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The generalized semi-normed difference of double gai sequence spaces defined by a modulus function

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Abstract. In this paper we introduce generalized semi normed difference of double gai sequence spaces defined by a modulus function. We study their different properties and obtain some inclusion relations involving these semi normed difference double gai sequence spaces.

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1. Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively.

We write w^2 for the set of all complex sequences $(x_{m,n})$, where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces were found in Bromwich [5]. Later on, they were investigated by Hardy [16], Moricz [24], Moricz and Rhoades [25], Basarir and Solankan [3], Tripathy [42], Colak and Turkmenoglu [8], Turkmenoglu [44], and many others.

Let us define the following sets of double sequences

$$\mathcal{M}_u(t) := \left\{ (x_{m,n}) \in w^2 : \sup_{m,n\in N} |x_{m,n}|^{t_{m,n}} < \infty \right\},$$

$$\mathcal{C}_p(t) := \left\{ (x_{m,n}) \in w^2 : p - \lim_{m,n\to\infty} |x_{m,n} - l|^{t_{m,n}} = 1 \text{ for some } l \in \mathbb{C} \right\},$$

$$\mathcal{C}_{0p}(t) := \left\{ (x_{m,n}) \in w^2 : p - \lim_{m,n\to\infty} |x_{m,n}|^{t_{m,n}} = 1 \right\},$$

$$\mathcal{L}_u(t) := \left\{ (x_{m,n}) \in w^2 : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{m,n}|^{t_{m,n}} < \infty \right\},$$
$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t),$$

where $t = (t_{m,n})$ is the sequence of strictly positive reals $t_{m,n}$ for all $m, n \in \mathbb{N}$ and $p - \lim_{m,n \to \infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{m,n} = 1$ for all $m, n \in \mathbb{N}$; $\mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [14,15] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [46] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [27] have recently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [28] and Mursaleen and Edely [29] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the $M-$core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{j,k})$ into one whose core is a subset of the $M-$core of x. More recently, Altay and Basar [2] have defined the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also have examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta(\vartheta)-$ duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Quite recently Basar and Sever [7] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and have examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [36,40] have studied the space $\chi^2_M(p, q, u)$ and the generalized gai of double sequences and have given some inclusion relations.

We need the following inequality in the sequel of the paper. For $a, b \geq 0$ and $0 < p < 1$, we have

$$(a + b)^p \leq a^p + b^p. \tag{1.1}$$

The double series $\sum_{m,n=1}^{\infty} x_{m,n}$ is called convergent if and only if the double sequence $(s_{m,n})$ is convergent, where $s_{m,n} = \sum_{i,j=1}^{m,n} x_{i,j} (m, n \in \mathbb{N})$ (see[1]).

A sequence $x = (x_{m,n})$ is said to be double analytic if

$$\sup_{m,n} |x_{m,n}|^{1/(m+n)} < \infty.$$

The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{m,n})$ is called double entire sequence if $|x_{m,n}|^{1/(m+n)} \to 0$ as $m, n \to \infty$. The double entire sequences will be denoted by Γ^2. A sequence $x = (x_{m,n})$ is called double gai sequence if $((m + n)! |x_{m,n}|)^{1/(m+n)} \to 0$ as

$m,n \to \infty$. The double gai sequences will be denoted by χ^2. Let ϕ denote the set of all finite sequences.

Consider a double sequence $x = (x_{i,j})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{i,j}\Im_{i,j}$ for all $m,n \in \mathbb{N}$, where $\Im_{ij}$ denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i,j \in \mathbb{N}$.

An FK-space (or a metric space) X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{m,n})$ $(m,n \in \mathbb{N})$ are also continuous.

Orlicz [32] used the idea of Orlicz function to construct the space $\left(L^M\right)$. Lindenstrauss and Tzafriri [21] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p\,(1 \le p < \infty)$. subsequently, different classes of sequence spaces were defined by Parashar and Choudhary [33], Mursaleen et al. [26], Bektas and Altin [4], Tripathy et al. [43], Rao and Subramanian [9], and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [17].

Recalling [32] and [17], an Orlicz function is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, non-decreasing, and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by subadditivity of M, then this function is called modulus function, defined by Nakano [31] and further discussed by Ruckle [34] and Maddox [23], and many others.

An Orlicz function M is said to satisfy the Δ_2- condition for all values of u if there exists a constant $K > 0$ such that $M(2u) \le KM(u)\,(u \ge 0)$. The Δ_2- condition is equivalent to $M(\ell u) \le K\ell M(u)$, for all values of u and for $\ell > 1$.

Lindenstrauss and Tzafriri [21] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty,\, for\, some\, \rho > 0\right\}.$$

The space ℓ_M with the norm

$$\|x\| = inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1\right\}$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p\,(1 \le p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.

If X is a sequence space, we give the following definitions

(i) $X^{'}=$ the continuous dual of X;

(ii) $X^{\alpha} = \left\{a = (a_{m,n}) : \sum_{m,n=1}^{\infty} |a_{m,n}x_{m,n}| < \infty, \text{ for each } x \in X\right\}$;

(iii) $X^{\beta} = \left\{a = (a_{m,n}) : \sum_{m,n=1}^{\infty} a_{m,n}x_{m,n} \text{ is convergent, for each } x \in X\right\}$;

(iv) $X^{\gamma} = \left\{a = (a_{m,n}) : \sup_{m,n} \ge 1 \left|\sum_{m,n=1}^{M,N} a_{m,n}x_{m,n}\right| < \infty, \text{ for each } x \in X\right\}$;

(v) let X be an FK-space $\supset \phi$; then $X^f = \left\{ f(\Im_{m,n}) : f \in X' \right\}$;

(vi) $X^\delta = \left\{ a = (a_{m,n}) : \sup_{m,n} |a_{m,n} x_{m,n}|^{1/(m+n)} < \infty, \text{ for each } x \in X \right\}$;

X^α, X^β, X^γ are called α- (or Köthe-Toeplitz) dual of X, β- (or generalized-Köthe-Toeplitz) dual of X, γ-dual of X, δ - dual of X respectively. X^α is defined by Gupta and Kamptan [18]. It is clear that $x^\alpha \subset X^\beta$ and $X^\alpha \subset X^\gamma$, but $X^\alpha \subset X^\gamma$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [19] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_o$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$. Here w, c, c_o and ℓ_∞ denote the classes of all, convergent,null and bounded scalar valued single sequences respectively. The above spaces are Banach spaces normed by

$$\|x\| = |x_1| + \sup_{k \geq 1} |\Delta x_k| .$$

The notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{ x = (x_{m,n}) \in w^2 : (\Delta x_{m,n}) \in Z \right\}$$

where $Z = \Lambda^2, \Gamma^2$ and χ^2 respectively.

$$\Delta x_{m,n} = (x_{m,n} - x_{m,n+1}) - (x_{m+1,n} - x_{m+1,n+1})$$

$$= x_{m,n} - x_{m,n+1} - x_{m+1,n} + x_{m+1,n+1}$$

for all $m, n \in \mathbb{N}$.

Let $r \in \mathbb{N}$ be fixed, then

$$Z(\Delta^r) = \{(x_{m,n}) : (\Delta^r x_{m,n}) \in Z\} \; for \, Z = \chi^2, \Gamma^2 \, and \, \Lambda^2$$

where $\Delta^r x_{m,n} = \Delta^{r-1} x_{m,n} - \Delta^{r-1} x_{m,n+1} - \Delta^{r-1} x_{m+1,n} + \Delta^{r-1} x_{m+1,n+1}$.

Now we introduced a generalized difference double operator as follows.

Let $r, \gamma \in \mathbb{N}$ be fixed. Then

$$Z\left(\Delta^r_\gamma\right) = \left\{ (x_{m,n}) : \left(\Delta^r_\gamma x_{m,n}\right) \in Z \right\} \; for \, Z = \chi^2, \Gamma^2 \, and \, \Lambda^2,$$

where $\Delta^r_\gamma x_{m.n} = \Delta^{r-1}_\gamma x_{m,n} - \Delta^{r-1}_\gamma x_{m,n+1} - \Delta^{r-1}_\gamma x_{m+1,n} + \Delta^{r-1}_\gamma x_{m+1,n+1}$ and $\Delta^0_\gamma x_{m,n} = x_{m,n}$ for all $m, n \in \mathbb{N}$.

The notion of a modulus function was introduced by Nakano [31]. We recall that a modulus f is a function from $[0, \infty) \to [0, \infty)$, such that

(1) $f(x) = 0$ if and only if $x = 0$

(2) $f(x+y) \leq f(x) + f(y)$, for all $x \geq 0$, $y \geq 0$,

(3) f is increasing,

(4) f is right-continuous at $x = 0$.

Since $|f(x) - f(y)| \leq f(|x-y|)$, it follows from condition (4) that f is continuous on $[0, \infty)$.

Also from condition (2), we have $f(nx) \leq nf(x)$ for all $n \in \mathbb{N}$ and $n^{-1} f(x) \leq f\left(xn^{-1}\right)$, for all $n \in \mathbb{N}$.

2. Remark

If f is a modulus function, then the composition $f^s = f \cdot f \cdots f$ $(s\, times)$ is also a modulus function, where s is a positive integer.

Let $p = (p_{m,n})$ be a sequence of positive real numbers. We have the following well known inequality, which will be used throughout this paper

$$|a_{m,n} + b_{m,n}|^{p_{m,n}} \leq D \left(|a_{m,n}|^{p_{m,n}} + |b_{m,n}|^{p_{m,n}}\right), \tag{2.1}$$

where $a_{m,n}$ and $b_{m,n}$ are complex numbers, $D = \max\left\{1, 2^{H-1}\right\}$ and $H = \sup_{m,n} p_{m,n} < \infty$.

Spaces of strongly summable sequences were studied at the initial stage by Kuttner [20], Maddox [30] and others. The class of sequences those are strongly Cesàro summable with respect to a modulus was introduced by Maddox [23] as an extension of the definition of strongly Cesàro summable sequences. Cannor [10] further extended this definition to a definition of strongly $A-$ summability with respect to a modulus when A is non-negative regular matrix.

Let $\eta = (\lambda_i)$ be a non-decreasing sequence of positive real numbers tending to infinity and $\lambda_1 = 1$ and $\lambda_{i+1} \leq \lambda_{i+1} + 1$, for all $i \in \mathbb{N}$.

The generalized de la Vallee-Poussin means is defined by $t_i(x) = \lambda_i^{-1} \sum_{k \in I_i} x_k$, where $I_i = [i - \lambda_i + 1, i]$. A sequence $x = (x_k)$ is said to be $(V, \lambda)-$ summable to a number L if $t_i(x) \to L$, as $i \to \infty$ (see [22]).

3. Definitions and preliminaries

Let w^2 denote the set of all complex double sequences. A sequence $x = (x_{m,n})$ is said to be double analytic if $\sup_{m,n} |x_{m,n}|^{1/(m+n)} < \infty$. The vector space of all prime sense double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{m,n})$ is called prime sense double entire sequence if $|x_{m,n}|^{1/(m+n)} \to 0$ as $m, n \to \infty$. The double entire sequences will be denoted by Γ^2. The spaces Λ^2 and Γ^2 are metric spaces with the metric

$$d(x, y) = \sup_{m,n} \left\{ |x_{m,n} - y_{m,n}|^{1/(m+n)} : m, n : 1, 2, 3, ... \right\}, \tag{3.1}$$

for all $x = (x_{m,n})$ and $y = (y_{m,n})$ in Γ^2.

A sequence $x = (x_{m,n})$ is called prime sense double gai sequence if $\left((m+n)!\, |x_{m,n}|\right)^{1/(m+n)} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by χ^2. The space χ^2 is a metric space with the metric

$$\widetilde{d}(x, y) = \sup_{m,n} \left\{ \left((m+n)!\, |x_{m,n} - y_{m,n}|\right)^{1/(m+n)} : m, n : 1, 2, 3, ... \right\}, \tag{3.2}$$

for all $x = (x_{m,n})$ and $y = (y_{m,n})$ in χ^2.

Throughout the article E will represent a semi normed space, semi normed by q. We define $w^2(E)$ to be the vector space of all $E-$ valued sequences. Let f be a modulus function $p = (p_{m,n})$ be any sequence of positive real numbers. Let $A = \left(a^{j,k}_{m,n}\right)$ be four dimensional infinite regular matrix of non-negative complex numbers such that $\sup_{j,k} \sum_{m,n=1}^{\infty} a^{j,k}_{m,n} < \infty$.

We define the following sets of sequences

$$\left[V^E_\lambda, A, \Delta^r_\gamma, f, p\right]_{\chi^2}$$

$$=\left\{x\in w^2(E): \lim_{p,q\to\infty} \lambda^{-1}_{pq} \sum_{m,n\in I_{p,q}} a^{j,k}_{m,n}[f(q((m+n)!\Delta^r_\gamma x_{m,n})^{1/(m+n)})]^{p_{m,n}} = 0\right\}$$

uniformly in m, n,

$$\left[V^E_\lambda, A, \Delta^r_\gamma, f, p\right]_{\Gamma^2}$$

$$= \left\{x \in w^2(E) : \lim_{p,q\to\infty} \lambda^{-1}_{pq} \sum_{m,n\in I_{p,q}} a^{j,k}_{m,n}[f(q(\Delta^r_\gamma x_{m,n})^{1/(m+n)})]^{p_{m,n}} = 0\right\}$$

uniformly in m, n,

$$\left[V^E_\lambda, A, \Delta^r_\gamma, f, p\right]_{\Lambda^2}$$

$$= \left\{x \in w^2(E) : \sup_{j,k} \sup_{p,q} \lambda^{-1}_{pq} \sum_{m,n\in I_{p,q}} a^{j,k}_{m,n}[f(q(\Delta^r_\gamma x_{m,n})^{1/(m+n)})]^{p_{m,n}} < \infty\right\}.$$

For $\gamma = 1$, these spaces are denoted by $\left[V^E_\lambda, A, \Delta^r, f, p\right]_Z$, for $Z = \chi^2$, Γ^2 and Λ^2 respectively. We define

$$\left[V^E_\lambda, A, \Delta^r_\gamma, f, p\right]_{\chi^2}$$

$$= \left\{x \in w^2(E) : \lim_{p,q\to\infty} \lambda^{-1}_{pq} \sum_{m,n\in I_{p,q}} [f(q((m+n)!\Delta^r_\gamma x_{mn})^{1/(m+n)})]^{p_{mn}} = 0\right\}.$$

Similarly $\left[V^E_\lambda, \Delta^r_\gamma, f, p\right]_{\Gamma^2}$ and $\left[V^E_\lambda, \Delta^r_\gamma, f, p\right]_{\Lambda^2}$ can be defined.

For $E = \mathbb{C}$, the set of complex numbers, $q(x) = |x|$; $f(x) = x^{1/(m+n)}$; $p_{m,n} = 1$, for all $m, n \in \mathbb{N}$. For $r = 0, \gamma = 0$ the spaces $\left[V^E_\lambda, \Delta^r_\gamma, f, p\right]_Z$, represent the spaces $[V, \lambda]_Z$, for $Z = \chi^2, \Gamma^2$ and Λ^2. These spaces are called as $\lambda-$ strongly gai to zero, $\lambda-$ strongly entire to zero and $\lambda-$ strongly analytic by the de la Vallée-Poussin method. In the special case, where $\lambda_{pq} = pq$, for all $p, q = 1, 2, 3, \cdots$ the sets $[V, \lambda]_{\chi^2}$, $[V, \lambda]_{\Gamma^2}$ and $[V, \lambda]_{\Lambda^2}$ reduce to the sets $w^2_{\chi^2}$, $w^2_{\Gamma^2}$ and $w^2_{\Lambda^2}$.

4. Main results

Theorem 4.1. *Let the sequence $p = (p_{m,n})$ be bounded. Then the set $\left[V^E_\lambda, A, \Delta^r_\gamma, f, p\right]_Z$ is linear space over the complex field $\mathbb{C}$, for $Z = \chi^2$ and Λ^2.*

The proof is easy, consequently we omit it.

Theorem 4.2. *Let f be a modulus function. One has $\left[V^E_\lambda, A, \Delta^r_\gamma, f, p\right]_{\chi^2} \subset \left[V^E_\lambda, A, \Delta^r_\gamma, f, p\right]_{\Lambda^2}$.*

Proof. Let $x = (x_{m,n}) \in \left[V_\lambda^E, A, \Delta_\gamma^r, f, p\right]_{\chi^2}$ will represent a semi normed space, semi normed by q. Here there exists a positive integer M_1 such that $q \leq M_1$. Then we have

$$\lambda_{p,q}^{-1} \sum_{m,n \in I_{p,q}} a_{m,n}^{j,k} \left[f\left(q\left(\Delta_\gamma^r x_{m,n}\right)^{1/(m+n)}\right)\right]^{p_{m,n}}$$

$$\leq D\lambda_{pq}^{-1} \sum_{m,n \in I_{pq}} a_{m,n}^{j,k} \left[f\left(q\left((m+n)!\Delta_\gamma^r x_{m,n}\right)^{1/(m+n)}\right)\right]^{p_{m,n}}$$

$$+D\left(M_1, f(1)\right)^H \lambda_{p,q}^{-1} \sum_{m,n \in I_{pq}} a_{m,n}^{j,k}.$$

Thus $x \in \left[V_\lambda^E, A, \Delta_\gamma^r, f, p\right]_{\Lambda^2}$. This completes the proof. □

Theorem 4.3. *Let $p = (p_{m,n}) \in \chi^2$, then $\left[V_\lambda^E, A, \Delta_\gamma^r, f, p\right]_{\chi^2}$ is a paranormed space with*

$$g(x) =$$

$$\sup_{p,q}\left(\lambda_{p,q}^{-1} \sum_{m,n \in I_{p,q}} a_{m,n}^{j,k} \left[f\left(q\left((m+n)!\Delta_\gamma^r x_{m,n}\right)^{1/(m+n)}\right)\right]^{p_{m,n}}\right)^{1/H},$$

where $H = \max\left(1, \sup_{m,n} p_{m,n}\right)$.

Proof. From Theorem 4.1, for each $x \in \left[V_\lambda^E, A, \Delta_\gamma^r, f, p\right]_{\chi^2}$, $g(x)$ exists. Clearly $g(-x) = g(x)$. It is trivial that $\left((m+n)!\Delta_\gamma^r x_{m,n}\right)^{1/(m+n)} = \theta$ for $x = \overline{\theta}$. Hence, we get $g\left(\overline{\theta}\right) = 0$. By Minkowski inequality, we have $g(x+y) \leq g(x) + g(y)$. Now we show that the scalar multiplication is continuous. Let α be any fixed complex number. By definition of f, we deduce that $x \to \theta$ implies $g(\alpha x) \to 0$. Similarly, we have x fixed and $\alpha \to 0$ implies $g(\alpha x) \to 0$. Finally $x \to \theta$ and $\alpha \to 0$ implies $g(\alpha x) \to 0$. This completes the proof. □

Theorem 4.4. *If $r \geq 1$, then the inclusion*

$$\left[V_\lambda^E, A, \Delta_\gamma^{r-1}, f, p\right]_{\chi^2} \subset \left[V_\lambda^E, A, \Delta_\gamma^r, f, p\right]_{\chi^2}$$

is strict. In general

$$\left[V_\lambda^E, A, \Delta_\gamma^j, f, p\right]_{\chi^2} \subset \left[V_\lambda^E, A, \Delta_\gamma^r, f, p\right]_{\chi^2}$$

for $j = 0, 1, 2, \cdots r-1$ and the inclusions are strict.

Proof. The result follows from the following inequality

$$\lambda_{pq}^{-1} \sum_{m,n \in I_{pq}} a_{m,n}^{j,k} \left[f\left(q\left((m+n)!\Delta_\gamma^r x_{m,n}\right)^{1/(m+n)}\right)\right]^{p_{m,n}}$$

$$\leq D\,\lambda_{pq}^{-1} \sum_{m,n \in I_{pq}} a_{m,n}^{j,k} \left[f\left(q\left((m+n)!x_{m,n}\right)^{1/(m+n)}\right)\right]^{p_{m,n}}$$

$$+D\,\lambda_{pq}^{-1} \sum_{m,n \in I_{p,q}} a_{m,n}^{j,k} \left[f\left(q\left((m+n+1)!x_{m,n+1}\right)^{1/(m+n+1)}\right)\right]^{p_{m,n}}$$

$$+D\,\lambda_{pq}^{-1}\sum_{m,n\in I_{p,q}} a_{m,n}^{j,k}\left[f\left(q\left((m+1+n)!x_{m+1,n}\right)^{1/(m+1+n)}\right)\right]^{p_{m,n}}$$

$$+D\,\lambda_{p,q}^{-1}\sum_{m,n\in I_{p,q}} a_{m,n}^{j,k}\left[f\left(q\left((m+n+2)!x_{m+1,n+1}\right)^{1/(m+n+2)}\right)\right]^{p_{m,n}}.$$

Proceeding inductively, we have

$$\left[V_\lambda^E, A, \Delta_\gamma^j, f, p\right]_{\chi^2} \subset \left[V_\lambda^E, A, \Delta_\gamma^r, f, p\right]_{\chi^2} \text{ for } j = 0,1,2,\cdots r-1.$$

The inclusion is strict follows from the following example.

Let $E = \mathbb{C}$, $q(x) = |x|$; $\lambda_{pq} = 1$ for all $p, q \in \mathbb{N}$, $p_{m,n} = 2$ for all $m, n \in \mathbb{N}$. Let $f(x) = x$, for all $x \in [0,\infty)$; $a_{m,n}^{j,k} = m^{-2}n^{-2}$ for all $m, n, j, k \in \mathbb{N}$; $\gamma = 1$, $r \geq 1$. Consider the sequence $x = (x_{m,n})$ defined by $x_{m,n} = \frac{1}{(m+n)!}(mn)^{r(m+n)}$ for all $m, n \in \mathbb{N}$. Hence $(x_{m,n}) \in \left[V_\lambda^C, A, \Delta^r, f, p\right]_{\chi^2}$ but $(x_{m,n}) \notin \left[V_\lambda^C, A, \Delta^{r-1}, f, p\right]_{\chi^2}$. □

Theorem 4.5. *Let f be a modulus function and s be a positive integer. Then,*

$$\left[V_\lambda^E, A, \Delta_\gamma^r, f, q\right]_{\Lambda^2} \subset \left[V_\lambda^E, A, \Delta_\gamma^r, f, p\right]_{\Lambda^2}.$$

Proof. Let $\epsilon > 0$ be given and choose δ with $0 < \delta < 1$ such that $f(t) < \epsilon$ for $0 \leq t \leq \delta$. Write $y_{m,n} = f^{s^{-1}}\left(q\left(\Delta_\gamma^r x_{m,n}\right)^{1/(m+n)} - M\right)$ and consider

$$\sum_{m,n\in I_r} a_{m,n}^{j,k}\left[f(y_{m,n})\right]^{p_{m,n}} = \sum_{m,n\in I_r,\, y_{m,n}\leq\delta} a_{m,n}^{j,k}\left[f(y_{m,n})\right]^{p_{m,n}}$$

$$+\sum_{m,n\in I_r} a_{m,n}^{j,k}\left[f(y_{m,n})\right]^{p_{m,n}}.$$

Since f is continuous, we have

$$\sum_{m,n\in I_r,\, y_{m,n}\leq\delta} a_{m,n}^{j,k}\left[f(y_{m,n})\right]^{p_{m,n}} \leq \epsilon^H \sum_{m,n\in I_r,\, y_{m,n}\leq\delta} a_{m,n}^{j,k} \tag{4.1}$$

and for $y_{m,n} > \delta$, we use the fact that, $y_{m,n} < \frac{y_{m,n}}{\delta} \leq 1 + \frac{y_{m,n}}{\delta}$ and so, by the definition of f, we have for $y_{m,n} > \delta$,

$$f(y_{m,n}) < 2f(1)\frac{y_{m,n}}{\delta}.$$

Hence

$$\frac{1}{\lambda_{pq}}\sum_{m,n\in I_r,\, y_{m,n}\leq\delta} a_{m,n}^{j,k}\left[f(y_{m,n})\right]^{p_{m,n}}$$

$$\leq \max\left(1, \left(2f(1)\delta^{-1}\right)^H\right)\frac{1}{\lambda_{pq}}\sum_{m,n\in I_r,\, y_{m,n}\leq\delta} a_{m,n}^{j,k} y_{m,n}^{p_{m,n}}. \tag{4.2}$$

From (4.1) and (4.2) we obtain $\left[V_\lambda^E, A, \Delta_\gamma^r, f, q\right]_{\Lambda^2} \subset \left[V_\lambda^E, A, \Delta_\gamma^r, f, p\right]_{\Lambda^2}$. This completes the proof. □

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