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Vanishing viscosity method for quasilinear variational inequalities

Tünde Zsuzsánna Szász

Abstract. In this paper we first define the notion of viscosity solution for the following partial differential quasilinear variational inequalities involving a subdifferential operator:

$$\frac{\partial u(t,x)}{\partial t} + F(t,x,u(t,x)) \cdot Du(t,x) + f(t,x,u(t,x)) \in \partial\varphi(u(t,x)) \text{ in } \mathcal{O}$$

$t \in [0,T]\,, x \in \mathbb{R}^d$, where $\partial\varphi$ is the subdifferential operator of the proper convex lower semicontinuous function $\varphi : \mathbb{R}^d \to (-\infty, +\infty]$. We prove the existence of a viscosity solution $u : \mathcal{O} \to \mathbb{R}^n$, where $\mathcal{O}$ an open set in $[0,T] \times \mathbb{R}^d$.

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1. Introduction

The viscosity solution was first introduced by M.G. Crandall and P.L. Lions [3] in 1983. These generalized solutions need not be differentiable anywhere, as the only regularity required in the definition is continuity (for example see [4]). M.G. Crandall, L.C. Evans, P.L. Lions in [2] give the existence theorem to use the vanishing viscosity method for the nonlinear scalar partial differential equation of the form $F(y, u(y), Du(y)) = 0$ for $y \in \mathcal{O}$, where $\mathcal{O}$ is an open set from $\mathbb{R}^n$, $F : \mathcal{O} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is continuous. The name viscosity comes from a traditional engineering application where a nonlinear first order PDE is approximated by quasilinear first order equations which are obtained from the initial PDE by adding a regularizing $\epsilon\Delta u_\epsilon$ term, which is called a 'viscosity term', and these approximate equations can be solved by classical or numerical methods and the limit of their solution hopefully solves the initial equation.

L. Maticiuc, E. Pardoux, A. Răşcanu, A. Zălinescu in [6] studied the existence of a viscosity solution of a system of parabolic variational inequalities involving a subdifferential operator. The authors use a stochastic approach in order to prove the existence result (see in [6] pg.6).

The aim of this paper is to give an existence for a viscosity solution $u : \mathcal{O} \to \mathbb{R}^n$, where $\mathcal{O}$ is an open set in $[0, T] \times \mathbb{R}^d$, by the classical vanishing viscosity method for the following partial differential quasilinear variational inequalities involving a subdifferential operator:

$$\frac{\partial u(t,x)}{\partial t} + F(t,x,u(t,x)) \cdot Du(t,x) + f(t,x,u(t,x)) \in \partial\varphi(u(t,x)) \text{ in } \mathcal{O} \tag{1.1}$$

$t \in [0, T], x \in \mathbb{R}^d$, where $\partial\varphi$ is the subdifferential operator of the proper convex lower semicontinuous function $\varphi : \mathbb{R}^d \to (-\infty, +\infty]$. This method can be used just in the quasilinear case .

2. Main results

Throughout this paper $\mathcal{O}$ is an open set in $[0, T] \times \mathbb{R}^d$, where T is a positive number.

We make the following assumptions:

(A.1) the functions

$$F : [0, T] \times \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^d, \quad f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n$$

are continuous.

(A.2) The functions $\varphi : \mathbb{R}^d \to (-\infty, +\infty]$ is proper (i.e. $\varphi \neq +\infty$), convex, lower semicontinuous.

We recall that the subdifferential $\partial\varphi$ is defined by

$$\partial\varphi(u) = \{u^* \in R^n : \langle u^*, v - u\rangle \le \varphi(v) - \varphi(u), \forall v \in R^n\}.$$

It is a common practice to regard sometimes $\partial\varphi$ as a subset of $R^n \times \mathbb{R}^n$ by writing $(u, u^*) \in \partial\varphi(u)$ instead of $u^* \in \partial\varphi(u)$.

We denote by

$$\begin{aligned} \mathrm{Dom}(\varphi) &= \{u \in \mathbb{R}^n : \varphi(u) < +\infty\} \\ \mathrm{Dom}(\partial\varphi) &= \{u \in \mathbb{R}^n : \partial\varphi(u) \neq \emptyset\} \end{aligned}$$

We recall some definitions and results which will be used in the following (see [1] for more details).

Theorem 2.1. *Let $\varphi : \mathbb{R}^d \to (-\infty, +\infty]$ be a convex function. Then, for all $u \in Dom(\varphi)$ and $z \in \mathbb{R}^n$, there exist*

$$\begin{aligned} \varphi'_{-}(u; z) &:= \lim_{t \nearrow 0} \frac{\varphi(u + tz) - \varphi(u)}{t} = \sup_{t<0} \frac{\varphi(u + tz) - \varphi(u)}{t} \\ \varphi'_{+}(u; z) &:= \lim_{t \searrow 0} \frac{\varphi(u + tz) - \varphi(u)}{t} = \inf_{t>0} \frac{\varphi(u + tz) - \varphi(u)}{t}. \end{aligned} \tag{2.1}$$

Moreover, the following hold:

(a) $\varphi'_-(u;z) \le \varphi'_+(u;z)$, $\forall u \in Dom(\varphi)$ *and* $z \in \mathbb{R}^n$,
(b) $\varphi'_-(u;-z) = -\varphi'_+(u;z)$, $\forall u \in Dom(\varphi)$ *and* $z \in \mathbb{R}^n$,
(c) $\varphi'_-(u,\cdot)$ *is superlinear and* $\varphi'_+(u,z)$ *is sublinear,*
(d) *if* u *and* z *are such that there exists* $\delta > 0$ *such that* $u + tz \in Dom(\varphi)$, $\forall t \in (-\delta, +\delta)$, *then* $\varphi'_-(u,z), \varphi'_+(u,z) \in \mathbb{R}$.

If we take $d = 1$, then we know that, in every point $u \in \mathrm{Dom}(\varphi)$,

$$\partial\varphi(u) \;=\; \mathbb{R} \cap \left[\varphi'_-(u), \varphi'_+(u)\right] \tag{2.2}$$

where $\varphi'_-(u)$ and $\varphi'_+(u)$ are respectively, the left and the right derivative of φ at the point u.

The following proposition generalizes the above characterization to the case of $d \ge 1$:

Proposition 2.2. *Let* $\varphi : \mathbb{R}^d \to (-\infty, +\infty]$ *be a proper, convex function and* $u \in Dom(\varphi)$. *The following statements are equivalent:*

(i) $u^\star \in \partial\varphi(u)$;
(ii) $\langle u^*, z\rangle \ge \varphi'_-(u;z), \forall z \in \mathbb{R}^n$;
(iii) $\langle u^*, z\rangle \le \varphi'_+(u;z), \forall z \in \mathbb{R}^n$.

Let us define, for $u \in \overline{\mathrm{Dom}(\varphi)}$ and $z \in \mathbb{R}^n$,

$$\varphi'_\star(u;z) \;=\; \liminf_{\substack{v \to u \\ v \in \mathrm{Dom}(\partial\varphi)}} \varphi'_-(v;z), \qquad \varphi'^{,\star}(u;z) \;=\; \limsup_{\substack{v \to u \\ v \in \mathrm{Dom}(\partial\varphi)}} \varphi'_+(v;z)$$

For $u \in \mathbb{R}^n$, let (with the usual convention $\inf \emptyset = +\infty$)

$$|\partial\varphi|_0(u) \;=\; \inf |\partial\varphi(u)|.$$

If $u \in \mathrm{Dom}(\partial\varphi)$, then there is a unique $u^\star \in \mathbb{R}^n$, denoted $(\partial\varphi)_0(u)$ such that $|\partial\varphi|_0(u) = |(\partial\varphi)_0(u)|$.

Let $u, v \in \mathbb{R}^d$. The notation $u \cdot v$ denotes the euclidean inner product (also known as the dot product) on $\mathbb{R}^d$. We denote by Du the gradient of u, and Δu the Laplace operator of u:

$$Du(x_1, ..., x_d) \;=\; \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \cdots & \frac{\partial u_n}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_2} \\ \vdots & \ddots & \vdots \\ \frac{\partial u_1}{\partial x_d} & \cdots & \frac{\partial u_n}{\partial x_d} \end{pmatrix}$$

$$\Delta u(x_1, ..., x_d) \;=\; (\Delta u_1, \Delta u_2, \ldots, \Delta u_n) \;=\; \left(\sum_{i=1}^{d} \frac{\partial^2 u_1}{\partial x_i^2}, \ldots, \sum_{i=1}^{d} \frac{\partial^2 u_n}{\partial x_i^2}\right)$$

We may now define the concept of viscosity solution of (1.1):

Definition 2.3. *Let* $u : \mathcal{O} \to \mathbb{R}^n$ *be a continuous function. We say the function* u *is a viscosity solution of (*1.1*), if:*

$$u(t,x) \in Dom(\partial\varphi), \;\; \forall (t,x) \in \;\; \mathcal{O}$$

and for all $\Psi : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ *continuous function, and* $z \in \mathbb{R}^n$*,*

$$\text{if } u \cdot z - \Psi \text{ attains a local maximum at } (t_0, x_0) \in \mathcal{O}, \quad \text{then}$$

we have

$$\frac{\partial \Psi(t_0, x_0)}{\partial t} + F(t_0, x_0, u(t_0, x_0)) \cdot D\Psi(t_0, x_0) + \\ + f(t_0, x_0, u(t_0, x_0)) \cdot z \le \varphi^{`,*}(u(t_0, x_0); z) \tag{2.3}$$

Remark 2.4. *Observe that the Definition* 2.3 *is the particular case of the definition given in (*[6]*) for the quasilinear case.*

The main result is the following:

Theorem 2.5. *Let* $\epsilon > 0$*, and* $F_\epsilon : [0,T] \times \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^d$*,* $f_\epsilon : [0,T] \times \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n$ *be a family of continuous functions such that* $F_\epsilon(t,x,p)$*,* $f_\epsilon(t,x,p)$ *converges uniformly on compact subsets of* $\mathcal{O} \times \mathbb{R}^n$ *to some function* $F(t,x,p)$ *and* $f(t,x,p)$*, as* ϵ *tends to* 0*. Finally, suppose that for all* $\epsilon > 0$ $u_\epsilon \in C^2(\mathcal{O})$ *is a solution of*

$$\frac{\partial u_\epsilon(t,x)}{\partial t} - \epsilon \Delta u_\epsilon(t,x) \\ + F_\epsilon(t,x,u_\epsilon(t,x)) \cdot Du_\epsilon(t,x) + f_\epsilon(t,x,u_\epsilon(t,x)) \in \partial\varphi(u_\epsilon(t,x)) \text{ in } \mathcal{O}. \tag{2.4}$$

Then if u_ϵ *converge uniformly on compact subsets of* $\mathcal{O}$ *to some* $u \in C(\mathcal{O})$*, we have*

$$u \text{ is a viscosity solution of (1.1).}$$

Remark 2.6. *By the Proposition* 2.2 *the inequation (*2.4*) can be written in the form*

$$\left(u'_{\epsilon_t} \cdot z\right)(t,x) - \epsilon \Delta u_\epsilon(t,x) \cdot z + F_\epsilon(t,x,u_\epsilon(t,x)) \cdot Du_\epsilon(t,x) \cdot z + \\ + + f_\epsilon(t,x,u_\epsilon(t,x)) \cdot z \le \varphi'_+(u_\epsilon(t,x); z) \text{ in } \mathcal{O}, \text{ for all } z \in \mathbb{R}^n \tag{2.5}$$

Proof. Let us check (2.3) first for $\Psi \in C^2(\mathcal{O})$. We assume that $\forall z \in R^n$, $u \cdot z - \Psi$ has a local maximum point at $(t_0, x_0) \in \mathcal{O}$.

Choose $\xi \in C^\infty(\mathcal{O})$, such that

$$0 \le \xi < 1, \text{ if } (t,x) \ne (t_0, x_0), \text{ and } \xi(t_0, x_0) = 1.$$

Obviously, $u \cdot z - (\Psi - \xi)$ has a strict local minimum point at $(t_0, x_0) \in \mathcal{O}$, and thus for ϵ small enough, $u_\epsilon \cdot z - (\Psi - \xi)$ has a local maximum point at some $(t_\epsilon, x_\epsilon) \in \mathcal{O}$, and $(t_\epsilon, x_\epsilon) \to (t_0, x_0)$ as $\epsilon \to 0$.

But at the point $(t_\epsilon, x_\epsilon) = (t_0, x_0)$, we have

$$D(u_\epsilon \cdot z - (\Psi - \xi))(t_\epsilon, x_\epsilon) = 0$$

$$\left(u'_{\epsilon_t} \cdot z\right)(t_\epsilon, x_\epsilon) = \Psi'_t(t_\epsilon, x_\epsilon) - \xi'_t(t_\epsilon, x_\epsilon) \tag{2.6}$$

$$(D_x u_\epsilon \cdot z)(t_\epsilon, x_\epsilon) = D_x\Psi(t_\epsilon, x_\epsilon) - D_x\xi(t_\epsilon, x_\epsilon) \tag{2.7}$$

By taking (2.6) and (2.7) in (2.4) we have

$$\Psi_t'(t_\epsilon, x_\epsilon) - \xi_t'(t_\epsilon, x_\epsilon) - \epsilon\Delta u_\epsilon(t_\epsilon, x_\epsilon)\cdot z$$
$$+F_\epsilon(t_\epsilon, x_\epsilon, u_\epsilon(t_\epsilon, x_\epsilon))\cdot(D_x\Psi(t_\epsilon, x_\epsilon) - D_x\xi(t_\epsilon, x_\epsilon))$$
$$+f_\epsilon(t_\epsilon, x_\epsilon, u_\epsilon(t_\epsilon, x_\epsilon))\cdot z \le \varphi_+'(u_\epsilon(t_\epsilon, x_\epsilon); z) \text{ in } \mathcal{O}, \quad \text{for all } z\in\mathbb{R}^n \tag{2.8}$$

Since, as $\epsilon \to 0$

$$u_\epsilon(t_\epsilon, x_\epsilon) \quad\to\quad u(t_0, x_0),$$
$$D\left(u_\epsilon\cdot z\right)(t_\epsilon, x_\epsilon) = D\left(\Psi - \xi\right)(t_\epsilon, x_\epsilon) \to D\left(\Psi - \xi\right)(t_0 x_0) = D\Psi(t_0 x_0)$$
$$\epsilon\Delta u_\epsilon(t_\epsilon, x_\epsilon)\cdot z \le \epsilon\Delta\left(\Psi - \xi\right)(t_\epsilon, x_\epsilon) \to 0$$

and F, f are continuous functions, φ is lower semicontinuous, we have

$$\frac{\partial\Psi(t_0, x_0)}{\partial t} + F(t_0, x_0, u(t_0, x_0))\cdot D\Psi(t_0, x_0)+$$
$$+ f(t_0, x_0, u(t_0, x_0))\cdot z \le \varphi^{`,*}(u(t_0, x_0); z) \tag{2.9}$$

However, we have to show this for test functions from $C^1(\mathcal{O})$. Let $\Psi\in C^1(\mathcal{O})$, and assume that $\forall z\in R^n$, $u\cdot z - \Psi$ has a local maximum point at $(t_0, x_0)\in\mathcal{O}$.

We have to show that

$$\frac{\partial\Psi(t_0, x_0)}{\partial t} + F(t_0, x_0, u(t_0, x_0))\cdot D\Psi(t_0, x_0) + f(t_0, x_0, u(t_0, x_0))\cdot z$$
$$\le \varphi^{`,*}(u(t_0, x_0); z)$$

Let $\Psi_n \in C^1(\mathcal{O})$ such that $\Psi_n \to \Psi$ in $C^1(\mathcal{O})$ and, as before, choose $\xi\in C^\infty(\mathcal{O})$ such that

$$0\le\xi<1, \text{ if } (t,x)\ne(t_0, x_0), \text{ and } \xi(t_0, x_0)=1.$$

For n large enough, $u_\epsilon\cdot z - (\Psi_n - \xi)$ has a local maximum point at some $(t_n, x_n)\in\mathcal{O}$, and $(t_n, x_n)\to(t_0, x_0)$ as $n\to\infty$.

It follows

$$\left(Du\cdot z\right)(t_n, x_n) = D\Psi_n(t_n, x_n) - D\xi(t_n, x_n) \tag{2.10}$$

Then as shown above, for each n we have

$$\frac{\partial\Psi_n(t_n, x_n)}{\partial t} - \frac{\partial\xi(t_n, x_n)}{\partial t} + F(t_n, x_n, u(t_n, x_n))\cdot(D_x\Psi_n(t_n, x_n) - D_x\xi(t_n, x_n))$$
$$+f(t_n, x_n, u(t_n, x_n))\cdot z \le \varphi_+'(u(t_n, x_n); z) \text{ in } \mathcal{O}, \quad \text{for all } z\in\mathbb{R}^n \tag{2.11}$$

Since, as $n\to\infty$

$$u(t_n, x_n) \quad\to\quad u(t_0, x_0),$$
$$D\left(u\cdot z\right)(t_n, x_n) = D\left(\Psi_n - \xi\right)(t_n, x_n) \to D\left(\Psi - \xi\right)(t_0 x_0) = D\Psi(t_0 x_0)$$

and F, f are continuous functions, φ lower semicontinuous, we have

$$\frac{\partial\Psi(t_0, x_0)}{\partial t} + F(t_0, x_0, u(t_0, x_0))\cdot D\Psi(t_0, x_0) + f(t_0, x_0, u(t_0, x_0))\cdot z$$
$$\le \varphi^{`,*}(u(t_0, x_0); z)$$

Therefore u is a viscosity solution of (1.1). □

References

[1] Borwein, J., Lewis, A., *Convex analysis and nonlinear optimization: Theory and Examples*, Springer-Verlag, New-York, 2000.

[2] Crandall, M. G., Evans, L. C., Lions, P. L., *Some properties of viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc., **282**(1984), 487-502.

[3] Crandall, M. G., Lions, P. L., *Viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc., **277**(1983), 1-42.

[4] Crandall, M. G., Ishii, H., Lions, P. L., *User's guide to the viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc., **27**(1992), 1-67.

[5] Lions, P. L., *Generalized solutions of Hamilton-Jacobi equations*, Pitman, London, 1982.

[6] Maticiuc, L., Pardoux, E., Rascanu, A., Zalinescu, A., *Viscosity solutions for systems of parabolic variational inequalities*, preprint, http://arxiv.org/abs/0807.4415, 2008.

Tünde Zsuzsánna Szász
Faculty of Mathematics and Computer Science
Babeş-Bolyai University
Str. Kogălniceanu nr. 1, 400084 Cluj-Napoca, Romania
e-mail: `bodizst@yahoo.com`