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BOOK REVIEWS

Alexandru Kristály, Vicenţiu Rădulescu and Csaba György Varga, ***Variational Principles in mathematics, Physics, Geometry, and Economics - Qualitative Analysis of Nonlinear Equations and Unilateral Problems,*** xv+368 pp, Encyclopedia of Mathematics and its Applications, vol. 136, Cambridge University Press, 2010, ISBN: 978-0-521-11782-1.

The use of variational principles has a long and fruitful history in mathematics and physics, both in solving problems and shaping theories, and it has been introduced recently in economics. The corresponding literature is enormous and several monographs are already classical. The present book, Variational Principles in Mathematical Physics, Geometry, and Economics, by Kristály, Rădulescu and Varga, is original in several ways.

In Part I, devoted to variational principles in mathematical physics, unavoidable classical topics such as the Ekeland variational principle, the mountain pass lemma, and the Ljusternik-Schnirelmann category, are supplemented with more recent methods and results of Ricceri, Brezis-Nirenberg, Szulkin, and Pohozaev. The chosen applications cover variational inequalities on unbounded strips and for area-type functionals, nonlinear eigenvalue problems for quasilinear elliptic equations, and a substantial study of systems of elliptic partial differential equations. These are challenging topics of growing importance, with many applications in natural and human sciences, such as demography.

Part II demonstrates the importance of variational problems in geometry. Classical questions concerning geodesics or minimal surfaces are not considered, but instead the authors concentrate on a less standard problem, namely the transformation of classical questions related to the Emden-Fowler equation into problems defined on some four-dimensional sphere. The combination of the calculus of variations with group theory provides interesting results. The case of equations with critical exponents, which is of special importance in geometrical problems since Yamabe's work, is also treated.

Part III deals with variational principles in economics. Some choice is also necessary in this area, and the authors first study the minimization of cost-functions on manifolds, giving special attention to the Finslerian-Poincar disc. They then consider best approximation problems on manifolds before approaching Nash equilibria through variational inequalities.

The high level of mathematical sophistication required in all three parts could be an obstacle for potential readers more interested in applications. However, several appendices recall in a precise way the basic concepts and results of convex analysis, functional analysis, topology, and set-valued analysis. Because the present in science depends upon its past and shapes its future, historical and bibliographical notes are complemented by perspectives. Some exercises are proposed as complements to the covered topics.

Among the wide recent literature on critical point theory and its applications, the authors have had to make a selection. Their choice has of course been influenced by their own tastes and contributions. It is a happy one, because of the interest and beauty of selected topics, because of their potential for applications, and because of the fact that most of them have not been covered in existing monographs. Hence I believe that the book by Kristály, Rădulescu, and Varga will be appreciated by all scientists interested in variational methods and in their applications.

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NOTE. This text was published as a Foreword to the above mentioned book. It is reproduced here with the kind permission of the author and with the agreement of Cambridge University Press.

Jaroslav Lukeš, Jan Malý, Ivan Netuka, Jiří Spurný, *Integral Representation Theory - Applications to Convexity, Banach Spaces and Potential Theory*, Walter de Gruyter, Berlin - New York, 2010,715 pages, ISBN: 978-3-11-020320-2.

The Krein-Milman theorem asserts that any compact convex subset X of a locally convex space agrees with the closed convex hull of the set ext X of its extreme points, and if $X =$cl conv(Z), for some $Z \subset X$, then ext $X \subset \overline{Z}$. G. Choquet gave a reformulation of this result, showing that any point of a compact convex set X is the barycenter of a Radon measure supported by the closure of the set of extreme points of the set X, initiating a fruitful area of investigation, known as the Choquet theory. The aim of the present book is to present a more general approach to integral representation theory based on the notion of function space, with applications to the study of convex sets, Banach spaces and potential theory.

The book starts with a chapter headed *Prologue* (there is not an *Epilogue* to the book), containing a proof of Korovkin approximation theorem in the case $C[0,1]$, a more general approach being proposed in Chapter 3. The second chapter, *Compact convex sets*, is devoted to the study of the extremal structure of compact convex sets and of representing measures with support contained in the closure of the set of extreme points. Exposed points, their connection with farthest points (Strasewicz theorem) as well as measure convex and measure extremal sets are also considered.

In Chapter 3, *The Choquet theory of function spaces,* the Choquet theory is developed within the framework of function spaces. A function space if a subspace $\mathcal{H}$ of the space $C(K)$ of all continuous functions on the compact topological space K, containing the constants and separating the points of K. Many examples of concrete function spaces fit in this general scheme - the space $C(K)$ itself, the space of quadratic polynomials in $C[0,1]$, the space of affine functions on K (K a compact convex set), spaces of harmonic functions ($K = \overline{U}$, $U \subset \mathbb{R}^n$ open and bounded), spaces of Markov operators on $C(K)$. The results of this chapter, including Choquet boundary, Choquet ordering, Bauer maximum principle, a.o., are essential for the whole book. The next chapter, 4, *Affine functions on compact convex sets,* presents the classical Choquet theory for spaces of affine functions on compact convex sets. Chapter 5, *Perfect classes of functions and representation of affine functions,* concerned with a hierarchy of Borel sets and functions, is crucial for subsequent applications to descriptive set theory.

Other results are contained in the chapters 6, *Simplicial function spaces* (Choquet and Bauer simplices, the Daugavet property for simplicial spaces are included), 7. *Choquet theory for function cones* (based on a key Lemma containing a Hahn-Banach type extension theorem for measures), 8, *Choquet-like sets,* 9, *Topologies and boundaries*, 10, *Deeper results on function spaces and compact convex sets* (James and Shilov boundaries, isometries of spaces of affine functions, embedding of ℓ^1 in Banach spaces, metrizability of compact convex sets, topological properties of the sets of extreme points), 11, *Continuous and measurable selectors* (Lazar selection theorem and applications, measurable selectors), 12, *Constructions of function spaces* (products and inverse limits of function spaces), 13, *Function spaces in potential theory and the Dirichlet problem* (Bauer and Cornea approaches to harmonic spaces and Dirichlet problem).

The reward for the effort done in reading the book comes in Chapter 14, *Applications*, where a lot of classical representation results are obtained using the tools of the generalized Choquet theory developed in the previous chapters - integral representations for convex and for concave functions (Bauer's results), doubly stochastic matrices (in the finite dimensional case Krein-Milman suffices), the Riesz-Herglotz theorem on the Poisson kernel, typically real holomorphic functions and holomorphic functions with positive real part, Bernstein's representation theorem for completely monotone functions, Lyapunov theorem on the convexity of the range of a nonatomic measure, the Stone-Weierstrass approximation theorem, the existence of invariant and ergodic measures. In all of these cases, more complicated objects are expressed as mixtures (or averages) of simpler ones, which turn to be the extreme points of some appropriate compact convex sets.

Each chapter contains a set of exercises completing the main text. All are accompanied by sufficiently detailed hints, so that the reader can handle them without excessive effort. Historical and bibliographical notes and comments, as well as references to further work, are supplied at the end of each chapter. A consistent Appendix (60 pages) collects the basic notion and results from functional analysis, measure theory, descriptive set theory, partial differential equations and axiomatic potential theory, used in the main body of the book.

The book is clearly written and very well organized - a list of symbols, a notion index and a well documented bibliography counting 479 items, help the reader to navigate through the book and to find further bibliographical information.

Incorporating many original results of the authors, the book is addressed both to students who can find a clear and thorough presentation of the basics of the integral representation theory, as well as for the advanced readers who find a substantial amount of recent results, appearing for the first time in book form.

S. Cobzaş

A. L. Dontchev and R. T. Rockafellar, ***Implicit Functions and Solution Mappings. A View from Variational Analysis,*** Springer Monographs in Mathematics, Springer, New York, 2009, xii + 375 pp, ISBN: 978-0-387-87820-1,

The implicit function theorem is in the gallery of basic theorems in analysis and its many variants are basic tools in various parts of mathematics, including partial differential equations, nonsmooth analysis, and numerical analysis. The book starts with the classical framework of implicit function theorem and then is largely focusing on properties of solution mapping of variational problems.

The goal of the authors was to provide a reference on the topic and to present a unified collection of results scattered throughout the literature. The authors fully realized their goal.

In the classical framework the main issue in the subject is whether the solution to an equation depending on parameters may be considered as a function of those parameters and if so, what properties that function might have. This is addressed by the classical theory of implicit functions, which began with the single real variable and developed through several variables to equations in infinite dimensions, such as equations associated with integral and differential operators.

An important feature of the book is to lay out lesser known variants, for instance where standard assumptions of differentiability are relaxed in different senses. On the same vain the book shows how the same circle of ideas, when articulated in a suitable framework, can deal successfully may other problems than just solving equations, as min or max problems with inequality constraints. In such a case a question is addressed if a solution can be expressed as a function of parameters of the problem. Mathematical models resting on equations are replaced by "variational inequality" models. At this level of generality the main concept is that of the set-valued solution mapping which assigns to each instance of the parameter element in the model all the corresponding solutions, if any. The main question is whether a solution mapping can be localized graphically in order to achieve single-valuedness and in that sense produce a function, the searched implicit function.

The book under review concentrates primarily on local properties of solution mappings that can be captured metrically, rather than on results derived from topological considerations or involving lesser used spaces.

The first chapter concerns with the implicit function paradigm in the classical case of the solution mapping associated with a parameterized equation. Two proofs of the classical inverse function theorem are given and then two equivalent forms of it are derived: the implicit function theorem and the correction function theorem. The differentiability assumption is gradually relaxed in various ways and even completely exit from it. Instead it is substituted by the Lipschitz condition.

The second chapter was inspired by optimization problems and models of competitive equilibrium. The questions are essentially the same as in the first chapter, namely, when a solution mapping can be localized to a function with some continuity properties. Here the authors are dealing with generalized equations which captures a more complicated dependence and covers, among others, variational inequality conditions formulated in terms of the set-valued normal cone mapping associated with a convex set.

In the third chapter there are introduced the notions of Painlevé-Kuratowski convergence and Pompeiu-Hausdorff convergence for sequences of set, and used them in proving properties of continuity and Lipschitz continuity for set-valued mappings. Important results are obtained regarding the solution mapping associated with constraint systems.

In the fourth chapter graphical differentiation of a set-valued mapping is defined through the variational geometry of the mapping's graph. A characterization of the Aubin property is derived and applied to the case of a solution mapping. Strong metric subregularity is characterized and applications are introduced to parameterized constraint systems and special features of solution mappings for variational inequalities.

The fifth chapter is devoted to the study of regularity in infinite dimensions. It presents extensions of the Banach open mapping theorem which are shown to fit infinite-dimensionally into the paradigm of the theory developed finite dimensionally in Chapter 3.

The sixth chapter contains applications in numerical variational analysis. It is illustrated how some of the implicit function/mapping theorems from the earlier of the book can be used in the study of problems in numerical analysis.

Each chapter ends with a section of commentary. By these sections the authors exhibits connections of the results just introduced with other results by other authors. The commentaries are deep, pertinent and very useful.

The book ends with a rich list of references, a glossary of notation, and a subject index. This references reflect from one side the authors's contribution to this topic and from other side the contributions of many other researchers all over the world.

Certainly this wonderful work will be included in many libraries all over the world.

Marian Mureşan

William J. Terrell, ***Stability and Stabilization. An Introduction***, Princeton University Press, 2009, XV+457 pp, ISBN-13: 978-0-691-13444-4.

William Terrell's book is a first intermediate textbook that covers stability and feedback stabilization of equilibria for both linear and nonlinear autonomous systems of ordinary differential equations. It covers a portion of the core of mathematical control theory, including the concepts of linear systems theory and Lyapunov stability theory for nonlinear systems, with applications to feedback stabilization of control systems. This book takes a unique modern approach that bridges the gap between linear and nonlinear systems.

The book is designed for advanced undergraduates and beginning graduate students in the sciences, engineering, and mathematics. The minimal prerequisite for reading it is a working knowledge of elementary ordinary differential equations and elementary linear algebra.

The author structured his book in seventeen chapters and six appendixes. There are five chapters on linear systems (2–7) and nine chapters on nonlinear systems (8–16); an introductory chapter (1); a mathematical background chapter (2); and a short final chapter on further reading (17). Appendixes cover notations, basic analysis, ordinary differential equations, manifolds and the Frobenius theorem, and comparison functions and their use in differential equations. The introduction to linear system theory presents the full framework of basic state-space theory, providing just enough detail to prepare students for the material on nonlinear systems.

Clear formulated definitions and theorems, correct proofs and many interesting examples and exercises make this textbook very attractive.

Ferenc Szenkovits

Pankaj Sharan, ***Spacetime, Geometry and Gravitation***, Progress in Mathematical Physics 56, Birkhäuser, Basel - Boston - Berlin, 2009, XIV+355 pp, ISBN: 978-3-7643-9970-24.

This readable introductory textbook on the general theory of relativity presents a solid foundation for those who want to learn about relativity. The author offers us a physically intuitive, but mathematically rigorous presentation of the subject.

The book is structured into three parts. The first part, containing four chapters, offers preliminary topics on general relativity. This part begins with a general background and introduction, followed by an introduction to curvature through Gauss's Theorema Egregium, continue with basics on general relativity, and presents also simpler topics in general relativity like the Newtonian limit, red shift, the Schwarzschild solution, precession of the perihelion and bending of line in a gravitational field.

The second part is a comprehensive and accurate incursion in the mathematical background of relativity theory. Separate chapters are dedicated to the next topics: vectors and tensors, inner product, elementary differential geometry, connection and curvature, Riemannian geometry and varied additional topics in geometry are also presented.

The last part, entitled Gravitation, contains six chapters dealing with: the Einstein equation, general features of spacetime, weak gravitational fields, Schwarzschild and Kerr solutions, cosmology and special topics. Interesting exercises with complete answers simplify the understanding of the presented material. This textbook is recommended to advanced graduates and researchers.

Ferenc Szenkovits

John J. Benedetto and Wojciech Czaja, ***Integration and Modern Analysis***, xix+575 pp,Birkhäuser Advanced Texts, Birkhäuser, Boston -Basel - Berlin, 2009, ISBN: 978-0-8176-4306-5, e-ISBN: 978-0-8176-4656-1

The aim of the present book is to emphasize how the modern integration theory evolved from some classical problems in function theory, related mainly to Fourier analysis. It is worth to mention that some problems that arose in the study of Fourier series in the nineteenth century lay at the basis of many modern branches of mathematics as, for instance, set theory. For this reason the first chapter of the book, Ch. 1, *Classical real variables,* contains some classical results related to differentiation (e.g., continuous nowhere differentiable functions) and its imperfect relations with the Riemann integral, culminating in the new theory of integration developed by Lebesgue, which put the things in their right places. In fact, one of the main ideas of the book is the study of the relations between integrals and the a.e. derivatives (the Fundamental Theorem of Calculus - FTC), realized by the key notion of absolute continuity, viewed by the authors as a unifying concept for the FTC, Lebesgue dominated convergence theorem (LDC) and Radon-Nikodym theorem.

Another major topic of the book is that of switching limits in various processes of analysis (differentiation, integration), considered as a fundamental problem in the whole analysis, being related to compactness and weak compactness in function spaces (Dunford-Pettis theorem) or in spaces of measures (Grothendieck's results).

The first five chapters of the book, the first one mentioned above, 2. *Lebesgue measure and general measure theory,* 3. *The Lebesgue integral,* 4. *The relationship between differentiation and integration,* and 5. *Spaces of measures and the Radon-Nikodym theorem*, can serve as a basic one-semester course in real analysis.

The second part, containing the chapters 6. *Weak convergence of measures,* 7. *Riesz representation theorem,* 8. *Lebesgue differentiation theorem on* $\mathbb{R}^d$*,* 9. *Self-similar sets and fractals,* contains finer points, providing material for the second semester of a full-year course. Each chapter in the first part ends with a set of exercises and problems, ranging from routine to challenging (dedicated by the authors to the *"mathochistic"* student). The second part of the book contains no problems.

The book has also two appendices, one on basic results in functional analysis, and a consistent survey (42 pages) on harmonic analysis, a field in which the authors are experts, called by them one of the goddesses of mathematics. In fact, the book is written in the idea to provide the reader with the basic concepts and results in measure theory and integration, as a basic acting tool in several branches of mathematics as potential theory, harmonic analysis, probability theory, nonlinear dynamics, etc.

Each chapter ends with a section entitled *Potpourri and titillation,* dedicated to historical remarks and references for further reading, meant to be informative and fun, providing some breadth without depths, but which are strongly recommended to be read by the students. Beside these sections, the book contains short biographical sketches of some of the great figures of the domain and some disputes (e.g. that between Borel and Lebesgue).

A special tribute is paid to Vitali for his outstanding contributions, somewhat neglected by the mathematical community, who gave 100 years ago modern proofs to some fundamental results in real-analysis - the notion of absolute continuity, the first example of a nonmeasurable set, the first proof of Luzin's measurability theorem (Luzin 1912, Vitali 1905).

The bibliography (550 items plus 23 on the history of integration, listed at the end of first chapter) contains references to original works, to recent papers and books, as well as various folklore topics collected from classical books or from the American Mathematical Monthly.

A name index, containing full names (a misprint - Chebyshev should be "Pafnutii" not "Patnutii") with references to specific pages where they are quoted, a subject index and a notation index make the book very well organized.

The result is a nice book, containing a lot of results in measure theory and integration theory, making a good connection between classical and modern ones. The live style of exposition make the reading both instructive and agreeable. It can be recommended to instructors for one-semester or two semester courses in real analysis, to students for self-study and to researchers in various domains of analysis as a reference text.

S. Cobzaş

Steven G. Krantz, ***Explorations in Harmonic Analysis- With Applications to Complex Function Theory and the Heisenberg Group***, xiv+360 pp, Applied and Numerical Harmonic Analysis, Birkhäuser, Boston -Basel - Berlin, 2009, ISBN: 978-0-8176-4668-4, e-ISBN: 978-0-8176-4669-1

This is a self-contained introduction to modern harmonic analysis, starting with the classical theory of Fourier series and culminating in the theory of pseudodifferential operators and analysis on Heisenberg group. The idea of studying harmonic analysis on Heisenberg group (fractional integrals and singular integrals) belongs to E. M. Stein and his school in the 1970s. This turned out to be a powerful device for developing sharp estimates for integral operators (the Bergman projections, the Szegö projections) that arise naturally in the several complex variable setting. (A chapter of the book is dedicated to a crash course in several complex variables).

The book starts with a short historical account and comments on the evolution of the idea of Fourier expansion and continues with an introduction to the Hilbert transform (considered by the author as the most important operator in analysis), the Fourier transform, fractional and singular integrals, canonical integral operators (Bergman and Szegö kernels).

Hardy spaces are discussed in Chapter 8, including a sketch of the real variable method that provides a bridge between classical holomorphic-functions Hardy spaces and the more modern real-variable approach to Hardy and BMO spaces.

The analysis on the Heisenberg group is presented in Chapters 9. *Introduction to Heisenberg group*, 10. *Analysis on the Heisenberg group,* and 11. *A coda on domains of finite type.*

Three appendices: A1. *Rudiments of Fourier series*, A2. *The Fourier transform,* and A3. *Pseudodifferential operators,* complete the main text, making the book as self-contained as possible (but not more).

The text is very well organized: each chapter begins with an introductory *Prologue*, each section with a *Capsule* giving a quick preview of the material, and each key theorem is preceded by a *Prelude* providing motivation and putting the result in an adequate context. Some details of the proofs are left as *Exercises for the reader.*

Written by an expert in analysis, understood in a broad and unitary sense, this new book gives the reader a panoramic and, at a same time, detailed, view of the subject, showing how basic analytic tools, ranging from real analysis, several complex variables, Lie theory, differential equations, differential geometry, dynamical systems and other parts of mathematics, interact into this modern area of investigation which is harmonic analysis.

The book will be useful for advanced courses on harmonic analysis, singular integrals, as well as reference text for researchers in various domains of analysis, both pure and applied.

Gabriela Kohr

Mariano Giaquinta and Giuseppe Modica, *Mathematical Analysis - An Introduction to Functions of Several Variables*, xii+348 pp, Birkhäuser, Boston -Basel - Berlin, 2010, ISBN: 978-0-8176-4509-0 (hardcover), 978-0-8176-4507-6 (softcover) e-ISBN: 978-0-8176-4612-7

This is a part of an ampler project of the authors:

GM1. *Mathematical Analysis - Functions of one variable,* Birkhäuser 2003;

GM2. *Mathematical Analysis - Approximation and discrete processes,* Birkhäuser 2004;

GM3. *Mathematical Analysis - Linear and metric structures and continuity,* Birkhäuser 2007.

Another volume, GM5. *Mathematical Analysis - Foundation and advanced techniques for functions of several variables,* is announced to be published with Birkhäuser, too (the present volume is numbered as GM4). All these volumes were first published in Italian by Pitagora Editors, Bologna, these ones being revised translations of the original Italian versions.

The first chapter of the book, *Differential calculus,* contains the standard results of the subject, including differential calculus in Banach spaces and two proofs for the inverse mapping theorem. The first section of the second chapter, *Integral calculus,* contains the basic definitions and results on Lebsegue measure in $\mathbb{R}^n$ and Lebesgue integral. This section contains no proofs, the reader being referred to GM5 for a complete treatment. Accepting these basic results, one proves then results on mollifiers and smooth approximation, Luzin's theorem on the characterization of measurability, the properties of the integrals depending on parameter, the Hausdorff measure, the area and co-area formulas and Green-Gauss formula.

The aim of Chapter 3, *Curves and differential forms*, is to prove Poincaré's Lemma and Stokes' Theorem. Chapter 4, *Holomorphic functions,* is a good introduction (with complete proofs) to holomorphic functions of one complex variable and their applications. Some more special topics, as, for instance, the Mellin transform, are also included.

Chapter 5, *Surfaces and level sets,* is concerned with r-dimensional surfaces in $\mathbb{R}^n$ - parameterizations of maximal rank, various versions of the implicit function theorem, Sard's theorem, Morse Lemma, gradient flows, curvature, the first and the second fundamental form of a surface.

The last chapter of the book, 6, *Systems of ordinary differential equations* (ODE) is concerned with first order linear systems of ODEs, higher-order linear differential equations, and a short discussion on stability - Lyapunov's method and Cantor-Bedixson's theorem.

The applications and the examples included in the book make it more attractive. There are also exercises at the end of each chapter. When referring to a classical result, a picture of the author as well as an excerpt (the first page or the cover) of the original publication are included.

A word must be said about the elegant layout of the book.

Undoubtedly that finished, this ambitious project will supply the reader with a fairly complete account of the fundamental results in mathematical analysis and applications, including Lebesgue integration in $\mathbb{R}^n$ and complex analysis of one variable.

The books can be used for courses in real or complex analysis and their applications.

Tiberiu Trif

Martin Schechter, ***Minimax Systems and Critical Point Theory,*** xiv+239 pp, Birkhäuser, Boston -Basel - Berlin, 2009, ISBN: 978-0-8176-4805-3, e-ISBN: 978-0-8176-4902-9

Many problems in differential equations come from variational considerations, involving a real-valued functional G defined on a Banach space E. This means to minimize or maximize the functional G, the original approach to this problem being the solving of the Euler-Lagrange equation (1): $G'(u) = 0$, i.e., finding the critical points of the functional G. This works well in the one-dimensional case, but in higher dimensions it is more difficult to solve the Euler-Lagrange equations than to find minima or maxima, and so this approach was abandoned for many years. It was reconsidered again, when nonlinear partial differential equations and systems arose in applications leading to the question whether these equations and systems are the Euler-Lagrange equations corresponding to some functionals.

The minima and maxima approach in finding critical points works only for functionals bounded from below or from above (semibounded). If not, then other conditions must be imposed which can substitute the semiboundedness of the functional G. One such notion is that of linking pair, meaning two subsets A, B of E such that

(2): $a_0 := \sup G(A) < \inf G(B) =: b_0$, a topic treated by the author in a previous book, *Linking Methods in Critical Point Theory,* Birkhäuser, Boston - Basel-Berlin, 1999.

A Palais-Smale sequence for a bounded from below C^1-functional G is a sequence (u_k) in E such that (PS): $G(u_k) \to a := \inf G$ and $G'(u_k) \to 0$. A sequence (u_k) such that $G(u_k) \to \inf G$ is called a minimizing sequence. One obtains the stronger notion of Cerami sequence by replacing the second condition in (PS) by the stronger one $(1 + \|u_k\|)G'(u_k) \to 0$. The interesting case is when a PS-sequence, or a Cerami sequence, contains a convergent subsequence, a fact easier to be shown for these kind of sequences than for arbitrary minimizing sequences.

The aim of the present book is to expose in a unified way some new methods and results which appeared since the publication of that book. One of the main notions used in the book is that of linking pair $A, B \subset E$: one says that A links B if (2) implies (3): $\exists u,\ G(u) \geq b_0,\ G'(u) = 0$. Also one says that G satisfies the PS-condition if (PS) always implies (3). In this case, every functional that satisfies the PS-condition and is separated by a pair of linking sets has a critical point satisfying (3).

Other important notion is that of minimax systems for a subset A of E, meaning a family $\mathcal{K}$ of subsets of E such that $\sup G(A) < \inf\{\sup G(K) : K \in \mathcal{K}\}$. Again, a minimax system has the same advantages as a semibounded functional. The role played by the linking pairs and the minimax systems in finding critical points is shown in Chapters 1 and 2. The proofs of these results, based on some existence theorems for differential equations in abstract spaces given in Chapter 4, are postponed to Chapter 5. Examples of linking pairs and minimax methods, as well as some abstract methods of finding them, are given in Chapters 3 and 6.

In Chapter 7 the author introduces the notion of sandwich pair, meaning a pair $A, B \subset E$ such that $-\infty < \inf G(B) \leq \sup G(A) < \infty$. An important particular case of sandwich pair is formed by two subspaces M, N of a Hilbert space, one of them being finite dimensional, such that $M = N^{\perp}$. The author shows that infinite dimensional subspaces qualify as well. Weak linking and weak sandwich pairs are considered in Chapters 10, respectively 11.

The rest of the book is dedicated to applications, many of which are more involved than those existing in the literature: semilinear problems (Ch. 9), semilinear wave equations (Ch. 13), the Fučik spectrum (Chapters 11 and 14), multiple solutions (Ch. 16), second-order periodic systems (Ch. 17).

The book is rather elementary, being accessible to students with a background in functional analysis.

Largely based on the results obtained by the author, alone or in cooperation with other mathematicians, this book, together with the 1999 volume, cover a broad spectrum of applications of critical point method in solving various nonlinear differential and partial differential equations. The author proposes some very general and unitary approaches to find critical points and exposes them in a clear and sequential way. The book can be recommended for researchers in applied functional analysis, partial differential equations and their applications.

It is worth to mention the spectacular evolution of author's family: the book by the author and Wenming Zou, *Critical Point Theory and its Applications,* Springer 2006, was dedicated to his 22 grandchildren and 1 great grandchild, the present volume records 24 grandchildren and 6 great grandchildren (so far). Let them all live in peace and happiness.

Cornel Pintea

Manuel González and Antonio Martínez-Abejón, *Tauberian Operators,* xii+245 pp, Birkhäuser, Boston -Basel - Berlin, 2010, ISBN: 978-3-7643-8997-0, e-ISBN: 978-3-7643-8998-7

Abel theorem asserts that

(1) $\sum_{n=0}^{\infty} a_n = \lambda$ implies (3) $\lim_{x\to 1-} \sum_{n=0}^{\infty} a_n x^n = \lambda$.

Tauber proved in 1897 a conditioned converse of this result: the condition (3) plus (2) $\lim_n a_n/n = 0$ imply (1). Tauberian operators were introduced by Kalton and Wilanski in 1976 as an abstract counterpart of some operators associated to conservative summability matrices. An infinite matrix $A = (a_{ij})$ is called conservative if $Ax := (\sum_j a_{ij}x_j)_i$ is well defined and convergent for every convergent sequence x. Based on the Toeplitz-Silverman characterization of conservative matrices, one can associate with every conservative matrix A two operators $S_A : c \to c$ and $T_A : \ell_\infty \to \ell_\infty$, both defined by the same value Ax. J. P. Crawford in his thesis (1966) proved that $T_A^{-1}(c) \subset c$ iff $S^{**-1}(c) \subset c$, paving the way to the abstract definition (given by Kalton and Wilanski) of tauberian operators as those continuous linear operators $T : X \to Y$, X, Y Banach spaces for which $T^{**-1}(Y) \subset X$. This turned out to be a very important and useful class of operators in Banach space theory: Davis-Figiel-Johnson-Pelczynski factorization theorem, the study of exact sequences of Banach spaces, some summability problems of tauberian type, the study of the equivalence between Krein-Milman and Radon-Nikodym theorems, some sequels of James' characterization of reflexivity, extensions of the principle of local reflexivity to operators, the study of Calkin algebras associated with the weakly compact operators. All these applications are presented in the third chapter of the book.

The historical roots of tauberian operators are explained in Chapter 1, *The origins of tauberian operators.*

The basic properties of tauberian operators, as well as some important characterizations and connections with weakly compact and semi-Fredholm operators, are given in Chapter 2. *Tauberian operators. Basic properties,* a study continued in chapters 3. *Duality and examples of tauberian operators*, and 4. *Tauberian operators on spaces of integrable functions.* Some generalizations of tauberian operators, as well as connections with operator ideals and ultrapower classes of operators are studied in Chapter 6. *Tauberian-like operators.*

The prerequisites are a basic course in functional analysis and some familiarity with Fredholm theory, ultraproducts, operator ideals is recommended. For the convenience of the reader, these more specialized topics are summarized in a consistent appendix (40 pages) at the end of the volume.

The book present in a clear and unified way the basic properties of tauberian operators and their applications in functional analysis scattered throughout the literature. Some open problems are included, too.

Written by to experts in the field, the book is addressed to graduate students and researchers in functional analysis and operator theory, but it can be used also as a basic text for advanced graduate courses.

V. Anisiu

Lluís Puig Editors, *Frobenius categories versus Brauer blocks. The Grothendieck group of the Frobenius category of a Brauer block*. Progress in Mathematics 274. Birkhäuser Basel-Boston-Berlin, 2009, 498 p., ISBN: 978-3-7643-9997-9/hbk; ISBN: 978-3-7643-9998-6/ebook.

The concept of Frobenius category (also called saturated fusion system) has been developed by Lluís Puig many years ago. Such a category has as objects the subgroups of a given p-group S (where p is a prime), the morphisms between two subgroups are injective, and contain all conjugations and inclusions of subgroups of S; all these satisfy certain axioms. This is a generalization the p-local structure of a finite group G, and it turns out this point of view is the correct framework for modular representation theory of finite groups. Indeed, the theory of Brauer pairs, due to J. Alperin and M. Broué associates to a subgroup of the defect group of a block a conjugacy relation similar to that between the subgroups of the Sylow p-subgroup. Moreover, group-theoretic notions, such as normalizers and centralizers, can be generalized to Frobenius categories. Later, Broto, Levi and Oliver realized that Puig's concepts are useful in p-local homotopy theory, particularly for the proof of the Martino-Priddy conjecture. Therefore, the study of fusion systems has been very active recently. Let me briefly present the content of the book. Chapter 1 introduces the general background, and Chapter 2 present the first definitions concerning Frobenius P-categories. Chapter 3 presents the definition of the Frobenius P-category of a p-block of a finite groups, by using Brauer pairs. Chapter 4 introduces self-centralizing objects, by mimicking the definition of selfcentralizing Brauer pairs, and also related concepts. Chapter 5 develops, in the context of Frobenius categories, a refinement of Alperin's fusion theorem , by introducing the so-called essential objects. Chapter 6 introduces the exterior quotient of the subcategory of selfcentralizing objects. Chapter 7 analyzes the selfcentralizing and nilcentralized objects of the Frobenius category of a block. Chapters 8, 9 and 10 discuss Dade P-algebras, introduce the concept of polarization and present a gluing theorem for Dade algebras. Chapter 11 provides a tool for defining later in Chapter 14 the Grothendieck group of a Frobenius category of a block, and proves a lifting theorem for the nil-centralizing subcategory. Chapter 12 discusses quotients and normal subcategories, mimicking quotients and normal subgroups of a group. Chapter 13 introduces the hyperfocal subcategory of a Frobenius category. Chapters 14, 15 and 16 define and study in detail the Grothendieck group of a Frobenius category, and develops a strategy towards reducing certain questions to quasi-simple groups. Chapters 17 and 18 are devoted to coherent perfect localities, which have important applications to group cohomology. Chapter 19 generalizes the notion of solvability to Frobenius categories. Chapter 20 investigates the extendability of partial perfect localities of the nil-centralized Frobenius system to perfect localities of the original Frobenius category. Chapter 21 gives a definition of Frobenius categories in terms of bisets, and the final three chapters investigate the basic locality of a Frobenius category in these terms. The book ends with Appendix developing the author's point of view on cohomology of small categories. This volume is a research monograph containing many original results which appear for the first time in print (although Puig's ideas have widely circulated in manuscript and preprint form). The book is, in principle, accessible to advanced graduate students, but the technicalities and the complicated notation make it quite difficult to read. Having said that, the book is a necessary tool for any researcher in the field.

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