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ON THE COMBINATORIAL IDENTITIES OF ABEL-HURWITZ TYPE AND THEIR USE IN CONSTRUCTIVE THEORY OF FUNCTIONS

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Abstract. This paper is concerned with the problem of approximation of multivariate functions by means of the Abel-Hurwitz-Stancu type linear positive operators. Inspired by the work of D. D. Stancu [13], we continue the discussions of the approximation of trivariate functions by a class of Abel-Hurwitz-Stancu operators in the case of trivariate variables, continues on the unit cub $K_3 = [0,1]^3$. In this paper there are three sections. In Section 1, which is the Introduction, is mentioned the generalization given by N. H. Abel [1] in 1826, for the Newton binomial formula and then the very important extension of this formula given by A. Hurwitz in 1902, in the paper [3]. Here is mentioned an interesting combinatorial significance in a cycle-free directed graphes given by D. E. Knuth [5]. Then is presented a main result given in 2002 by D. D. Stancu [13], where is used a variant of the Hurwitz identity in order to construct and investigate a new linear positive operator, which was used in the theory of approximation univariate functions. In Section 2 is discussed in detail the trivariate polynomial operator of Stancu-Hurwitz type $S_{m,n,r}^{(\beta),(\gamma),(\delta)}$ associated to a function $f \in C(K_3)$, where K_3 is the unit cub $[0,1]^3$. Section 3 is devoted to the evaluation of the remainder term of the approximation formula (3.1) of the function $f(x,y,z)$ by means of the Stancu-Hurwitz type operator $S_{m,n,r}^{(\beta),(\gamma),(\delta)}$. Firstly is presented an integral form of this remainder, based on the Peano-Milne-Stancu result [12]. Then we give a Cauchy type form for this remainder. By using a theorem of T. Popoviciu [8] we gave an expression, using the divided differences of the first three orders. When the coordinates of the vectors (β), (γ), (δ) have respectively the same values we are in the case of the second operator of Cheney-Sharma [2]. In this case we obtain an extension of the results from the papers [13] and [17].

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1. **Introduction**

By means of the D. D. Stancu [13] trivariate polynomial operator of Stancu-Hurwitz type $S_{m,n,r}^{(\beta),(\gamma),(\delta)}$ associated to a function $f \in (K_3)$, where K_3 is the unit cub $K_3 = [0,1]^3$, we construct some linear positive operators, useful in constructive theory of functions.

It is known that by using the celebrated generalization of Newton binomial formula, given in 1826 by the outstanding Norvegian Niels Henrik Abel [1], namely

$$(u+v)^m = \sum_{k=0}^{m} \binom{m}{k} u(u-k\beta)^{k-1}(v+k\beta)^{m-k}, \tag{1.1}$$

where β is a nonnegative parameter, the German mathematician Adolf Hurwitz has given in 1902, in the paper [3], a generalization of the Abel identity (1.1), represented by the equality

$$(x+y)(x+y+z_1+\ldots+z_m)^{m-1} \tag{1.2}$$

$$=\sum x(x+\varepsilon_1 z_1+\ldots+\varepsilon_m z_m)^{\varepsilon_1+\ldots+\varepsilon_m-1} y(y+(1-\varepsilon_1)z_1+\ldots+(1-\varepsilon_m)z_m)^{m-1-\varepsilon_1-\ldots-\varepsilon_m}$$

summed over all 2^m choices of $\varepsilon_1, \ldots, \varepsilon_m$ independently taking the values 0 and 1. This is an identity in $2m+2$ variables, and Abel's binomial formula is the special case $z_1 = z_2 = \ldots = z_m$.

The famous specialist in computer science D. E. Knuth has given in [5] an interesting combinatorial significance in cycle-free directed graphs.

By using the following variant of Hurwitz identity

$$(u+v)(u+v+\beta_1+\ldots+\beta_m)^{m-1}$$

$$=\sum u(u+\beta_{i_1}+\ldots+\beta_{i_k})v(b+\beta_{j_1}+\ldots+\beta_{j_{m-k}}),$$

professor D. D. Stancu has constructed in the paper [13] a general linear positive operator useful for uniform approximation of continuous functions, namely

$$(S^{(\beta_1,\ldots,\beta_m)}f(x) = \frac{1}{1+\beta_1+\ldots+\beta_m} \sum_{k=0}^{m} w_{m,k}^{(\beta_1,\ldots,\beta_m)}(x) f\left(\frac{k}{m}\right), \tag{1.3}$$

where

$$w_{m,k}^{(\beta_1,\ldots,\beta_m)}(x) = \sum x(x+\beta_1+\ldots+\beta_m)^{k-1}(1-x)(1-x+\beta_{j_1}+\ldots+\beta_{j_{m-k}})^{m-k-1}.$$

Because we can write

$$(1+\beta_1+\ldots+\beta_m)^{m-1}(S_m^{(\beta_1,\ldots,\beta_m)}f)(x)=(1-x)(1-x+\beta_1+\ldots+\beta_m)^m f(0)$$

$$+x(1-x)\sum_{k=1}^{m-1} w_{m,k}^{(\beta_1,\ldots,\beta_m)}(x)f\left(\frac{k}{m}\right)+x(x+\beta_1+\ldots+\beta_m)^{m-1}f(1),$$

it is easy to see that the polynomial defined at (1.3) is interpolatory at both sides of the interval $[0,1]$, for any nonnegative values of the parameters $\beta,\ldots,\beta_m$.

Consequently we can write

$$(S_m^{(\beta_1,\ldots,\beta_m)}f)(0)=f(0),\quad (S_m^{(\beta_1,\ldots,\beta_m)}f)(1)=f(1)$$

and we conclude that the operator of Stancu-Hurwitz defined at (1.3) reproduces the linear functions.

2. The trivariate polynomial of Stancu-Hurwitz type

For simplicity we restrict ourselves here to the case of the space of real-valued functions $f(x,y,z)$, continuous on the unit cub $K_3=[0,1]^3$. We associate to the function $f\in(K_3)$ the Stancu-Hurwitz trivariate polynomial $S_m^{(\beta),(\gamma),(\delta)}$, defined by the formula

$$(S_{m,n,r}^{(\beta),(\gamma),(\delta)})(x,y,z)=\sum_{k=0}^{m}\sum_{j=0}^{n}\sum_{\nu=0}^{r} u_{m,k}^{(\beta)}(x)v_{m,j}^{(\gamma)}(y)w_{r,\nu}^{(\delta)}(z)f\left(\frac{k}{m},\frac{j}{n},\frac{\nu}{r}\right), \tag{2.1}$$

where we have the vectors with nonnegative coordinates $\beta=(\beta_1,\ldots,\beta_m)$, $\gamma=(\gamma_1,\ldots,\gamma_n)$, $\delta=(\delta_1,\ldots,\delta_r)$, while the basic polynomials are given by the formulas

$$(1+\beta_1+\ldots+\beta_m)^{m-1}u_{m,k}^{(\beta)}(x)$$

$$=\sum x(x+\beta_1+\ldots+\beta_{i_k})^{k-1}(1-x)(1-x+\beta_{j_1}+\ldots+\beta_{j_{m-k}})^{m-k-1},$$

$$(1+\gamma_1+\ldots+\gamma_n)^{n-1}v_{n,j}^{(\gamma)}(y)$$

$$=\sum y(y+\gamma_1+\ldots+\gamma_{s_\nu})^{j-1}(1-y)(1-y+\gamma_{t_1}+\ldots+\gamma_{t_{n-\nu}})^{n-j-1},$$

$$(1+\delta_1+\ldots+\delta_r)^{r-1}w_{r,\nu}^{(\delta)}(z)$$

$$=\sum z(z+\delta_1+\ldots+\delta_{\gamma_\tau})^{\nu-1}(1-z)(1-z+\delta_{\tau_1}+\ldots+\delta_{\tau_{r-1}})^{r-\nu-1}.$$

In the special cases $\beta_i = \beta$ $(i = \overline{1,m})$, $\gamma_j = \gamma$ $(j = \overline{1,n})$, $\delta_s = \delta$ $(s = \overline{1,r})$, we obtain the Cheney-Sharma-Stancu type trivariate linear positive operator defined by the following formula

$$(S_{m,n,r}^{(\beta,\gamma,\delta)} f)(x,y,z) = \sum_{k=0}^{m}\sum_{j=0}^{n}\sum_{\nu=0}^{r} u_{m,k}^{(\beta)}(x) v_{m,j}^{(\gamma)}(y) w_{r,\nu}^{(\delta)}(z) f\left(\frac{k}{m}, \frac{j}{n}, \frac{\nu}{r}\right),$$

where now we have

$$(1+m\beta)^{m-1} u_{m,k}^{(\beta)}(x) = \binom{m}{k} x(x+k\beta)^{k-1}(1-x)(1-x+(m-k)\beta)^{m-k-1},$$

$$(1+n\gamma)^{n-1} v_{n,j}^{(\gamma)}(y) = \binom{n}{j} y(y+j\gamma)^{j-1}(1-y+(n-j)\gamma)^{n-j-1},$$

$$(1+r\delta) w_{r,\nu}^{(\delta)}(z) = \binom{r}{\nu} z(z+\nu\delta)^{\nu-1}(1-z+(r-\nu)\delta)^{r-\nu-1}.$$

This operator $S_{m,n,r}^{(\beta,\gamma,\delta)}$ represents an extension to three variables of the second operator of Cheney-Sharma [2].

3. The remainder of the approximation formula of the function $f(x,y,z)$ by means of the operator of Stancu-Hurwitz type $S_{m,n,r}^{(\beta),(\gamma),(\delta)}$

If we use a theorem of Peano-Milne-Stancu type, given in the paper of D. D. Stancu [12], we can present an integral representation of the remainder term of the approximation formula

$$f(x,y,z) = (S_{m,n,r}^{(\beta),(\gamma),(\delta)} f)(x,y,z) + (R_{m,n,r}^{(\beta),(\gamma),(\delta)} f)(x,y,z), \tag{3.1}$$

having the degree of exactness $(1,1,1)$.

We can state the following

Theorem 3.1. *If the function f of three variables has continuous second-order partial derivatives on the unit cub K_3, then the remainder of the above approximation formula can be represented under the following integral form*

$$(R_{m,n,r}^{(\beta),(\gamma),(\delta)} f)(x,y,z) = \int_0^1 L_m^{(\beta)}(u,y,z) f^{(2,0,0)}(u,y,z)du$$

$$+ \int_0^1 M_n^{(\gamma)}(x,v,z) f^{(0,2,0)}(x,v,z)dv + \int_0^1 N_r^{(\delta)}(x,y,w) f^{(0,0,2)}(x,y,w)dw$$

$$- \int_0^1\int_0^1 L_m^{(\beta)}(u,v,z) N_r^{(\delta)}(u,y,w) f^{(2,0,2)}(\xi,y,\zeta)dudw$$

$$-\int_0^1\int_0^1 L_m^{(\beta)}(u,v,z)M_n^{(\gamma)}(u,v,z)f^{(2,2,0)}(\xi,\eta,z)dudv$$

$$-\int_0^1\int_0^1 L_m^{(\beta)}(u,y,v)N_r^{(\delta)}(u,y,w)f^{(0,2,2)}(x,v,w)dudw$$

$$+\int_0^1\int_0^1\int_0^1 L_m^{(\beta)}(u,v,w)M_n^{(\gamma)}(u,v,w)N_r^{(\delta)}(u,v,w)f^{(2,2,2)}(u,v,w)dudvdw$$

where the Peano kernels are

$$L_m^{(\beta)}(u,y,z)=(R_m^{(\beta)}\varphi_x)(u),$$

$$M_n^{(\gamma)}(x,v,z)=(R_n^{(\gamma)}\psi_y)(v),$$

$$N_r^{(\delta)}(x,y,w)=(R_r^{(\delta)}\tau_z)(w),$$

with

$$\varphi_x(u)=\frac{x-u+|x-u|}{2}=(x-u)_+,$$

$$\psi_y(v)=\frac{y-v+|y-v|}{2}=(y-v)_+,$$

$$\tau_z(w)=\frac{z-w+|z-w|}{2}=(z-w)_+.$$

For the partial derivatives we have used the notation

$$F^{(r,s,t)}(u,v,w)=\frac{\partial^{r+s+t}F(u,v,w)}{\partial u^r\partial v^s\partial w^t}.$$

It follows that we can write explicitly

$$L_m^{(\beta)}(u,y,z)=(x-u)_+-\sum_{k=0}^{m}w_{m,k}^{(\beta)}(x)\left(\frac{k}{m}-u\right)_+,$$

$$M_n^{(\gamma)}(x,v,z)=(y-v)_+-\sum_{j=0}^{n}v_{n,j}^{(\gamma)}(y)\left(\frac{j}{n}-v\right)_+,$$

$$N_r^{(\delta)}(x,y,w)=(z-w)_+-\sum_{\nu=0}^{r}w_{r,\nu}^{(\delta)}(z)\left(\frac{i}{r}-w\right)_+.$$

By using these explicit expressions for the partial Peano kernels, we can see that they represent polygonal lines situated beneath the u axis, respectively the v axis and the w axis, which joins the points $(0,0,0)$ and $(0,1,0)$, respectively $(0,0,1)$.

Now if we take into account that on the cub K_3 we have $L_m^{(\beta)} \leq 0$, $M_n^{(\gamma)} \leq 0$ and $N_r^{(\delta)} \leq 0$, we can apply the first law of the mean to the integrals and we can find that

$$(R_{m,n,r}^{(\beta),(\gamma),(\delta)} f)(x,y,z) = f^{(2,0,0)}(\xi,y,z) \int_0^1 L_m^{(\beta)}(u,y,z)du$$

$$+f^{(0,2,0)}(x,\eta,z) \int_0^1 M_n^{(\gamma)}(x,v,z)dv + f^{(0,0,2)}(x,y,\zeta) \int_0^1 N_r^{(\delta)}(x,y,w)dw$$

$$-f^{(2,2,0)}(\xi,\eta,z) \int_0^1 L_m^{(\beta)}(u,v,z) M_n^{(\gamma)}(u,v,z)dudx$$

$$-f^{(2,0,2)}(\xi,y,\zeta) \int_0^1 \int_0^1 L_m^{(\beta)}(u,y,w) N_r^{(\delta)}(u,y,w)dudw$$

$$-f^{(0,2,2)}(x,\eta,\zeta) \int_0^1 \int_0^1 L_m^{(\beta)}(u,v,z) N_r^{(\delta)}(u,y,w)dvdw$$

$$+f^{(2,2,2)}(\xi,\eta,\zeta) \int_0^1 \int_0^1 \int_0^1 L_m^{(\beta)}(u,v,w) M_n^{(\gamma)}(u,v,w) N_r^{(\delta)}(u,v,w)dudvdw,$$

where ξ, η and ζ are certain points from the cub K_3.

It is easy to see that we have

$$\int_0^1 L_m^{(\beta)}(u,y,z)du = \frac{1}{2}(R_m^{(\beta)} e_{2,0,0})(x),$$

$$\int_0^1 M_n^{(\gamma)}(x,v,z)dv = \frac{1}{2}(R_n^{(\gamma)} e_{0,2,0})(y),$$

$$\int_0^1 N_r^{(\delta)}(x,y,w)dw = \frac{1}{2}(R_r^{(\delta)} e_{2,0,2})(z),$$

where we have considered the univariate remainders

$$R_m^{(\beta)} = I - S_m^{(\beta)}, \quad R_n^{(\gamma)} = I - S_m^{(\gamma)}, \quad R_r^{(\delta)} = I - S_r^{(\delta)}.$$

Now we can state the following result concerning the remainder of the approximation formula (3.1).

Theorem 3.2. *If $f \in C^{(2,2,2)}(K_3)$, then the remainder of the approximation formula (3.1) can be represented under the following Cauchy type form*

$$(R_{m,n,r}^{(\beta),(\gamma),(\delta)} f)(x,y,z) \tag{3.2}$$

$$= \frac{1}{2}(R_m^{(\beta)} e_{2,0,0})(x) f^{(2,0,0)}(\xi,y,z)$$

$$+\frac{1}{2}(R_n^{(\gamma)} e_{0,2,0})(y) f^{(0,2,0)}(x,\eta,z)$$

$$+\frac{1}{2}(R_r^{(\delta)}e_{0,0,2})(z)f^{(0,0,2)}(x,y,\zeta)$$
$$-\frac{1}{4}(R_m^{(\beta)}e_{2,0,0})(x)(R_n^{(\gamma)}e_{0,2,0})(y)f^{(2,2,0)}(\xi,\eta,z)$$
$$-\frac{1}{4}(R_m^{(\beta)}e_{2,0,0})(x)(R_r^{(\delta)}e_{0,0,2})(z)f^{(2,0,2)}(\xi,\eta,\zeta)$$
$$-\frac{1}{4}(R_m^{(\beta)}e_{2,0,0})(x)(R_r^{(\delta)}e_{2,0,2})(z)f^{(0,2,2)}(\xi,y,\zeta)$$
$$+\frac{1}{8}(R_m^{(\beta)}e_{2,0,0})(x)(R_n^{(\gamma)}e_{0,2,0})(z)(R_r^{(\delta)}e_{0,0,2})(z)f^{(2,2,2)}(\xi,\eta,\zeta).$$

Since $(S_m^{(\beta)}f)(x)$, $(S_n^{(\gamma)}f)(y)$, $(S_r^{(\delta)}f)(z)$ are interpolatory at both sides of the interval $[0,1]$, we can conclude that $(R_m^{(\beta)}e_{2,0,0})(x)$ contains the factor $x(x-1)$, then $(R_n^{(\gamma)}e_{0,2,0})(y)$ contains the factor $\eta(\eta-1)$, while $(R_r^{(\delta)}e_{0,0,2})(z)$ contains the factor $z(z-1)$.

Because $(R_{m,n,r}^{(\beta),(\gamma),(\delta)}e_{0,0,0})(x,y,z)=0$ and the remainder is different from zero for any convex function f of the first order, we can apply a criterion of T. Popoviciu [8] and we can conclude that the remainder of the approximation formula (3.1) is of simple form and we can state

Theorem 3.3. *If the second-order divided differences of the function $f\in C(K_3)$ are bounded on the unit cub K_3, then we can give an expression in terms of divided differences of the remainder of the approximation formula (3.1), namely*

$$(R_{m,n,r}^{(\beta),(\gamma),(\delta)}f)(x,y,z)$$
$$=(R_m^{(\beta)}e_{2,0,0}(x)[x_{m,1},x_{m,2},x_{m,3};f(t_1,y,z)]$$
$$+(R_n^{(\gamma)}e_{0,2,0})(y)[y_{n,1},y_{n,2},y_{n,3};f(x,t_2,z)]$$
$$+(R_r^{(\delta)}e_{0,0,2})(z)[z_{r,1},z_{r,r},z_{r,3};f(x,y,t_3)]$$
$$-(R_m^{(\beta)}e_{2,0,0})(x)(R_n^{(\gamma)}e_{0,2,0})(y)\left[\begin{matrix}x_{m,1},x_{m,2},x_{m,3}\\ y_{n,1},y_{n,2},y_{n,3}\end{matrix};f(t_1,t_2,z)\right]$$
$$-(R_m^{(\beta)}e_{2,0,0})(x)(R_r^{(\delta)}e_{0,0,2})(y)\left[\begin{matrix}x_{m,1},x_{m,2},x_{m,3}\\ z_{r,1},z_{r,2},z_{r,3}\end{matrix};f(t_1,y,t_3)\right]$$
$$-(R_n^{(\gamma)}e_{0,2,0})(y)(R_r^{(\delta)}e_{0,0,2})(z)\left[\begin{matrix}y_{n,1},y_{n,2},y_{n,3}\\ z_{r,1},z_{r,2},z_{r,3}\end{matrix};f(x,t_2,t_3)\right]$$

$$+(R_m^{(\beta)}e_{2,0,0})(x)(R_n^{(\gamma)}e_{0,2,0})(y)(R_r^{(\delta)}e_{0,0,2})(z)\left[\begin{array}{l} x_{m,1},x_{m,2},x_{m,3} \\ y_{n,1},y_{n,2},y_{n,3} \quad ;f(t_1,t_2,t_3) \\ z_{r,1},z_{r,2},z_{r,3} \end{array}\right],$$

where $x_{m,1}, x_{m,2}, x_{m,3}$*, respectively* $y_{n,1}, y_{n,2}, y_{n,3}$ *and* $z_{n,1}, z_{n,2}, z_{n,3}$ *are certain points in the interval* $[0,1]$.

Now if we consider that $f \in C^{2,2,2}(K_3)$, then we can apply the mean value theorems to the divided differences which occur above and we arrive at the expression (3.2) for the remainder of the approximation formula (3.1).

Clearly, the results of this paper can be extended to functions of more than three variables without any difficulty.

References

[1] Abel, N. H., *Démonstration d'une expression de laquelle la formule binôme est un cas particulier*, Journ. für Reine und Angewandte Mathematick, Oeuvres Complètes, Christiania, Groendahls, 1839. **1** (1826), 159-160.

[2] Cheney, E. W., Sharma, A., *On a generalization of Bernstein polynomials*, Riv. Mat. Univ., Parma, **5** (1964), 77-84.

[3] Hurwitz, A., *Über Abel's Verallgemeinerung der binomischen Formel*, Acta Mathematica, **26** (2002), 199-203.

[4] Jensen, J. L. W., *Sur une identité d'Abel et sur d'autre formules analogues*, Acta Mathematica, **26** (1902), 307-318.

[5] Knuth, D. E., *The Art of Computer-Programming*, vol. 1, *Fundamental Algorithms*, Addison-Wesley Publ. Comp., Reading, Massachusetts, USA, 1968, p. 75.

[6] Milne, W. E., *The remainder in linear methods of approximation*, J. Res. Mat. Bur. Standards, **43** (1949), 501-511.

[7] Peano, G., *Resto nelle formule di quadratura expresso con un integrale definito*, Atti Acad. Naz. Lincei Rend., **22** (1913), 562-569.

[8] Popoviciu, T., *Sur le reste dans certaines formules linéaires d'approximation de l'analyse*, Mathematica, Cluj, **1**(**24**) (1959), 95-142.

[9] Riordan, J., *An Introduction to Combinatorial Analysis*, Wiley, New York, 1958.

[10] Riordan, J., *Combinatorial Identities*, Wiley, New York, 1968.

[11] Stancu, D. D., *Evaluation of the remainder term in approximation formulas by Bernstein polynomials*, Math. Comp., **17** (1963), 137-163.

[12] Stancu, D. D., *The remainder of certain linear approximation formulas in two variables*, J. SIAM Numer. Anal. B, **1** (1964), 137-163.

[13] Stancu, D. D., *Use of an identity of A. Hurwitz for construction of a linear positive operator of approximation*, Rev. Analyse Numér. Théorie de l'Approximation, **31** (2002), 115-118.

[14] Stancu, D. D., *Methods for construction of linear positive operators of approximation*, In: Numerical Analysis and Approximation Theory, Proc. Internat. Symposium (R. T. Trîmbiţaş, ed.), Cluj University Press, 2002, 23-45.

[15] Stancu, D. D., Cismaşiu, C., *On an approximating linear positive operator of Cheney-Sharma*, Rev. Anal. Numér. Théor. Approx., **26** (1997), 221-227.

[16] Stoica-Laze, E. I., *On the use of Abel-Jensen type combinatorial formulas for construction and investigation of some algebraic polynomial operators of approximation*, Studia Univ. "Babeş-Bolyai", Mathematica, **54** (2009), no. 4, 167-182.

[17] Taşcu, I., *Approximation of bivariate functions by operators of Stancu-Hurwitz type*, Facta Math. Inform., **20** (2005), 33-39.

[18] Toader, S., *An approximation operator of Stancu type*, Rev. Anal. Numér. Théorie de l'Approx., **34** (2005), 115-121.

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