b}{a\sqrt{\pi}}\arctan\frac{aY}{bX}\right);$$

- For $b|X| \leq a|Y|$,

$$(X,Y) \longmapsto (x,y) = \operatorname{sign}(Y)\sqrt{\frac{b^2}{a^2}X^2 + Y^2}\left(\frac{2a}{b\sqrt{\pi}}\arctan\frac{bX}{aY},\ \frac{\sqrt{\pi}}{2}\right).$$

3. Uniform and refinable grids

The area preserving maps constructed in the previous section can be used for the construction of UR grids on elliptic domains, by mapping any UR rectangular grid.

Figure 1 shows the image of horizontal lines by an application $T^{a,b}_{L_1,L_2}$. In Figures 2 and 3 we show two grids on an elliptic domain and its refinement, both images of a rectangular grid.

Of course, other 2D uniform grids on a rectangle can be constructed, including triangular grids with different types of refinements.

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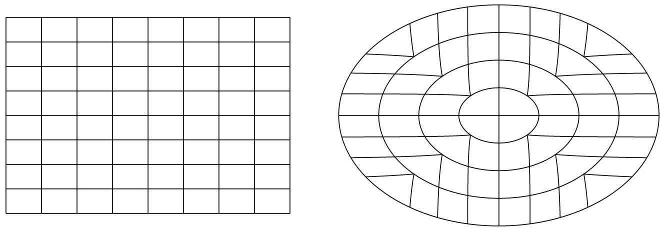

FIGURE 2. A uniform grid on a rectangle and its image - a uniform grid on the elliptic domain.

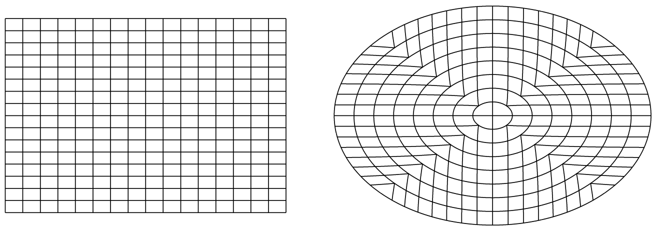

FIGURE 3. A refinement of the grid in Figure 2 and its image - a refinement of the elliptic grid in Figure 2.

References

[1] Roşca, D., *New uniform grids on the sphere*, Astronomy and Astrophysics, vol. 520, A63, 2010.

[2] Roşca, D., *Uniform and refinable grids on elliptic domains and on some surfaces of revolution*, Appl. Math. Comput, accepted doi:10.1016/j.amc.2011.02.095

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