Jet geometrical objects produced by linear ODEs systems and superior order ODEs

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Abstract. The aim of this paper is to construct a Riemann-Lagrange geometry on 1-jet spaces, in the sense of d-connections, d-torsions, d-curvatures, electromagnetic d-field and geometric electromagnetic Yang-Mills energy, starting from a given linear ODEs system or a given superior order ODE. The case of a non-homogenous linear ODE of superior order is discussed.

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1. Introduction

According to Olver's opinion expressed in [7] and in private discussions, we point out that the 1-jet spaces are main mathematical models necessary for the study of classical or quantum field theories. In such a context, the contravariant differential geometry of the 1-jet spaces was intensively studied by authors like Asanov [1] or Saunders [9].

Situated in the direction initiated by Asanov [1], it has been recently developed the *Riemann-Lagrange geometry of 1-jet spaces* [2], [4], which is a geometrical theory on 1-jet spaces analogous with the well known *Lagrange geometry of the tangent bundle* developed by Miron and Anastasiei [3].

It is important to note that the Riemann-Lagrange geometry of the 1-jet spaces allows the regarding of the solutions of a given ODEs (respectively, PDEs) system as *geodesics* [10] (respectively, *generalized harmonic maps* [6] or *potential maps* [11]) in a convenient Riemann-Lagrange geometrical structure on 1-jet spaces. In this way, it was given a final solution for an open problem suggested by Poincaré [8] (*find the geometric structure which transforms the field lines of a given vector field into geodesics*) and generalized by

Udrişte [10] (*find the geometrical structure which converts the solutions of a given first order PDEs system into harmonic maps*).

In this context, using the Riemann-Lagrange geometrical methods, it was constructed an entire contravariant differential geometry on 1-jet spaces, in the sense of d-connections, d-torsions, d-curvatures, electromagnetic d-field and geometric electromagnetic Yang-Mills energy, starting only with a given ODEs [5] (respectively, PDEs [6]) system of order one and a pair of Riemannian metrics.

2. Jet Riemann-Lagrange geometry produced by a non-linear ODEs system of order one and a pair of Riemannian metrics

In this Section we present the main jet Riemann-Lagrange geometrical ideas used for the geometrical study of a given non-linear first order ODEs system. For more details, the reader is invited to consult the works [4], [5] and [11].

Let $T = [a, b] \subset \mathbb{R}$ be a compact interval of the set of real numbers and let us consider the jet fibre bundle of order one

$$J^1(T, \mathbb{R}^n) \to T \times \mathbb{R}^n, \ n \geq 2,$$

whose local coordinates (t, x^i, x_1^i), $i = \overline{1, n}$, transform by the rules

$$\widetilde{t} = \widetilde{t}(t), \ \widetilde{x}^i = \widetilde{x}^i(x^j), \ \widetilde{x}_1^i = \frac{\partial \widetilde{x}^i}{\partial x^j} \frac{dt}{d\widetilde{t}} \cdot x_1^j.$$

Remark 2.1. *From a physical point of view, in the 1-jet space of* **physical events** *the coordinate t has the physical meaning of* **relativistic time**, *the coordinates* $(x^i)_{i=\overline{1,n}}$ *represent* **spatial coordinates** *and the coordinates* $(x_1^i)_{i=\overline{1,n}}$ *have the physical meaning of* **relativistic velocities***.*

Let $X = \left(X_{(1)}^{(i)}(t, x^k) \right)$ be an arbitrary given d-tensor field on the first order jet space $J^1(T, \mathbb{R}^n)$, which produces the jet non-linear ODEs system of order one (*jet dynamical system*)

$$x_1^i = X_{(1)}^{(i)}(t, x^k(t)), \ \forall \ i = \overline{1, n}, \tag{2.1}$$

where $c(t) = (x^i(t))$ is an unknown curve on $\mathbb{R}^n$ and we use the notations

$$x_1^i \stackrel{not}{=} \dot{x}^i = \frac{dx^i}{dt}, \ \forall \ i = \overline{1, n}.$$

Suppose now that we fixed *a priori* two Riemannian structures $(T, h_{11}(t))$ and $(\mathbb{R}^n, \varphi_{ij}(x))$, where $x = (x^k)_{k=\overline{1,n}}$, together with their attached Christoffel symbols $H_{11}^1(t)$ and $\gamma_{jk}^i(x)$. Automatically, the jet non-linear ODEs system of order one (2.1), together with the pair of Riemannian metrics

$$\mathcal{P} = (h_{11}(t), \varphi_{ij}(x)),$$

produce the *jet least squares Lagrangian function*

$$JLS_{\mathcal{P}}^{\text{ODEs}} : J^1(T, \mathbb{R}^n) \to \mathbb{R}_+,$$

expressed by

$$JLS_{\mathcal{P}}^{\text{ODEs}}(t, x^k, x_1^k) = h^{11}(t)\varphi_{ij}(x)\left[x_1^i - X_{(1)}^{(i)}(t, x)\right]\left[x_1^j - X_{(1)}^{(j)}(t, x)\right].$$

It is obvious that the *global minimum points* of the *jet least squares energy action*

$$\mathbb{E}_{\mathcal{P}}^{\text{ODEs}}(c(t)) = \int_a^b JLS_{\mathcal{P}}^{\text{ODEs}}(t, x^k(t), \dot{x}^k(t))\sqrt{h_{11}(t)}dt$$

are exactly the solutions of class C^2 of the jet non-linear ODEs system of order one (2.1). In other words, we have

Theorem 2.2. *The solutions of class C^2 of the first order ODEs system (2.1) verify the second order Euler-Lagrange equations produced by the jet least squares Lagrangian function $JLS_{\mathcal{P}}^{ODEs}$, namely (**jet geometric dinamics**)*

$$\frac{\partial\left[JLS_{\mathcal{P}}^{ODEs}\right]}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial\left[JLS_{\mathcal{P}}^{ODEs}\right]}{\partial \dot{x}^i}\right) = 0, \ \forall \ i = \overline{1, n}. \tag{2.2}$$

Remark 2.3. *Conversely, the above statement does not hold good because there exist solutions for the second order Euler-Lagrange ODEs system (2.2) which are not global minimum points for the jet least squares energy action $\mathbb{E}_{\mathcal{P}}^{ODEs}$, that is which are not solutions for the jet first order ODEs system (2.1).*

As a conclusion, we believe that we may regard $JLS_{\mathcal{P}}^{\text{ODEs}}$ as a natural geometrical substitute on $J^1(T, \mathbb{R}^n)$ for the jet first order ODEs system (2.1).

But, we point out that a Riemann-Lagrange geometry on $J^1(T, \mathbb{R}^n)$ produced by the jet least squares Lagrangian function $JLS_{\mathcal{P}}^{\text{ODEs}}$, via its second order Euler-Lagrange equations (2.2), geometry in the sense of non-linear connection, generalized Cartan connection, d-torsions and d-curvatures, is now completely done in the papers [4], [5] and [6]. Moreover, a distinguished jet electromagnetic 2-form, characterized by some natural generalized Maxwell equations and a geometric jet Yang-Mills energy [5], is constructed from the jet least squares Lagrangian function $JLS_{\mathcal{P}}^{\text{ODEs}}$.

Definition 2.4. *Any geometrical object on $J^1(T, \mathbb{R}^n)$, which is produced by the jet least squares Lagrangian function $JLS_{\mathcal{P}}^{ODEs}$, via the Euler-Lagrange equations (2.2), is called* **geometrical object produced by the jet first order ODEs system (2.1) and the pair of Riemannian metrics $\mathcal{P}$.**

In this context, we give the following jet Riemann-Lagrange geometrical result, which is proved in [5] and, for the multi-time general case, in [6]. For more details, the reader is invited to consult the book [4].

Theorem 2.5. *(i) The* **canonical non-linear connection on $J^1(T, \mathbb{R}^n)$ produced by the jet first order ODEs system (2.1) and the pair of Riemannian metrics $\mathcal{P}$ is**

$$\Gamma_{\mathcal{P}}^{ODEs} = \left(M_{(1)1}^{(i)}, N_{(1)j}^{(i)}\right),$$

whose local components are given by

$$M^{(i)}_{(1)1} = -H^1_{11}x^i_1 \text{ and } N^{(i)}_{(1)j} = \gamma^i_{jk}x^k_1 - \frac{1}{2}\left[X^{(i)}_{(1)||j} - \varphi^{ir}X^{(s)}_{(1)||r}\varphi_{sj}\right],$$

where

$$X^{(i)}_{(1)||j} = \frac{\partial X^{(i)}_{(1)}}{\partial x^j} + X^{(m)}_{(1)}\gamma^i_{mj}.$$

(ii) The **canonical generalized Cartan connection** $C\Gamma^{ODEs}_{\mathcal{P}}$ **produced by the jet first order ODEs system (2.1) and the pair of Riemannian metrics** $\mathcal{P}$ *has the adapted components*

$$C\Gamma^{ODEs}_{\mathcal{P}} = (H^1_{11}, 0, \gamma^i_{jk}, 0).$$

(iii) The effective adapted components of the **torsion** *d-tensor* $\mathbf{T}^{ODEs}_{\mathcal{P}}$ *of the canonical generalized Cartan connection* $C\Gamma^{ODEs}_{\mathcal{P}}$ **produced by the jet first order ODEs system (2.1) and the pair of Riemannian metrics** $\mathcal{P}$ *are*

$$R^{(i)}_{(1)1j} = \frac{1}{2}\left[X^{(i)}_{(1)||j//1} - \varphi^{ir}X^{(s)}_{(1)||r//1}\varphi_{sj}\right]$$

and

$$R^{(i)}_{(1)jk} = r^i_{jkm}x^m_1 - \frac{1}{2}\left[X^{(i)}_{(1)||j||k} - \varphi^{ir}X^{(s)}_{(1)||r||k}\varphi_{sj}\right],$$

where $r^l_{ijk}(x)$ *are the components of the curvature tensor of the Riemannian metric* $\varphi_{ij}(x)$ *and*

$$X^{(i)}_{(1)||j//1} = \frac{\partial X^{(i)}_{(1)||j}}{\partial t} - X^{(i)}_{(1)||j}H^1_{11},$$

$$X^{(i)}_{(1)||j||k} = \frac{\partial X^{(i)}_{(1)||j}}{\partial x^k} + X^{(m)}_{(1)||j}\gamma^i_{mk} - X^{(i)}_{(1)||m}\gamma^m_{jk}.$$

(iv) The effective adapted components of the **curvature** *d-tensor* $\mathbf{R}^{ODEs}_{\mathcal{P}}$ *of the canonical generalized Cartan connection* $C\Gamma^{ODEs}_{\mathcal{P}}$ **produced by the jet first order ODEs system (2.1) and the pair of Riemannian metrics** $\mathcal{P}$ *are only* $R^l_{ijk} = r^l_{ijk}.$

(v) The **geometric electromagnetic distinguished 2-form produced by the jet first order ODEs system (2.1) and the pair of Riemannian metrics** $\mathcal{P}$ *has the expression*

$$F^{ODEs}_{\mathcal{P}} = F^{(1)}_{(i)j}\delta x^i_1 \wedge dx^j,$$

where

$$\delta x^i_1 = dx^i_1 + M^{(i)}_{(1)1}dt + N^{(i)}_{(1)k}dx^k$$

and, if $h^{11} = 1/h_{11}$, *then*

$$F^{(1)}_{(i)j} = \frac{h^{11}}{2}\left[\varphi_{im}X^{(m)}_{(1)||j} - \varphi_{jm}X^{(m)}_{(1)||i}\right].$$

(vi) The adapted components of the electromagnetic d-form $F_{\mathcal{P}}^{ODEs}$ produced by the jet first order ODEs system (2.1) and the pair of Riemannian metrics $\mathcal{P}$ verify the **generalized Maxwell equations**

$$\begin{cases} F_{(i)j//1}^{(1)} = \dfrac{1}{4}\mathcal{A}_{\{i,j\}}\left\{h^{11}\varphi_{im}\left[X_{(1)||j//1}^{(m)} - \varphi^{mr}X_{(1)||r//1}^{(s)}\varphi_{sj}\right]\right\} \\ \sum_{\{i,j,k\}} F_{(i)j||k}^{(1)} = 0, \end{cases}$$

where $\mathcal{A}_{\{i,j\}}$ represents an alternate sum, $\sum_{\{i,j,k\}}$ means a cyclic sum and

$$F_{(i)j//1}^{(1)} = \frac{\partial F_{(i)j}^{(1)}}{\partial t} + F_{(i)j}^{(1)}H_{11}^1 \ \text{and} \ F_{(i)j||k}^{(1)} = \frac{\partial F_{(i)j}^{(1)}}{\partial x^k} - F_{(m)j}^{(1)}\gamma_{ik}^m - F_{(i)m}^{(1)}\gamma_{jk}^m$$

have the geometrical meaning of the horizontal local covariant derivatives ''$_{//1}$'' *and* ''$_{||k}$'' *produced by the Berwald linear connection $B\Gamma_0$ on $J^1(T,\mathbb{R}^n)$. For more details, please consult [4].*

(vii) The **geometric jet Yang-Mills energy produced by the jet first order ODEs system (2.1) and the pair of Riemannian metrics $\mathcal{P}$** *is defined by the formula*

$$EYM_{\mathcal{P}}^{ODEs}(t,x) = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left[F_{(i)j}^{(1)}\right]^2.$$

Now, let us consider on $T \times \mathbb{R}^n$ the particular pair of Euclidian metrics

$$\Delta = (h_{11}(t) = 1, \varphi_{ij}(x) = \delta_{ij}),$$

where δ_{ij} are the Kronecker symbols. Then we obtain the particular jet least squares Lagrangian function

$$JLS_{\Delta}^{\text{ODEs}} : J^1(T,\mathbb{R}^n) \to \mathbb{R}_+,$$

defined by

$$\begin{aligned} JLS_{\Delta}^{\text{ODEs}}(t,x^k,x_1^k) &= \delta_{ij}\left[x_1^i - X_{(1)}^{(i)}(t,x)\right]\left[x_1^j - X_{(1)}^{(j)}(t,x)\right] = \\ &= \sum_{i=1}^{n}\left[x_1^i - X_{(1)}^{(i)}(t,x)\right]^2. \end{aligned}$$

In this new context, we introduce the following concept:

Definition 2.6. *Any geometrical object on $J^1(T,\mathbb{R}^n)$, which is produced by the jet least squares Lagrangian function JLS_{Δ}^{ODEs}, via its attached second order Euler-Lagrange equations, is called* **geometrical object produced by the jet first order ODEs system (2.1)**.

As a consequence, particularizing the Theorem 2.5 for the pair of Euclidian metrics $\mathcal{P} = \Delta$ and taking into account that we have $H_{11}^1(t) = 0$ and $\gamma_{ij}^k(x) = 0$, we immediately get the following jet geometrical result:

Corollary 2.7. *(i) The* **canonical non-linear connection on $J^1(T,\mathbb{R}^n)$ produced by the jet first order ODEs system (2.1)** *has the local components*

$$\Gamma_{\Delta}^{ODEs} = \left(\bar{M}_{(1)1}^{(i)}, \bar{N}_{(1)j}^{(i)}\right),$$

where

$$\bar{M}^{(i)}_{(1)1} = 0 \ and \ \bar{N}^{(i)}_{(1)j} = -\frac{1}{2}\left[\frac{\partial X^{(i)}_{(1)}}{\partial x^j} - \frac{\partial X^{(j)}_{(1)}}{\partial x^i}\right], \ \forall \ i,j = \overline{1,n}.$$

(ii) *All adapted components of the* **canonical generalized Cartan connection** $C\Gamma^{ODEs}_\Delta$ *produced by the jet first order ODEs system* (2.1) *vanish.*

(iii) *The effective adapted components of the* **torsion** *d-tensor* $\mathbf{T}^{ODEs}_\Delta$ *of the canonical generalized Cartan connection* $C\Gamma^{ODEs}_\Delta$ **produced by the jet first order ODEs system** (2.1) *are*

$$\bar{R}^{(i)}_{(1)1j} = \frac{1}{2}\left[\frac{\partial^2 X^{(i)}_{(1)}}{\partial t \partial x^j} - \frac{\partial^2 X^{(j)}_{(1)}}{\partial t \partial x^i}\right], \ \forall \ i,j = \overline{1,n},$$

and

$$\bar{R}^{(i)}_{(1)jk} = -\frac{1}{2}\left[\frac{\partial^2 X^{(i)}_{(1)}}{\partial x^k \partial x^j} - \frac{\partial^2 X^{(j)}_{(1)}}{\partial x^k \partial x^i}\right], \ \forall \ i,j,k = \overline{1,n}.$$

(iv) *All adapted components of the* **curvature** *d-tensor* $\mathbf{R}^{ODEs}_\Delta$ *of the canonical generalized Cartan connection* $C\Gamma^{ODEs}_\Delta$ **produced by the jet first order DEs system** (2.1) *vanish.*

(v) *The* **geometric electromagnetic distinguished 2-form produced by the jet first order ODEs system** (2.1) *has the form*

$$F^{ODEs}_\Delta = \bar{F}^{(1)}_{(i)j}\delta x^i_1 \wedge dx^j,$$

where

$$\delta x^i_1 = dx^i_1 + \bar{N}^{(i)}_{(1)k}dx^k, \ \forall \ i = \overline{1,n},$$

and

$$\bar{F}^{(1)}_{(i)j} = \frac{1}{2}\left[\frac{\partial X^{(i)}_{(1)}}{\partial x^j} - \frac{\partial X^{(j)}_{(1)}}{\partial x^i}\right], \ \forall \ i,j = \overline{1,n}.$$

(vi) *The adapted components* $\bar{F}^{(1)}_{(i)j}$ *of the electromagnetic d-form* F^{ODEs}_Δ *produced by the jet first order ODEs system* (2.1) *verify the* **generalized Maxwell equations**

$$\begin{cases} \bar{F}^{(1)}_{(i)j//1} = \frac{1}{4}\mathcal{A}_{\{i,j\}}\left[\frac{\partial^2 X^{(i)}_{(1)}}{\partial t \partial x^j} - \frac{\partial^2 X^{(j)}_{(1)}}{\partial t \partial x^i}\right] = \frac{1}{2}\left[\frac{\partial^2 X^{(i)}_{(1)}}{\partial t \partial x^j} - \frac{\partial^2 X^{(j)}_{(1)}}{\partial t \partial x^i}\right] \\ \sum_{\{i,j,k\}} \bar{F}^{(1)}_{(i)j||k} = 0, \end{cases}$$

where $\mathcal{A}_{\{i,j\}}$ *represents an alternate sum,* $\sum_{\{i,j,k\}}$ *means a cyclic sum and*

$$\bar{F}^{(1)}_{(i)j//1} = \frac{\partial \bar{F}^{(1)}_{(i)j}}{\partial t} \ and \ \bar{F}^{(1)}_{(i)j||k} = \frac{\partial \bar{F}^{(1)}_{(i)j}}{\partial x^k}, \ \forall \ i,j,k = \overline{1,n}.$$

(vii) The **geometric jet Yang-Mills energy produced by the jet first order ODEs system** (2.1) *has the expression*

$$EYM_{\Delta}^{ODEs}(t,x) = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left[\bar{F}_{(i)j}^{(1)}\right]^2 = \frac{1}{4}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left[\frac{\partial X_{(1)}^{(i)}}{\partial x^j} - \frac{\partial X_{(1)}^{(j)}}{\partial x^i}\right]^2.$$

Remark 2.8. *If we use the matrix notations*

- $J\left(X_{(1)}\right) = \left(\dfrac{\partial X_{(1)}^{(i)}}{\partial x^j}\right)_{i,j=\overline{1,n}}$ *- the* **Jacobian matrix,**

- $\bar{N}_{(1)} = \left(\bar{N}_{(1)j}^{(i)}\right)_{i,j=\overline{1,n}}$ *- the* **non-linear connection matrix,**

- $\bar{R}_{(1)1} = \left(\bar{R}_{(1)1j}^{(i)}\right)_{i,j=\overline{1,n}}$, *- the* **temporal torsion matrix,**

- $\bar{R}_{(1)k} = \left(\bar{R}_{(1)jk}^{(i)}\right)_{i,j=\overline{1,n}}$, $\forall\, k=\overline{1,n}$, *- the* **spatial torsion matrices,**

- $\bar{F}^{(1)} = \left(\bar{F}_{(i)j}^{(1)}\right)_{i,j=\overline{1,n}}$ *- the* **electromagnetic matrix,**

then the following matrix geometrical relations attached to the jet first order ODEs system (2.1) hold good:

1. $\bar{N}_{(1)} = -\dfrac{1}{2}\left[J\left(X_{(1)}\right) - {}^{T}J\left(X_{(1)}\right)\right]$;

2. $\bar{R}_{(1)1} = -\dfrac{\partial}{\partial t}\left[\bar{N}_{(1)}\right]$;

3. $\bar{R}_{(1)k} = \dfrac{\partial}{\partial x^k}\left[\bar{N}_{(1)}\right]$, $\forall\, k=\overline{1,n}$;

4. $\bar{F}^{(1)} = -\bar{N}_{(1)}$;

5. $EYM_{\Delta}^{ODEs}(t,x) = \dfrac{1}{2}\cdot Trace\left[\bar{F}^{(1)}\cdot {}^{T}\bar{F}^{(1)}\right]$, *that is the jet electromagnetic Yang-Mills energy coincides with the square of the norm of the skew-symmetric electromagnetic matrix* $\bar{F}^{(1)}$ *in the Lie algebra* $o(n) = L(O(n))$.

Remark 2.9. *Note that the spatial torsion matrix* $\bar{R}_{(1)k}$ *does not coincide for* $k=1$ *with the temporal torsion matrix* $\bar{R}_{(1)1}$. *We have only an overlap of notations.*

3. Jet Riemann-Lagrange geometry produced by a non-homogenous linear ODEs system of order one

In this Section we apply the preceding jet Riemann-Lagrange geometrical results for a non-homogenous linear ODEs system of order one. In this way,

let us consider the following non-homogenous linear first order ODEs system locally described, in a convenient chart on $J^1(T, \mathbb{R}^n)$, by the differential equations

$$\frac{dx^i}{dt} = \sum_{k=1}^{n} a_{(1)k}^{(i)}(t)x^k + f_{(1)}^{(i)}(t), \ \forall \ i = \overline{1, n}, \tag{3.1}$$

where the local components $a_{(1)k}^{(i)}$ and $f_{(1)}^{(i)}$ transform after the tensorial rules

$$a_{(1)k}^{(i)} = \frac{\partial x^i}{\partial \widetilde{x}^j} \frac{d\widetilde{t}}{dt} \cdot \widetilde{a}_{(1)k}^{(j)}, \ \forall \ k = \overline{1, n},$$

and

$$f_{(1)}^{(i)} = \frac{\partial x^i}{\partial \widetilde{x}^j} \frac{d\widetilde{t}}{dt} \cdot \widetilde{f}_{(1)}^{(j)}.$$

Remark 3.1. *We suppose that the product manifold $T \times \mathbb{R}^n \subset J^1(T, \mathbb{R}^n)$ is endowed **a priori** with the pair of Euclidian metrics $\Delta = (1, \delta_{ij})$, with respect to the coordinates (t, x^i).*

It is obvious that the non-homogenous linear ODEs system (3.1) is a particular case of the jet first order non-linear ODEs system (2.1) for

$$X_{(1)}^{(i)}(t, x) = \sum_{k=1}^{n} a_{(1)k}^{(i)}(t)x^k + f_{(1)}^{(i)}(t), \ \forall \ i = \overline{1, n}. \tag{3.2}$$

In order to expose the main jet Riemann-Lagrange geometrical objects that characterize the non-homogenous linear ODEs system (3.1), we use the matrix notation

$$A_{(1)} = \left(a_{(1)j}^{(i)}(t) \right)_{i,j=\overline{1,n}}.$$

In this context, applying our preceding jet geometrical Riemann-Lagrange theory to the non-homogenous linear ODEs system (3.1) and the pair of Euclidian metrics $\Delta = (1, \delta_{ij})$, we get:

Theorem 3.2. *(i) The **canonical non-linear connection on $J^1(T, \mathbb{R}^n)$** produced by the non-homogenous linear ODEs system (3.1) has the local components*

$$\hat{\Gamma} = \left(0, \hat{N}_{(1)j}^{(i)} \right),$$

where $\hat{N}_{(1)j}^{(i)}$ are the entries of the matrix

$$\hat{N}_{(1)} = \left(\hat{N}_{(1)j}^{(i)} \right)_{i,j=\overline{1,n}} = -\frac{1}{2} \left[A_{(1)} - {}^T A_{(1)} \right].$$

*(ii) All adapted components of the **canonical generalized Cartan connection** $C\hat{\Gamma}$ produced by the non-homogenous linear ODEs system (3.1) vanish.*

*(iii) The effective adapted components $\hat{R}_{(1)1j}^{(i)}$ of the **torsion** d-tensor $\hat{\mathbf{T}}$ of the canonical generalized Cartan connection $C\hat{\Gamma}$ produced by the non-homogenous linear ODEs system (3.1) are the entries of the matrices*

$$\hat{R}_{(1)1} = \left(\hat{R}_{(1)1j}^{(i)} \right)_{i,j=\overline{1,n}} = \frac{1}{2} \left[\dot{A}_{(1)} - {}^T \dot{A}_{(1)} \right],$$

where

$$\dot{A}_{(1)} = \frac{d}{dt}\left[A_{(1)}\right].$$

(iv) All adapted components of the **curvature** *d-tensor* $\hat{\mathbf{R}}$ *of the canonical generalized Cartan connection* $C\hat{\Gamma}$ **produced by the non-homogenous linear ODEs system** (3.1) *vanish.*

(v) The **geometric electromagnetic distinguished 2-form produced by the non-homogenous linear ODEs system** (3.1) *is given by*

$$\hat{F} = \hat{F}^{(1)}_{(i)j}\delta x^i_1 \wedge dx^j,$$

where

$$\delta x^i_1 = dx^i_1 - \frac{1}{2}\left[a^{(i)}_{(1)k} - a^{(k)}_{(1)i}\right]dx^k, \ \forall \ i = \overline{1,n},$$

and the adapted components $\hat{F}^{(1)}_{(i)j}$ *are the entries of the matrix*

$$\hat{F}^{(1)} = \left(\hat{F}^{(1)}_{(i)j}\right)_{i,j=\overline{1,n}} = -\hat{N}_{(1)} = \frac{1}{2}\left[A_{(1)} - {}^T A_{(1)}\right],$$

that is

$$\hat{F}^{(1)}_{(i)j} = \frac{1}{2}\left[a^{(i)}_{(1)j} - a^{(j)}_{(1)i}\right].$$

(vi) The **jet Yang-Mills energy produced by the non-homogenous linear ODEs system** (3.1) *is given by the formula*

$$EYM^{NHLODEs}(t) = \frac{1}{4}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left[a^{(i)}_{(1)j} - a^{(j)}_{(1)i}\right]^2.$$

Proof. Using the relations (3.2), we easily deduce that we have the Jacobian matrix

$$J\left(X_{(1)}\right) = A_{(1)}.$$

Consequently, applying the Corollary 2.7 to the non-homogenous linear ODEs system (3.1), together with the Remark 2.8, we obtain the required results. $\qquad\square$

Remark 3.3. *The entire jet Riemann-Lagrange geometry produced by the non-homogenous linear ODEs system (3.1) does not depend on the non-homogeneity terms* $f^{(i)}_{(1)}(t)$.

Remark 3.4. *The jet Yang-Mills energy produced by the non-homogenous linear ODEs system (3.1) vanishes if and only if the matrix* $A_{(1)}$ *is a symmetric one. In this case, the entire jet Riemann-Lagrange geometry produced by the non-homogenous linear ODEs system (3.1) vanish, so it does not offer geometrical informations about the system (3.1). However, it is important to note that in this particular situation we have the symetry of the matrix* $A_{(1)}$, *which implies that the matrix* $A_{(1)}$ *is diagonalizable.*

Remark 3.5. *All torsion adapted components of a non-homogenous linear ODEs system with constant coefficients* $a^{(i)}_{(1)j}$ *are zero.*

4. Jet Riemann-Lagrange geometry produced by a superior order ODE

Let us consider the superior order ODE expressed by

$$y^{(n)}(t) = f(t, y(t), y'(t), ..., y^{(n-1)}(t)), \ n \geq 2, \tag{4.1}$$

where $y(t)$ is an unknown function, $y^{(k)}(t)$ is the derivative of order k of the unknown function $y(t)$ for each $k \in \{0, 1, ..., n\}$ and f is a given differentiable function depending on the distinct variables t, $y(t)$, $y'(t)$, ..., $y^{(n-1)}(t)$.

It is well known the fact that, using the notations

$$x^1 = y, \ x^2 = y', \ ..., \ x^n = y^{(n-1)},$$

the superior order ODE (4.1) is equivalent with the non-linear ODEs system of order one

$$\begin{cases} \dfrac{dx^1}{dt} = x^2 \\[2mm] \dfrac{dx^2}{dt} = x^3 \\[1mm] \quad . \\ \quad . \\ \quad . \\ \dfrac{dx^{n-1}}{dt} = x^n \\[2mm] \dfrac{dx^n}{dt} = f(t, x^1, x^2, ..., x^n). \end{cases} \tag{4.2}$$

But, the first order non-linear ODEs system (4.2) can be regarded, in a convenient local chart, as a particular case of the jet non-linear ODEs system of order one (2.1), taking

$$X_{(1)}^{(1)}(t, x) = x^2, \quad X_{(1)}^{(2)}(t, x) = x^3, \qquad \cdots$$
$$\cdots \qquad X_{(1)}^{(n-1)}(t, x) = x^n, \quad X_{(1)}^{(n)}(t, x) = f(t, x^1, x^2, ..., x^n), \tag{4.3}$$

where we suppose that the geometrical object $X = \left(X_{(1)}^{(i)}(t, x) \right)$ behaves like a d-tensor on $J^1(T, \mathbb{R}^n)$.

Remark 4.1. *We assume that the product manifold $T \times \mathbb{R}^n \subset J^1(T, \mathbb{R}^n)$ is endowed* **a priori** *with the pair of Euclidian metrics $\Delta = (1, \delta_{ij})$, with respect to the coordinates (t, x^i).*

Definition 4.2. *Any geometrical object on $J^1(T, \mathbb{R}^n)$, which is produced by the* **first order non-linear ODEs system** (4.2) *is called* **geometrical object produced by the superior order ODE** (4.1).

In this context, the Riemann-Lagrange geometrical behavior on the 1-jet space $J^1(T, \mathbb{R}^n)$ of the superior order ODE (4.1) is described in the following result:

Theorem 4.3. *(i) The* **canonical non-linear connection on** $J^1(T, \mathbb{R}^n)$ **produced by the superior order ODE** (4.1) *has the local components*

$$\check{\Gamma} = \left(0, \check{N}^{(i)}_{(1)j}\right),$$

where $\check{N}^{(i)}_{(1)j}$ *are the entries of the matrix* $\check{N}_{(1)} = \left(\check{N}^{(i)}_{(1)j}\right)_{i,j=\overline{1,n}} =$

$$= -\frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & \cdot & \cdot & 0 & 0 & -\dfrac{\partial f}{\partial x^1} \\ -1 & 0 & 1 & \cdot & \cdot & 0 & 0 & -\dfrac{\partial f}{\partial x^2} \\ 0 & -1 & 0 & \cdot & \cdot & 0 & 0 & -\dfrac{\partial f}{\partial x^3} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & 0 & 1 & -\dfrac{\partial f}{\partial x^{n-2}} \\ 0 & 0 & 0 & \cdot & \cdot & -1 & 0 & 1 - \dfrac{\partial f}{\partial x^{n-1}} \\ \dfrac{\partial f}{\partial x^1} & \dfrac{\partial f}{\partial x^2} & \dfrac{\partial f}{\partial x^3} & \cdot & \cdot & \dfrac{\partial f}{\partial x^{n-2}} & -1 + \dfrac{\partial f}{\partial x^{n-1}} & 0 \end{pmatrix}.$$

(ii) All adapted components of the **canonical generalized Cartan connection** $C\check{\Gamma}$ *produced by the superior order ODE* (4.1) *vanish.*

(iii) The effective adapted components of the **torsion** *d-tensor* $\check{T}$ *of the canonical generalized Cartan connection* $C\check{\Gamma}$ *produced by the superior order* **ODE** (4.1) *are the entries of the matrices*

$$\check{R}_{(1)1} = \frac{1}{2}\begin{pmatrix} 0 & 0 & \cdot & \cdot & 0 & -\dfrac{\partial^2 f}{\partial t \partial x^1} \\ 0 & 0 & \cdot & \cdot & 0 & -\dfrac{\partial^2 f}{\partial t \partial x^2} \\ \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 0 & -\dfrac{\partial^2 f}{\partial t \partial x^{n-1}} \\ \dfrac{\partial^2 f}{\partial t \partial x^1} & \dfrac{\partial^2 f}{\partial t \partial x^2} & \cdot & \cdot & \dfrac{\partial^2 f}{\partial t \partial x^{n-1}} & 0 \end{pmatrix}$$

and

$$\check{R}_{(1)k} = -\frac{1}{2} \begin{pmatrix} 0 & 0 & \cdot & \cdot & 0 & -\dfrac{\partial^2 f}{\partial x^k \partial x^1} \\ 0 & 0 & \cdot & \cdot & 0 & -\dfrac{\partial^2 f}{\partial x^k \partial x^2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 0 & -\dfrac{\partial^2 f}{\partial x^k \partial x^{n-1}} \\ \dfrac{\partial^2 f}{\partial x^k \partial x^1} & \dfrac{\partial^2 f}{\partial x^k \partial x^2} & \cdot & \cdot & \dfrac{\partial^2 f}{\partial x^k \partial x^{n-1}} & 0 \end{pmatrix},$$

where $k \in \{1, 2, ..., n\}$.

(iv) All adapted components of the **curvature** *d-tensor $\check{\mathbf{R}}$ of the canonical generalized Cartan connection $C\check{\Gamma}$* **produced by the superior order ODE** (4.1) *vanish.*

(v) The **geometric electromagnetic distinguished 2-form produced by the superior order ODE** (4.1) *has the form*

$$\check{F} = \check{F}_{(i)j}^{(1)} \delta x_1^i \wedge dx^j,$$

where

$$\delta x_1^i = dx_1^i + \check{N}_{(1)k}^{(i)} dx^k, \ \forall \ i = \overline{1, n},$$

and the adapted components $\check{F}_{(i)j}^{(1)}$ are the entries of the matrix

$$\check{F}^{(1)} = \left(\check{F}_{(i)j}^{(1)} \right)_{i,j=\overline{1,n}} = -\check{N}_{(1)}.$$

(vi) The **jet geometric Yang-Mills energy produced by the superior order ODE** (4.1) *is given by the formula*

$$EYM^{SODE}(t,x) = \frac{1}{4} \left[n - 1 - 2\frac{\partial f}{\partial x^{n-1}} + \sum_{j=1}^{n-1} \left(\frac{\partial f}{\partial x^j} \right)^2 \right].$$

Proof. By partial derivatives, the relations (4.3) lead to the Jacobian matrix

$$J\left(X_{(1)}\right) = \begin{pmatrix} 0 & 1 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 1 & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & 0 & 1 \\ \dfrac{\partial f}{\partial x^1} & \dfrac{\partial f}{\partial x^2} & \dfrac{\partial f}{\partial x^3} & \cdot & \cdot & \dfrac{\partial f}{\partial x^{n-1}} & \dfrac{\partial f}{\partial x^n} \end{pmatrix}.$$

In conclusion, the Corollary 2.7, together with the Remark 2.8, applied to first order non-linear ODEs system (4.2), give what we were looking for.
$\square$

5. Riemann-Lagrange geometry produced by a non-homogenous linear ODE of superior order

If we consider the non-homogenous linear ODE of order $n \in \mathbb{N}$, $n \geq 2$, expressed by

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + ... + a_{n-1}(t)y' + a_n(t)y = b(t), \qquad (5.1)$$

where $b(t)$ and $a_i(t)$, $\forall\ i = \overline{0,n}$, are given differentiable real functions and $a_0(t) \neq 0$, $\forall\ t \in [a,b]$, then we recover the superior order ODE (4.1) for the particular function

$$f(t,x) = \frac{b(t)}{a_0(t)} - \frac{a_n(t)}{a_0(t)} \cdot x^1 - \frac{a_{n-1}(t)}{a_0(t)} \cdot x^2 - ... - \frac{a_1(t)}{a_0(t)} \cdot x^n, \qquad (5.2)$$

where we recall that we have

$$y = x^1, y' = x^2, ..., y^{(n-1)} = x^n.$$

Consequently, we can derive the jet Riemann-Lagrange geometry attached to the non-homogenous linear superior order ODE (5.1).

Corollary 5.1. *(i) The* **canonical non-linear connection on** $J^1(T, \mathbb{R}^n)$ **produced by the non-homogenous linear superior order ODE** (5.1) *has the local components*

$$\tilde{\Gamma} = \left(0, \tilde{N}^{(i)}_{(1)j}\right),$$

where $\tilde{N}^{(i)}_{(1)j}$ are the entries of the matrix

$$\tilde{N}_{(1)} = \left(\tilde{N}^{(i)}_{(1)j}\right)_{i,j=\overline{1,n}} =$$

$$= -\frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & \cdot & \cdot & 0 & 0 & \dfrac{a_n}{a_0} \\ -1 & 0 & 1 & \cdot & \cdot & 0 & 0 & \dfrac{a_{n-1}}{a_0} \\ 0 & -1 & 0 & \cdot & \cdot & 0 & 0 & \dfrac{a_{n-2}}{a_0} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & 0 & 1 & \dfrac{a_3}{a_0} \\ 0 & 0 & 0 & \cdot & \cdot & -1 & 0 & 1 + \dfrac{a_2}{a_0} \\ -\dfrac{a_n}{a_0} & -\dfrac{a_{n-1}}{a_0} & -\dfrac{a_{n-2}}{a_0} & \cdot & \cdot & -\dfrac{a_3}{a_0} & -1 - \dfrac{a_2}{a_0} & 0 \end{pmatrix}.$$

(ii) All adapted components of the **canonical generalized Cartan connection** $C\tilde{\Gamma}$ *produced by the* **non-homogenous linear superior order ODE** (5.1) *vanish.*

(iii) All adapted components of the **torsion** d-tensor $\tilde{\mathbf{T}}$ of the canonical generalized Cartan connection $C\tilde{\Gamma}$ **produced by the non-homogenous linear superior order ODE** (5.1) are zero, except the temporal components

$$\tilde{R}^{(i)}_{(1)1n} = -\tilde{R}^{(n)}_{(1)1i} = \frac{a'_{n-i+1}a_0 - a_{n-i+1}a'_0}{2a_0^2}, \; \forall \; i = \overline{1, n-1},$$

where we denoted by " ′ " the derivatives of the functions $a_k(t)$.

(iv) All adapted components of the **curvature** d-tensor $\tilde{\mathbf{R}}$ of the canonical generalized Cartan connection $C\tilde{\Gamma}$ **produced by the non-homogenous linear superior order ODE** (5.1) vanish.

(v) The **geometric electromagnetic distinguished 2-form produced by the non-homogenous linear superior order ODE** (5.1) has the expression

$$\tilde{F} = \tilde{F}^{(1)}_{(i)j}\delta x_1^i \wedge dx^j,$$

where

$$\delta x_1^i = dx_1^i + \tilde{N}^{(i)}_{(1)k}dx^k, \; \forall \; i = \overline{1, n},$$

and the adapted components $\tilde{F}^{(1)}_{(i)j}$ are the entries of the matrix

$$\tilde{F}^{(1)} = \left(\tilde{F}^{(1)}_{(i)j} \right)_{i,j=\overline{1,n}} = -\tilde{N}_{(1)}.$$

(vi) The **jet geometric Yang-Mills electromagnetic energy produced by the non-homogenous linear superior order ODE** (5.1) has the form

$$EYM^{NHLSODE}(t) = \frac{1}{4}\left[n - 1 + 2\frac{a_2}{a_0} + \sum_{j=2}^{n} \frac{a_j^2}{a_0^2} \right].$$

Proof. We apply the Theorem 4.3 for the particular function (5.2) and we use the relations

$$\frac{\partial f}{\partial x^j} = -\frac{a_{n-j+1}}{a_0}, \; \forall \; j = \overline{1, n}.$$

□

Remark 5.2. *The entire jet Riemann-Lagrange geometry produced by the non-homogenous linear superior order ODE (5.1) is independent by the term of non-homogeneity b(t). In author's opinion, this fact emphasizes that the most important role in the study of the ODE (5.1) is played by its attached homogenous linear superior order ODE.*

Example 5.3. *The law of motion without friction (**harmonic oscillator**) of a material point of mass m > 0, which is placed on a spring having the constant of elasticity k > 0, is given by the homogenous linear ODE of order two*

$$\frac{d^2y}{dt^2} + \omega^2 y = 0, \tag{5.3}$$

where the coordinate y measures the distance from the mass's equilibrium point and $\omega^2 = k/m$. It follows that we have

$$n = 2, \; a_0(t) = 1, \; a_1(t) = 0 \; and \; a_2(t) = \omega^2,$$

that is the **harmonic oscillator** *second order ODE (*5.3*) provides the* **jet geometric Yang-Mills electromagnetic energy**

$$EYM^{Harmonic\ Oscillator} = \frac{1}{4}\left(1 + \omega^2\right)^2.$$

Open problem. There exists a real physical interpretation for the previous jet geometric Yang-Mills electromagnetic energy attached to the harmonic oscillator?

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