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WEIGHTED CONVERGENCE OF SOME POSITIVE LINEAR OPERATORS ON THE REAL SEMIAXIS

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Abstract. Some Shepard and Grünwald type operators are introduced on the semiaxis and their convergence, in suitable weighted spaces equipped with the uniform norm, is investigated.

1. **Introduction**

The purpose of this paper is the approximation of functions defined on $(0, \infty)$ by means of some positive linear operators based on the zeros of Laguerre polynomials. We will examine functions continuous on $(0, \infty)$, having singularities at the origin and increasing exponentially for $x \to +\infty$. In definitive functions belonging to the weighted space $L^{\infty}_{w_\gamma}$, $w_\gamma(x) = x^\gamma e^{-x}$, $\gamma \geq 0$ equipped with the uniform norm (see (2.4)) will be considered.

First we will examine the Shepard operator defined as

$$\mathcal{S}_m(f, x) = \frac{\displaystyle\sum_{k=1}^{m} (x - x_k)^{-2} f(x_k)}{\displaystyle\sum_{k=1}^{m} (x - x_k)^{-2}}, \quad x_k = -1 + \frac{2k}{m}$$

and introduced by D. Shepard in [20]. It has been widely used in approximation theory and is simple to implement in applications like interpolation of scattered data, curves and surfaces, fluid dynamics etc. (see, for instance, [2], [1] and references therein). For this reason, it has been several papers (see, for instance, [19], [4], [8], [6], [7], [15]) investigating on $\mathcal{S}_m$ on the equidistant knots and on the zeros of Jacobi polynomials.

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In this paper we consider the following

$$\mathcal{S}_m^{(\alpha)}(f,x) := \frac{\sum_{k=1}^{j}(x-x_k)^{-2}f(x_k)}{\sum_{k=1}^{j}(x-x_k)^{-2}} \tag{1.1}$$

where $f \in L^\infty_{w_\gamma}$, $x_1 < x_2 < ... < x_m$ are the zeros of the Laguerre polynomial $p_m(w_\alpha)$ with $w_\alpha(x) = x^\alpha e^{-x}$, $\alpha > -1$ and

$$x_j = \min_{1\le k\le m}\{x_k : x_k \ge 2m\}. \tag{1.2}$$

Then, in Theorem 3.1, the convergence of $\mathcal{S}_m^{(\alpha)}(f)$ to f is showed and an estimate of the error $|f(x)-S_m^{(\alpha)}(f,x)|$ in the norm of the space $L^\infty_{w_\gamma}$ is given. Moreover, Theorem 3.2 proves that this error does not improve for smoother functions.

We will also consider the following Hermann-Vertési type operator

$$\mathcal{V}_m^{(\alpha)}(f,x) = \phi_m(x)\sum_{k=1}^{j} l_k^2(x)f(x_k), \quad \phi_m(x) = \left[\sum_{k=1}^{j} l_k^2(x)\right]^{-1}, \tag{1.3}$$

where j is defined in (1.2), x_k are the Laguerre zeros and l_k are the fundamental Lagrange polynomials based on the zeros of $p_m(w_\alpha)$. This operator, introduced in a more general form by T. Hermann and P. Vértesi in [11], was investigated in the case when $x_k \in (-1,1)$ (see, for instance, [11], [4]). Here we examine $\mathcal{V}_m^{(\alpha)}$ at the Laguerre zeros and in Theorem 3.3 we will show that, $\mathcal{V}_m^{(\alpha)}$ has a behavior similar to the Shepard operator.

Finally we will examine the Grünwald operator (see (3.11))

$$G_{m+1}^{*(\alpha)}(f,x) = \sum_{k=1}^{j} \tilde{l}_k^2(x)f(x_k),$$

where j is defined in (1.2) and $\tilde{l}_k(x) = l_k(x)\frac{4m-x}{4m-x_k}$, $\forall k = 1,...,j$ are the fundamental Lagrange polynomials based on $m+1$ points. The Grünwald operator was firstly introduced by G. Grünwald in [10] and was investigated by several authors in the case when x_k are the zeros of polynomials which are orthonormal with respect to Jacobi weights or some Freud-type weights (see, for instance, [10], [22]). In this paper, the Grünwald operator $G_{m+1}^{*(\alpha)}(f)$ based on the Laguerre zeros is considered and in Theorem 3.4 the convergence of $G_{m+1}^{*(\alpha)}(f)$ to f in $L^\infty_{w_\gamma}$ is showed and an error estimate is given.

The paper is structured as follows. In Section 2 some basic facts are given. In Section 3 the main results are presented. Then in Subsection 3.1 the Shepard operator $\mathcal{S}_m^{(\alpha)}$ is considered, in Subsection 3.2 the Hermann-Vertési operator $\mathcal{V}_m^{(\alpha)}$ is examined and in Subsection 3.3 the Grünwald operator $G_{m+1}^{*(\alpha)}$ is investigated . In Section 4 the proofs of the main results conclude the paper.

2. **Basic facts**

First we give some notations and preliminary results. In the following we will denote by $\mathcal{C}$ a positive constant which may assume different values in different formulae and we shall write $\mathcal{C} \neq \mathcal{C}(a, b, ...)$ to indicate that $\mathcal{C}$ is independent of the parameters $a, b, ...$ Moreover if A and B are two positive quantities depending on some parameters, we will write $A \sim B$ if and only if $(A/B)^{\pm 1} \leq \mathcal{C}$, where $\mathcal{C}$ is a positive constant independent of the above parameters.

Now let $w_\alpha(x) = x^\alpha e^{-x}$, $x > 0$, $\alpha > -1$ a Laguerre weight and let $\{p_m(w_\alpha)\}$ the sequence of orthonormal Laguerre polynomials defined as

$$p_m(w_\alpha, x) = \gamma_m x^m + ..., \quad \gamma_m > 0,$$
$$\int_0^\infty p_m(w_\alpha, x) p_n(w_\alpha, x) w_\alpha(x) dx = \delta_{m,n}.$$

If we denote by $x_k := x_{m,k}(w_\alpha)$ the zeros of $p_m(w_\alpha)$, $m \geq 1$ then we have (see, for instance, [23], [9], [12]),

$$\frac{\mathcal{C}}{m} < x_1 < ... < x_m = 4m + 2\alpha + 2 - \mathcal{C}(4m)^{\frac{1}{3}}, \quad x_k \sim \frac{k^2}{m} \tag{2.1}$$

and $\forall k = 1, 2, ..., m-1$

$$\Delta x_k := x_{k+1} - x_k \sim \sqrt{\frac{x_k}{4m - x_k}}, \qquad \Delta x_k \sim \Delta x_{k+1} \tag{2.2}$$

where $\mathcal{C}$ and the constants involved in $\sim$ are independent of m and k.

Moreover, denoted by x_d a node closest to x, and by

$$x_j = \min_{1 \leq k \leq m} \{x_k : x_k \geq 2m\},$$

it results that if $0 \leq x \leq 4m$, $k \neq \{d, d \pm 1\}$, $k = 1, ...j$, $d \leq m$ and $x \neq x_k$, (see, for instance, [16, Lemma 5]) then

$$|x - x_k| \geq (|d - k| + 1) \min(\Delta x_d, \Delta x_k) \geq \mathcal{C}(|d - k| + 1)\Delta x_k \tag{2.3}$$

where $\mathcal{C} \neq \mathcal{C}(m, k)$.

Setting $w_\gamma(x) = x^\gamma e^{-x}$, $\gamma \geq 0$ and denoting by $C^\circ(B)$, $B \subseteq [0, \infty)$ the set of all continuous functions on B, we introduce the space $L^\infty_{w_\gamma}$ as follows

$$L^\infty_{w_\gamma} = \left\{ f \in C^\circ((0, \infty)) \ : \ \lim_{\substack{x \to 0 \\ x \to +\infty}} (f w_\gamma)(x) = 0 \right\} \tag{2.4}$$

equipped with the norm

$$\|f\|_{L^\infty_{w_\gamma}} := \|fw_\gamma\|_\infty = \sup_{x\geq 0} |(fw_\gamma)(x)|.$$

If $\gamma > 0$, then $L^\infty_{w_\gamma}$ denotes the set of all continuous functions on $[0,\infty)$ such that $\lim_{x\to+\infty}(fw_\gamma)(x) = 0$. In other words, when $\gamma > 0$, the functions $f \in L^\infty_{w_\gamma}$ could take very large values, with algebraic growth, as x approaches zero from the right, and could have an exponential growth as $x \to \infty$.

In order to characterize these type of functions we introduce the following modulus of smoothness (see, for instance, [5], [13, p. 175])

$$\omega_\varphi(f,t)_{w_\gamma} = \Omega_\varphi(f,t)_{w_\gamma} + \inf_{\mathcal{C}} \|[f-\mathcal{C}]w_\gamma\|_{L^\infty((0,4t^2))} + \inf_{\mathcal{C}} \|[f-\mathcal{C}]w_\gamma\|_{L^\infty((\frac{1}{t^2},\infty))},$$

where

$$\Omega_\varphi(f,t)_{w_\gamma} = \sup_{0<h\leq t} \sup_{x\in\left[4h^2,\frac{1}{h^2}\right]} |f(x+h\varphi(x)) - f(x)|w_\gamma(x), \quad \varphi(x) = \sqrt{x}$$

and we recall the following properties (see, for instance, [13, p. 169], [17], [5])

$$\Omega_\varphi(f,\lambda\ \delta_1)_{w_\gamma} \leq ([\lambda]+1)\ \Omega_\varphi(f,\delta_1)_{w_\gamma}, \tag{2.5}$$

$$\Omega_\varphi(f,\delta)_{w_\gamma} \leq \omega_\varphi(f,\delta)_{w_\gamma}, \tag{2.6}$$

$$\frac{\Omega_\varphi(f,\delta_1)}{\delta_1} \leq \frac{\Omega_\varphi(f,\delta_2)}{\delta_2}, \quad \delta_1 > \delta_2 \tag{2.7}$$

where $[\lambda]$ denotes the integer part of λ.

Finally, we define the error of best approximation of $f \in L^\infty_{w_\gamma}$ by means of polynomials of degree at most m ($P_m \in \mathbb{P}_m$) as

$$E_m(f)_{w_\gamma} = \inf_{P\in\mathbb{P}_m} \|(f-P)w_\gamma\|_\infty,$$

and we recall the following Jackson's inequality (see, for instance, [17], [5])

$$E_m(f)_{w_\gamma} \leq \mathcal{C}\omega_\varphi\left(f, \frac{1}{\sqrt{m}}\right)_{w_\gamma} \tag{2.8}$$

where $\mathcal{C} \neq \mathcal{C}(m,f)$.

3. **Main Results**

3.1. **Shepard-type operator.** Let us consider the Shepard-type operator defined in (1.1)

$$\mathcal{S}_m^{(\alpha)}(f,x) := \frac{\displaystyle\sum_{k=1}^{j}(x-x_k)^{-2}f(x_k)}{\displaystyle\sum_{k=1}^{j}(x-x_k)^{-2}}.$$

By the definition, it follows that $\mathcal{S}_m^{(\alpha)}$ is a rational linear positive average operator such that

$$\begin{aligned}\mathcal{S}_m^{(\alpha)}(e_0,x) &= e_0, \quad e_0 = 1 \\ \mathcal{S}_m^{(\alpha)}(f,x_i) &= f(x_i), \quad \forall i = 1,...,j.\end{aligned} \tag{3.1}$$

By considering $\mathcal{S}_m^{(\alpha)} : L^\infty_{w_\gamma} \to L^\infty_{w_\gamma}$ we can state the following.

Theorem 3.1. *For every* $f \in L^\infty_{w_\gamma}$*, we have*

$$\|[f-\mathcal{S}_m^{(\alpha)}(f)]w_\gamma\|_\infty \leq \mathcal{C}\left[\sum_{i=1}^{j}\frac{1}{i^2}\omega_\varphi\left(f,\frac{i}{\sqrt{m}}\right)_{w_\gamma} + e^{-Am}E_0(f)_{w_\gamma}\right] \tag{3.2}$$

where $\mathcal{C} \neq \mathcal{C}(m,f)$ *and* $A \neq A(m,f)$*.*

Note that both members of (3.2) vanish if f is a constant function.

Moreover by Theorem 3.1 it follows that, if $f \in Z_r(w_\gamma)$ with

$$Z_r(w_\gamma) = \left\{ f \in L^\infty_{w_\gamma} \ : \ \|f\|_{Z_r(w_\gamma)} = \|fw_\gamma\|_\infty + \sup_{t>0}\frac{\omega_\varphi(f,t)_{w_\gamma}}{t^r} < \infty \right\}, \tag{3.3}$$

it results

$$\|[f-\mathcal{S}_m^{(\alpha)}(f)]w_\gamma\|_\infty = \begin{cases} \mathcal{O}\left(\frac{1}{m^{r/2}}\right), & 0<r<1; \\ \\ \mathcal{O}\left(\frac{\log m}{\sqrt{m}}\right), & \text{r=1.} \end{cases} \tag{3.4}$$

In other words $\mathcal{S}_m^{(\alpha)}$ converges to f with the same order of the polynomial of best approximation of functions in Zygmund spaces in $(0,+\infty)$.

Furthermore we mention that, if in the definition (1.1) of the Shepard operator we consider the sums until m and not until j, estimate (3.2) is not true (see relation (4.11)).

The following theorem shows that the error $|f(x) - \mathcal{S}_m^{(\alpha)}(f,x)|$ does not improve for smoother functions.

Theorem 3.2. *The asymptotic relation*

$$\frac{\sqrt{m}}{\log m}[\mathcal{S}_m^{(\alpha)}(f,x) - f(x)]w_\gamma(x) = o(1), \quad m \to \infty$$

is not valid for every x and for every non-constant function with continue first derivative.

Finally we underline that, thanks to the interpolatory character of $\mathcal{S}_m^{(\alpha)}(f)$, there exists a subsequence $\{m_k\}$ of natural numbers and a nonconstant function f (see, for instance, [8]) such that

$$\|[f - \mathcal{S}_{m_k}^{(\alpha)}(f)]w_\gamma\|_\infty \le \epsilon_{m_k} \tag{3.5}$$

where $\{\epsilon_m\}_{m=1}^\infty$ is an arbitrary positive fixed sequence.

3.2. **Hermann-Vertési type operator.** Let us consider the following Hermann-Vertési type operator

$$\mathcal{V}_m^{(\alpha)}(f,x) = \frac{\displaystyle\sum_{k=1}^{j} l_k^2(x) f(x_k)}{\displaystyle\sum_{k=1}^{j} l_k^2(x)}$$

where as before the index j is defined in (1.2) and $l_k(x) := l_{m,k}(x) = \frac{p_m(w_\alpha,x)}{(x-x_k)p'_m(w_\alpha,x)}$ are the fundamental Lagrange polynomials based on the zeros of $p_m(w_\alpha)$.

By definition one can easily deduce some properties of $\mathcal{V}_m^{(\alpha)}(f)$. It is a positive linear operator having degree of exactness 1 i.e. $\mathcal{V}_m^{(\alpha)}(1,x) = 1$ interpolating the function at the nodes x_i $\forall i = 1,...,j$. Moreover, we underline that, on the contrary of the Shepard operator introduced in the previous subsection, it is a polynomial operator.

Next theorem shows that, under suitable conditions on the parameter γ of the weight of the space, $\mathcal{V}_m^{(\alpha)}(f)$ converges to f in $L^\infty_{w_\gamma}$.

Theorem 3.3. *Let $f \in L^\infty_{w_\gamma}$ with*

$$\max\left\{0, \alpha + \frac{1}{2}\right\} \le \gamma \le \alpha + 1.$$

Then

$$\|[f - \mathcal{V}_m^{(\alpha)}(f)]w_\gamma\|_\infty \le \mathcal{C}\left[\sum_{i=1}^{j} \frac{1}{i^2}\omega_\varphi\left(f, \frac{i}{\sqrt{m}}\right)_{w_\gamma} + e^{-Am}E_0(f)_{w_\gamma}\right]$$

where $\mathcal{C} \ne \mathcal{C}(m,f)$ and $A \ne A(m,f)$.

Thus, by the previous result, we can deduce that $\mathcal{V}_m^{(\alpha)}$ has the same behavior of the Shepard operator $\mathcal{S}_m^{(\alpha)}$. Consequently, if f belongs to the Zygmund space defined in (3.3), estimate (3.4) still holds true with $\mathcal{V}_m^{(\alpha)}(f)$ in place of $\mathcal{S}_m^{(\alpha)}(f)$.

Moreover, we mention that, in virtue of the interpolating character of $\mathcal{V}_m^{(\alpha)}$, relation (3.5) still satisfied with $\mathcal{V}_{m_k}^{(\alpha)}$ in place of $\mathcal{S}_{m_k}^{(\alpha)}(f)$.

3.3. **Grünwald operator.** In 1942 in [10] G. Grünwald introduced the following operator

$$G_m(f,x) = \sum_{k=1}^{m} l_k^2(x) f(x_k)$$

where f is a continuous function in $(-1,1)$, x_k are the zeros of Jacobi polynomials and l_k are the fundamental Lagrange polynomials based on the nodes x_k.

In 1999 in [22] V. E. S. Szabó investigated on $G_m(f)$ in the case when x_k are the roots of orthogonal polynomials with respect to any weight w belonging to the class of Freud weights W defined in [3]. Thus he proved that if $f \in C_{w^2}$ with

$$C_{w^2} = \left\{ f \;:\; f \text{ is continuous on } \mathbb{R} \mid \lim_{|x|\to\infty} (fw^2)(x) = 0 \right\},$$

then

$$\lim_{m\to\infty} \|[f - G_m(f)]w^2\|_\infty = 0$$

and this is the only case in which we can have an homogeneous error estimate in $\mathbb{R}$ for the Grünwald operator. Indeed, if we consider more general weights on $\mathbb{R}$ (see, for instance [21]) then it possible to have the following

$$\forall f \in C_{w_2} \quad \lim_{m\to\infty} \|[f - G_m(f)]w_1\|_\infty = 0$$

with $w_1 \neq w_2$.

In this subsection we will introduce a Grünwald operator at the Laguerre zeros and we will prove that an homogeneous error estimate can be given.

To this end let us consider the zeros of the orthonormal Laguerre polynomial $p_m(w_\alpha)$, $x_1 < ... < x_m$, and let $x_{m+1} := 4m$. We denote by $\tilde{l}_k$ the fundamental Lagrange polynomials based on the zeros $x_1 < x_2 < ... < x_m < x_{m+1}$

$$\tilde{l}_k(x) := \tilde{l}_{m,k}(x) = \frac{4m - x}{4m - x_k}\, l_{m,k}(x), \quad k \leq m, \tag{3.6}$$

$$\tilde{l}_{m+1}(x) := \tilde{l}_{m,m+1}(x) = \frac{p_m(w_\alpha, x)}{p_m(w_\alpha, 4m)} \tag{3.7}$$

and we introduce the Grünwald operator

$$G_{m+1}^{(\alpha)}(f,x) = \sum_{k=1}^{m+1} \tilde{l}_k^2(x) f(x_k). \tag{3.8}$$

By the definition it easily results

$$G_{m+1}^{(\alpha)}(1,x)=1, \quad G_{m+1}^{(\alpha)}(f,x_i)=f(x_i), \quad \forall i=1,...,m+1.$$

Now if

$$x_j=\min_{1\le k\le m}\{x_k : x_k\ge 2m\}, \tag{3.9}$$

for any $f\in C^{\circ}((0,\infty))$ we introduce the following function

$$f_j(x)=\begin{cases} f(x), & x\le x_j,\\ 0, & x>x_j\end{cases} \tag{3.10}$$

and we define the operator

$$G_{m+1}^{*(\alpha)}(f,x):=G_{m+1}^{(\alpha)}(f_j,x)=\sum_{k=1}^{j}\tilde{l}_k^2(x)f(x_k). \tag{3.11}$$

Next theorem gives an error estimate for $G_{m+1}^{*(\alpha)}(f)$.

Theorem 3.4. *Let $f\in L^\infty_{w_\gamma}$ with*

$$\max\left\{0,\alpha+\frac{1}{2}\right\}\le\gamma\le\alpha+1.$$

Then

$$\|[f-G_{m+1}^{*(\alpha)}(f)]w_\gamma\|_\infty\le\mathcal{C}\left[\sum_{i=1}^{j}\frac{1}{i^2}\omega_\varphi\left(f,\frac{i}{\sqrt{m}}\right)_{w_\gamma}+\frac{\|fw_\gamma\|_\infty}{m}\right] \tag{3.12}$$

where $\mathcal{C}\ne\mathcal{C}(m,f)$.

4. **Proofs**

In order to prove Theorem 3.1 we need the following lemma.

Lemma 4.1. *For every $f\in L^\infty_{w_\gamma}$ we have*

$$\|[f-\mathcal{S}_m^{(\alpha)}(f)]w_\gamma\|_{L^\infty((0,x_j])}\le\mathcal{C}\sum_{i=1}^{j}\frac{1}{i^2}\omega_\varphi\left(f,\frac{i}{\sqrt{m}}\right)_{w_\gamma} \tag{4.1}$$

where $\mathcal{C}\ne\mathcal{C}(m,f)$.

Proof. Since $\mathcal{S}_m^{(\alpha)}(e_0,x)=e_0$ with $e_0=1$ we can write

$$|[f(x)-\mathcal{S}_m^{(\alpha)}(f,x)]w_\gamma(x)| = |[f(x)\mathcal{S}_m^{(\alpha)}(e_0,x)-\mathcal{S}_m^{(\alpha)}(f,x)]w_\gamma(x)| \le \frac{\sum_{k=1}^{j}(x-x_k)^{-2}|f(x)-f(x_k)|}{\sum_{k=1}^{j}(x-x_k)^{-2}}.$$

Now we denote by x_d a node closest to x and assume $x_{d-1}<x<x_d<x_{d+1}$, $d\ge 2$. Then since

$$\sum_{k=1}^{j}\frac{1}{(x-x_k)^2} > \frac{1}{(x-x_d)^2}$$

we have

$$\begin{aligned}|[f(x)-\mathcal{S}_m^{(\alpha)}(f,x)]w_\gamma(x)| &\le \sum_{k=1}^{j}\left(\frac{x-x_d}{x-x_k}\right)^2|f(x)-f(x_k)|w_\gamma(x)\\ &= |f(x)-f(x_d)|w_\gamma(x)+\sum_{k=1}^{d-1}\left(\frac{x-x_d}{x-x_k}\right)^2|f(x)-f(x_k)|w_\gamma(x)\\ &\quad+\sum_{k=d+1}^{j}\left(\frac{x-x_d}{x-x_k}\right)^2|f(x)-f(x_k)|w_\gamma(x)\\ &:= A_1(x)+A_2(x)+A_3(x).\end{aligned}\tag{4.2}$$

In order to estimate $A_1(x)$ it is sufficient to observe that, being $x_{d-1}<x<x_d$ and $x\in(0,x_j]$, by (2.2) we have

$$x_d-x<x_d-x_{d-1}=\Delta x_{d-1}\le \mathcal{C}\sqrt{\frac{x_d}{m}}.\tag{4.3}$$

Therefore, setting $\varphi(y)=\sqrt{y}$ and taking into account (2.5) we get

$$\begin{aligned}A_1(x) &= |f(x+(x_d-x))-f(x)|w_\gamma(x)\\ &\le \sup_{0<h\le\frac{1}{\sqrt{2m}}}\ \sup_{y\in\left(4h^2,\frac{1}{h^2}\right)}|f(y+h\varphi(y))-f(y)|w_\gamma(y)\\ &\le \mathcal{C}\ \Omega_\varphi\left(f,\frac{1}{\sqrt{m}}\right)_{w_\gamma}.\end{aligned}\tag{4.4}$$

Now we estimate $A_2(x)$. Since $x_k<x<x_d$, $\forall k$ then $w_\gamma(x)\le \mathcal{C}w_\gamma(x_k)$ and

$$x-x_k<x_d-x_k<x_{d+1}-x_k=\sum_{i=k}^{d}\Delta x_i\le(d-k+1)\Delta x_d.$$

Thus, by using (4.3),(2.7) and (2.5) we can write

$$\begin{aligned}
A_2(x) &\leq \mathcal{C}\sum_{k=1}^{d-1}\left(\frac{x-x_d}{x-x_k}\right)^2 |f(x_k+(x-x_k))-f(x_k)|w_\gamma(x_k)\\
&\leq \mathcal{C}\sum_{k=1}^{d-1}\left(\frac{x-x_d}{x-x_k}\right)^2 \Omega_\varphi\left(f,\frac{d-k+1}{\sqrt{m}}\right)_{w_\gamma}\\
&\leq \mathcal{C}\left[\Omega_\varphi\left(f,\frac{1}{\sqrt{m}}\right)_{w_\gamma}+\sum_{k=1}^{[\frac{d}{2}]}\left(\frac{x-x_d}{x-x_k}\right)^2\Omega_\varphi\left(f,\frac{d-k}{\sqrt{m}}\right)_{w_\gamma}\right.\\
&\qquad\left.+\sum_{k=[\frac{d}{2}]+1}^{d-2}\left(\frac{x-x_d}{x-x_k}\right)^2\Omega_\varphi\left(f,\frac{d-k}{\sqrt{m}}\right)_{w_\gamma}\right].
\end{aligned} \tag{4.5}$$

If $1 \leq k \leq [\frac{d}{2}]$ since in general it results

$$x_i - x_r = \sum_{\ell=r}^{i-1}\Delta x_\ell > \Delta x_r(i-r), \quad \forall i > r \tag{4.6}$$

we have

$$\begin{aligned}
(x-x_k) > (x_{d-1}-x_k) &= (x_{d-1}-x_{[\frac{d}{2}]})+(x_{[\frac{d}{2}]}-x_k)\\
&> \frac{d}{2}\Delta x_{[\frac{d}{2}]}+\left(\left[\frac{d}{2}\right]-k\right)\Delta x_k\\
&> \frac{d}{2}\Delta x_{[\frac{d}{2}]} > \mathcal{C}\frac{d}{2}\Delta x_d.
\end{aligned} \tag{4.7}$$

Therefore

$$\begin{aligned}
\sum_{k=1}^{[\frac{d}{2}]}\left(\frac{x-x_d}{x-x_k}\right)^2\Omega_\varphi\left(f,\frac{d-k}{\sqrt{m}}\right)_{w_\gamma} &\leq \mathcal{C}\sum_{k=1}^{[\frac{d}{2}]}\frac{1}{d^2}\Omega_\varphi\left(f,\frac{d-k}{\sqrt{m}}\right)_{w_\gamma}\\
&\leq \mathcal{C}\frac{1}{d}\Omega_\varphi\left(f,\frac{d}{\sqrt{m}}\right)_{w_\gamma}\\
&\leq \mathcal{C}\Omega_\varphi\left(f,\frac{1}{\sqrt{m}}\right)_{w_\gamma},
\end{aligned} \tag{4.8}$$

in virtue of (2.5).
If $[\frac{d}{2}]+1 \le k \le d-2$, then by applying (4.6) and by using (2.2) and (2.1) we have

$$\begin{aligned}\sum_{k=[\frac{d}{2}]+1}^{d-2}\left(\frac{x-x_d}{x-x_k}\right)^2\Omega_\varphi\left(f,\frac{d-k}{\sqrt{m}}\right)_{w_\gamma} &\le \sum_{k=[\frac{d}{2}]+1}^{d-2}\left(\frac{\Delta x_d}{(d-k)\Delta x_k}\right)^2\Omega_\varphi\left(f,\frac{d-k}{\sqrt{m}}\right)_{w_\gamma}\\ &\le \mathcal{C}\sum_{k=[\frac{d}{2}]+1}^{d-2}\frac{d^2}{(d-k)^2k^2}\Omega_\varphi\left(f,\frac{d-k}{\sqrt{m}}\right)_{w_\gamma}\\ &\le \mathcal{C}\sum_{k=[\frac{d}{2}]+1}^{d-2}\frac{1}{(d-k)^2}\Omega_\varphi\left(f,\frac{d-k}{\sqrt{m}}\right)_{w_\gamma}. \qquad (4.9)\end{aligned}$$

Thus by replacing (4.8) and (4.9) in (4.5) we obtain

$$A_2(x) \le \mathcal{C}\left[\Omega_\varphi\left(f,\frac{1}{\sqrt{m}}\right)_{w_\gamma}+\sum_{k=[\frac{d}{2}]+1}^{d-2}\frac{1}{(d-k)^2}\Omega_\varphi\left(f,\frac{d-k}{\sqrt{m}}\right)_{w_\gamma}\right]. \qquad (4.10)$$

It remains to estimate $A_3(x)$. To this end it is sufficient to note that

$$x_k-x < x_k-x_{d-1} < (k-d)\Delta x_k \le \mathcal{C}(k-d)\sqrt{\frac{x_k}{m}}, \quad k>d \qquad (4.11)$$

$$x_k-x_d > (k-d)\Delta x_d,$$

and to proceed as done for $A_2(x)$. Hence we have

$$A_3(x) \le \mathcal{C}\left[\Omega_\varphi\left(f,\frac{1}{\sqrt{m}}\right)_{w_\gamma}+\sum_{k=d+2}^{j}\frac{1}{(k-d)^2}\Omega_\varphi\left(f,\frac{k-d}{\sqrt{m}}\right)_{w_\gamma}\right]. \qquad (4.12)$$

Thus by replacing (4.4), (4.10) and (4.12) in (4.2) we get

$$\|[f-\mathcal{S}_m^{(\alpha)}(f)]w_\gamma\|_{L^\infty((0,x_j])} \le \mathcal{C}\sum_{i=1}^{j}\frac{1}{i^2}\Omega_\varphi\left(f,\frac{i}{\sqrt{m}}\right)_{w_\gamma}$$

from which the thesis follows by using (2.6). □

Proof. [Proof of Theorem 3.1] Let $\Psi \in C^\infty(\mathbb{R})$ non decreasing such that

$$\Psi(x)=\begin{cases}1, & \text{if } x\ge 1,\\ 0, & \text{if } x\le 0,\end{cases} \qquad (4.13)$$

and we introduce the function

$$\Psi_j(x)=\Psi\left(\frac{x-x_j}{x_{j+1}-x_j}\right) \qquad (4.14)$$

where $x_j = \min\{x_k : x_k \geq 2m\}$. Moreover for any function $f \in C^\circ((0,\infty))$ we define $f_j = (1-\Psi_j)f$. Obviously $f_j = f$ in $(0, x_j]$ and $f_j = 0$ in $[x_{j+1}, \infty)$. Now since $\mathcal{S}_m^{(\alpha)}(f) = \mathcal{S}_m^{(\alpha)}(f_j)$ we can write

$$\begin{aligned}
\|[f-\mathcal{S}_m^{(\alpha)}(f)]w_\gamma\|_\infty &\leq \|[f-f_j]w_\gamma\|_\infty + \|[f_j - \mathcal{S}_m^{(\alpha)}(f_j)]w_\gamma\|_\infty \\
&\leq \mathcal{C}\Big[\|fw_\gamma\|_{L^\infty((x_j,+\infty))} + \|[f-\mathcal{S}_m^{(\alpha)}(f)]w_\gamma\|_{L^\infty((0,x_j])} \\
&\qquad + \|\mathcal{S}_m^{(\alpha)}(f)w_\gamma\|_{L^\infty((x_j,+\infty))}\Big] \\
&:= \mathcal{C}[N_1 + N_2 + N_3]. \qquad (4.15)
\end{aligned}$$

In order to estimate N_1 we denote by Q_M the near best approximant polynomial of f i.e. $\|[f-Q_M]w_\gamma\|_\infty \leq \mathcal{C}E_M(f)_{w_\gamma}$ with $M = \frac{1}{3}m$ and we write

$$\begin{aligned}
N_1 &\leq \|[f-Q_M]w_\gamma\|_{L^\infty((x_j,+\infty))} + \|Q_M w_\gamma\|_{L^\infty((x_j,+\infty))} \\
&\leq \mathcal{C}E_M(f)_{w_\gamma} + \|Q_M w_\gamma\|_{L^\infty((x_j,+\infty))} \\
&\leq \mathcal{C}E_M(f)_{w_\gamma} + \|Q_M w_\gamma\|_{L^\infty([2m,+\infty))} \qquad (4.16)
\end{aligned}$$

being $x_j \geq 2m$. Now we recall that (see, for instance, [17]) $\forall P_m \in \mathbb{P}_m$ and $\forall \delta > 0$ it results

$$\|P_m w_\gamma\|_{L^\infty([4m(1+\delta),+\infty))} \leq \mathcal{C}e^{-Am}\|P_m w_\gamma\|_\infty \qquad (4.17)$$

where $\mathcal{C}$ and A are positive constants independent of m and P_m.
Therefore, since for the choice of M, it results $2m = 4M(1+\frac{1}{2})$, by applying (4.17) to the last term of (4.16) we have

$$\begin{aligned}
N_1 &\leq \mathcal{C}[E_M(f)_{w_\gamma} + e^{-Am}\|Q_M w_\gamma\|_\infty] \\
&\leq \mathcal{C}[E_M(f)_{w_\gamma} + e^{-Am}\|fw_\gamma\|_\infty] \qquad (4.18)
\end{aligned}$$

Consequently, by using (2.8) we deduce

$$N_1 \leq \mathcal{C}\left[\omega_\varphi\left(f, \frac{1}{\sqrt{m}}\right)_{w_\gamma} + e^{-Am}\|fw_\gamma\|_\infty\right]. \qquad (4.19)$$

The term N_2 is estimated in Lemma 4.1 and then it remains only to bound N_3. Then let $x \in (x_j, +\infty)$, by the definition we get

$$|\mathcal{S}_m^{(\alpha)}(f,x)w_\gamma(x)|$$
$$\le \sum_{k=1}^{j} \left(\frac{x-x_j}{x-x_k}\right)^2 |(fw_\gamma)(x_k)| \frac{w_\gamma(x)}{w_\gamma(x_k)}$$
$$= \left\{ \sum_{x_1 \le x_k \le x_{[\frac{j}{2}]}} + \sum_{x_{[\frac{j}{2}]+1} \le x_k \le x_j} \right\} \left(\frac{x-x_j}{x-x_k}\right)^2 \left(\frac{x}{x_k}\right)^\gamma e^{-(x-x_k)} |(fw_\gamma)(x_k)|.$$

The first sum tends to zero exponentially. In fact, since in virtue of (2.1) it results $x_k \le x_{[\frac{j}{2}]} \le \frac{1}{4}\frac{j^2}{m} \le \frac{1}{4}x_j$, we deduce $e^{-(x-x_k)} \le e^{-\frac{3}{4}x_j} \le e^{-\frac{3}{2}m}$ being $x > x_j > 2m$. It remains to estimate the second sum. We first assume $x_j < x \le x_{[\frac{3}{2}j]}$. In virtue of (2.1) we have $(\frac{x}{x_k})^\gamma \le \mathcal{C}$. Moreover, taking into account that $\sup_x (x-x_j)^2 e^{-(x-x_j)} \le \mathcal{C}$, we have

$$\sum_{x_{[\frac{j}{2}]+1} \le x_k \le x_j} \left(\frac{x-x_j}{x-x_k}\right)^2 \left(\frac{x}{x_k}\right)^\gamma e^{-(x-x_k)} |(fw_\gamma)(x_k)|$$
$$\le \mathcal{C}\ \|fw_\gamma\|_{L^\infty([x_{[\frac{j}{2}]+1}, x_j])} \sum_{x_{[\frac{j}{2}]+1} \le x_k \le x_j} \frac{1}{(x-x_k)^2}$$
$$\le \mathcal{C}\ \|fw_\gamma\|_{L^\infty((\frac{m}{2},\infty))} \sum_{x_{[\frac{j}{2}]+1} \le x_k \le x_j} \frac{1}{(j-k+1)^2}$$

being $x_{[\frac{j}{2}]+1} > \frac{x_j}{2} > \frac{m}{2}$ and by applying (4.6). Hence by estimating the norm $\|fw_\gamma\|_{L^\infty((\frac{m}{2},\infty))}$ as already done for the term N_1 with $M = \frac{m}{12}$ we get

$$\sum_{x_{[\frac{j}{2}]+1} \le x_k \le x_j} \left(\frac{x-x_j}{x-x_k}\right)^2 \left(\frac{x}{x_k}\right)^\gamma e^{-(x-x_k)} |(fw_\gamma)(x_k)| \le \mathcal{C} e^{-Am} \|fw_\gamma\|_\infty.$$

Finally if $x > x_{[\frac{3}{2}j]}$ the sum tends to zero as e^{-Am}, $A > 0$. Thus we deduce

$$N_3 \le \mathcal{C} e^{-Am} \|fw_\gamma\|_\infty. \tag{4.20}$$

Hence by replacing (4.19), (4.1) and (4.20) in (4.15) we obtain

$$\|[f - \mathcal{S}_m^{(\alpha)}(f)]w_\gamma\|_\infty \le \mathcal{C} \left[\sum_{i=1}^{j} \frac{1}{i^2} \omega_\varphi \left(f, \frac{i}{\sqrt{m}}\right)_{w_\gamma} + e^{-Am} \|fw_\gamma\|_\infty \right]. \tag{4.21}$$

Now we consider $G(x) = f(x) - \mathcal{C}$ where $\mathcal{C}$ is a positive constant. By using (3.1), we can write

$$\|[f - \mathcal{S}_m^{(\alpha)}(f)]w_\gamma\|_\infty = \|[G - \mathcal{S}_m^{(\alpha)}(G)]w_\gamma\|_\infty.$$

Then by applying (4.21) to G we get

$$\|[f-\mathcal{S}_m^{(\alpha)}(f)]w_\gamma\|_\infty \le \mathcal{C}\left[\sum_{i=1}^{j}\frac{1}{i^2}\omega_\varphi\left(f-\mathcal{C},\frac{i}{\sqrt{m}}\right)_{w_\gamma}+e^{-Am}\|[f-\mathcal{C}]w_\gamma\|\right].$$

Thus being $\omega_\varphi\left(f-\mathcal{C},\frac{i}{\sqrt{m}}\right)_{w_\gamma} \le \omega_\varphi\left(f,\frac{i}{\sqrt{m}}\right)_{w_\gamma}$ and taking the infimum on $\mathcal{C}$ at the right-hand side we have

$$\|[f-\mathcal{S}_m^{(\alpha)}(f)]w_\gamma\|_\infty \le \mathcal{C}\left[\sum_{i=1}^{j}\frac{1}{i^2}\omega_\varphi\left(f,\frac{i}{\sqrt{m}}\right)_{w_\gamma}+e^{-Am}E_0(f)_{w_\gamma}\right]$$

and the theorem follows. □

Proof. [Proof of Theorem 3.2] Let f be a function with continue first derivative such that $f'(x)>0$ and $\int_{1/m}^{1}\frac{\Omega_\varphi(f',\sqrt{m}t)_{w_\gamma\varphi}}{t^2}dt<\infty$. Since by the Taylor formula, we have

$$f(x_k)-f(x)=(x_k-x)f'(x)+G_x(x_k) \tag{4.22}$$

where $G_x(x_k)$ is the error, we can write

$$\begin{aligned}\mathcal{S}_m^{(\alpha)}(f,x)-f(x) &= \frac{\displaystyle\sum_{k=1}^{j}(x-x_k)^{-2}[f(x_k)-f(x)]}{\displaystyle\sum_{k=1}^{j}(x-x_k)^{-2}}\\ &= \frac{\displaystyle\sum_{k=1}^{j}(x-x_k)^{-2}[f'(x)(x_k-x)+G_x(x_k)]}{\displaystyle\sum_{k=1}^{j}(x-x_k)^{-2}},\end{aligned}$$

and then in virtue of the linearity of $\mathcal{S}_m^{(\alpha)}$

$$[\mathcal{S}_m^{(\alpha)}(f,x)-f(x)]w_\gamma(x)=w_\gamma(x)f'(x)\mathcal{S}_m^{(\alpha)}(g_x,x)+w_\gamma(x)\mathcal{S}_m^{(\alpha)}(G_x,x) \tag{4.23}$$

where $g_x(t)=t-x$.

We estimate the last term of (4.23). We denote by x_d a node closest to x and assume $x_{d-1}<x<x_d$. By the definition of G_x we have

$$|G_x(x_k)|\le|x_k-x||f'(x+(\xi-x))-f'(x)|,\quad \xi\in(x_k,x)$$

and since

$$|\xi-x|\le|x_k-x|\le\Delta x_k(|k-d|+1)\le\sqrt{\frac{y}{m}}(|k-d|+1),\quad y=\sup x_k$$

we get

$$\begin{aligned}&|\mathcal{S}_m^{(\alpha)}(G_x,x)w_\gamma(x)|\\ &\leq \mathcal{C}\sum_{k=1}^{j}\frac{(x-x_d)^2}{|x_k-x|}\sup_{0<h\leq\frac{k-d+1}{\sqrt{m}}}\ \sup_{y\in\left(4h^2,\frac{1}{h^2}\right)}|f'(y+h\varphi(y))-f'(y)|w_\gamma(y).\end{aligned}$$

Now, being $x-x_d\leq\frac{\varphi(x)}{\sqrt{m}}$, we have

$$\begin{aligned}&|\mathcal{S}_m^{(\alpha)}(G_x,x)w_\gamma(x)|\\ &\leq\frac{\mathcal{C}}{\sqrt{m}}\sum_{k=1}^{j}\left|\frac{x-x_d}{x_k-x}\right|\sup_{0<h\leq\frac{k-d+1}{\sqrt{m}}}\ \sup_{y\in\left(4h^2,\frac{1}{h^2}\right)}|f'(y+h\varphi(y))-f'(y)|(w_\gamma\varphi)(y)\\ &\leq\frac{\mathcal{C}}{\sqrt{m}}\sum_{k=1}^{j}\left|\frac{x-x_d}{x_k-x}\right|\Omega_\varphi\left(f',\frac{k-d+1}{\sqrt{m}}\right)_{w_\gamma\varphi}.\end{aligned}$$

Consequently, by proceeding as in the proof of Theorem 3.1, we found

$$\begin{aligned}|\mathcal{S}_m^{(\alpha)}(G_x,x)w_\gamma(x)|&\leq\frac{\mathcal{C}}{\sqrt{m}}\sum_{i=1}^{j}\frac{1}{i}\Omega_\varphi\left(f',\frac{i}{\sqrt{m}}\right)_{w_\gamma\varphi}\\ &\leq\frac{\mathcal{C}}{\sqrt{m}}\int_{\frac{1}{m}}^{1}\frac{\Omega_\varphi(f',\sqrt{m}t)_{w_\gamma\varphi}}{t^2}dt\end{aligned}$$

from which in virtue of the assumptions on f', we can deduce

$$\lim_{m\to\infty}\frac{\sqrt{m}}{\log m}\mathcal{S}_m^{(\alpha)}(G_x,x)w_\gamma(x)=0.$$

Then by (4.23) we have

$$\lim_{m\to\infty}\frac{\sqrt{m}}{\log m}[\mathcal{S}_m^{(\alpha)}(f,x)-f(x)]w_\gamma(x)=\lim_{m\to\infty}\frac{\sqrt{m}}{\log m}\mathcal{S}_m^{(\alpha)}(g_x,x)w_\gamma(x)f'(x).$$

Now choose $x\in I:=(2m-(2m)^{1/4},2m]$. It results

$$\mathcal{S}_m^{(\alpha)}(g_x,x)=\frac{\displaystyle\sum_{k=1}^{j}\frac{1}{|x-x_k|}}{\displaystyle\sum_{k=1}^{j}\frac{1}{(x-x_k)^2}}\geq\mathcal{C}\frac{\log m}{\sqrt{m}},\quad\forall x\in I$$

from which

$$\lim_{m\to\infty}\frac{\sqrt{m}}{\log m}\mathcal{S}_m^{(\alpha)}(g_x,x)w_\gamma(x)f'(x)>w_\gamma(x)f'(x)>0$$

and the theorem follows. □

In order to prove Theorem 3.3 we recall that denoting by $\lambda_m(w_\alpha, x) = [\sum_{k=0}^{m-1} p_k^2(w_\alpha, x)]^{-1}$ the m-th Christoffel functions, it results (see, for instance, [23] and [18])

$$l_k^2(x) := l_{m,k}^2(w_\alpha, x) = p_m^2(w_\alpha, x)\frac{x_k\lambda_m(w_\alpha, x_k)}{(x-x_k)^2}, \quad \forall k = 1, ..., m. \tag{4.24}$$

and (see, for instance, [9], [12], [18])

$$\lambda_m(w_\alpha, x_k) \sim w_\alpha(x_k)\Delta x_k \sim w_\alpha(x_k)\sqrt{\frac{x_k}{4m - x_k}}. \tag{4.25}$$

Proof. [Proof of Theorem 3.3] We proceed as in the proof of Theorem 3.1. By estimate (4.15) with $\mathcal{V}_m^{(\alpha)}$ in place of $\mathcal{S}_m^{(\alpha)}$ we get

$$\begin{aligned}
\|[f - \mathcal{V}_m^{(\alpha)}(f)]w_\gamma\|_\infty &\le \|[f - f_j]w_\gamma\|_\infty + \|[f_j - \mathcal{V}_m^{(\alpha)}(f_j)]w_\gamma\|_\infty \\
&\le \mathcal{C}\Big[\|fw_\gamma\|_{L^\infty((x_j,+\infty))} + \|[f - \mathcal{V}_m^{(\alpha)}(f)]w_\gamma\|_{L^\infty((0,x_j])} \\
&\qquad + \|\mathcal{V}_m^{(\alpha)}(f)w_\gamma\|_{L^\infty((x_j,+\infty))}\Big] \\
&:= \mathcal{C}[N_1 + N_2 + N_3].
\end{aligned} \tag{4.26}$$

The term N_1 can be estimated by proceeding as in the proof of Theorem 3.1 (see estimate (4.18)).
Now we consider N_2. Since $\mathcal{V}_m^{(\alpha)}(f,1) = 1$ we have

$$|[f(x) - \mathcal{V}_m^{(\alpha)}(f,x)]w_\gamma(x)| = \frac{\displaystyle\sum_{k=1}^{j} l_k^2(x)w_\gamma(x)|f(x) - f(x_k)|}{\displaystyle\sum_{k=1}^{j} l_k^2(x)}.$$

We denote by x_d a node closest to x and we assume $x_{d-1} < x < x_d < x_{d+1}$, $d \ge 2$ $x \in (0, x_j]$. Then we write

$$\begin{aligned}
|[f(x) - \mathcal{V}_m^{(\alpha)}(f,x)]w_\gamma(x)| &\le \sum_{k=1}^{j}\frac{l_k^2(x)}{l_d^2(x)}|f(x) - f(x_k)|w_\gamma(x) \\
&= |f(x) - f(x_d)|w_\gamma(x) + \sum_{\substack{k=1\\k\ne d}}^{j}\frac{l_k^2(x)}{l_d^2(x)}|f(x) - f(x_k)|w_\gamma(x) \\
&\le |f(x) - f(x_d)|w_\gamma(x) \\
&\quad + \sum_{\substack{k=1\\k\ne d}}^{j}\left(\frac{x - x_d}{x - x_k}\right)^2 \frac{x_k}{x_d}\frac{\lambda_m(w_\alpha, x_k)}{\lambda_m(w_\alpha, x_d)}w_\gamma(x)|f(x) - f(x_k)|
\end{aligned}$$

by using (4.24). Moreover, by applying (4.25) we found

$$\begin{aligned}
&|[f(x) - \mathcal{V}_m^{(\alpha)}(f,x)]w_\gamma(x)| \\
&\le \mathcal{C}\left[|[f(x) - f(x_d)]w_\gamma(x)| \right.\\
&\left. + \left\{\sum_{k=1}^{d-1} + \sum_{k=d+1}^{j}\right\}\left(\frac{x-x_d}{x-x_k}\right)^2\left(\frac{x_k}{x}\right)^{1+\alpha-\gamma}\frac{\Delta x_k}{\Delta x_d}w_\gamma(x_k)|f(x)-f(x_k)|\right] \\
&:= \mathcal{C}\left[|[f(x) - f(x_d)]w_\gamma(x)| + S_1(x) + S_2(x)\right]. \qquad (4.27)
\end{aligned}$$

The first term is equal to that considered in (4.4) and it results

$$|f(x) - f(x_d)|w_\gamma(x) \le \mathcal{C}\Omega_\varphi\left(f, \frac{1}{\sqrt{m}}\right)_{w_\gamma}. \qquad (4.28)$$

About S_1, we note that since $x_k < x$ for all k, then $\Delta x_k \le \Delta x_d$ and taking into account that $\gamma \le \alpha + 1$ we have

$$S_1(x) \le \mathcal{C}\sum_{k=1}^{d-1}\left(\frac{x-x_d}{x-x_k}\right)^2 w_\gamma(x_k)|f(x)-f(x_k)|.$$

This sum was already estimated in the proof of Theorem 3.1 (see, estimates (4.5)-(4.10)) and it results

$$S_1(x) \le \mathcal{C}\left[\Omega_\varphi\left(f, \frac{1}{\sqrt{m}}\right)_{w_\gamma} + \sum_{k=[\frac{d}{2}]+1}^{d-2}\frac{1}{(d-k)^2}\Omega_\varphi\left(f, \frac{d-k}{\sqrt{m}}\right)_{w_\gamma}\right]. \qquad (4.29)$$

Now we give a bound for S_2. To this end we apply (2.2) getting

$$S_2(x) \le \sum_{k=d+1}^{j}\left(\frac{x-x_d}{x-x_k}\right)^2\left(\frac{x_k}{x}\right)^{\frac{1}{2}+\alpha-\gamma}\frac{\Delta^2 x_k}{\Delta^2 x_d}\left(\frac{4m-x_k}{4m-x_d}\right)^{1/2} w_\gamma(x_k)|f(x)-f(x_k)|.$$

Now by observing that $\left(\frac{4m-x_k}{4m-x_d}\right) \le \mathcal{C}$, taking into account that $\frac{x_k}{x} \le 1$ being $\gamma \ge \alpha + \frac{1}{2}$, and by using (4.6) we have

$$\begin{aligned}
S_2 &\le \mathcal{C}\sum_{k=d+1}^{j}\left(\frac{\Delta x_k}{x-x_k}\right)^2 w_\gamma(x_k)|f(x)-f(x_k)| \\
&\le \mathcal{C}\sum_{k=d+1}^{j}\frac{1}{(k-d)^2}w_\gamma(x_k)|f(x)-f(x_k)|.
\end{aligned}$$

Thus by proceeding as done in Theorem 3.1 for the term A_3 we get

$$S_2(x) \le \mathcal{C}\left[\Omega_\varphi\left(f, \frac{1}{\sqrt{m}}\right)_{w_\gamma} + \sum_{k=d+2}^{j}\frac{1}{(k-d)^2}\Omega_\varphi\left(f, \frac{k-d}{\sqrt{m}}\right)_{w_\gamma}\right]. \qquad (4.30)$$

Then in definitive by applying (4.28), (4.29) and (4.30) in (4.27) we found

$$N_2 \leq \mathcal{C} \sum_{i=1}^{j} \frac{1}{i^2} \Omega_\varphi \left(f, \frac{i}{\sqrt{m}} \right)_{w_\gamma}. \tag{4.31}$$

Finally we consider N_3. By using (4.24),(4.25) and taking into account that $\gamma \leq \alpha+1$ we get

$$\begin{aligned} \|\mathcal{V}_m^{(\alpha)}(f) w_\gamma\|_{L^\infty((x_j,+\infty))} &\leq \sup_{x>x_j} w_\gamma(x) \sum_{k=1}^{j} \frac{l_k^2(x)}{l_j^2(x)} |f(x_k)| \\ &\leq \mathcal{C} \sup_{x>x_j} \sum_{k=1}^{j} \left(\frac{x-x_j}{x-x_k} \right)^2 e^{-(x-x_j)} |(f w_\gamma)(x_k)| \end{aligned}$$

and this sum can be estimated as already done for the term N_3 in the proof of Theorem 3.1 (see estimate (4.20)). Thus by replacing (4.18), (4.31) and (4.20) in (4.26) and by applying (2.8) we deduce

$$\|[f - \mathcal{V}_m^{(\alpha)}(f)] w_\gamma\|_\infty \leq \mathcal{C} \left[\sum_{i=1}^{j} \frac{1}{i^2} \omega_\varphi \left(f, \frac{i}{\sqrt{m}} \right)_{w_\gamma} + e^{-Am} \|f w_\gamma\|_\infty \right].$$

Now the thesis follows by introducing the function $G = f - \mathcal{C}$ where $\mathcal{C}$ is a positive constant and by proceeding as done in Theorem 3.1. □

In order to prove Theorem 3.4 we need the following Lemma.

Lemma 4.2. *Let* $f \in L^\infty_{w_\gamma}$ *with*

$$\max\left\{0, \alpha + \frac{1}{2}\right\} \leq \gamma \leq \alpha + 1.$$

Then

$$\|[f - G_{m+1}^{*(\alpha)}(f)] w_\gamma\|_{L^\infty((0,x_j])} \leq \mathcal{C} \left[\sum_{i=1}^{j} \frac{1}{i^2} \omega_\varphi \left(f, \frac{i}{\sqrt{m}} \right)_{w_\gamma} + \frac{\|f w_\gamma\|_\infty}{m} \right] \tag{4.32}$$

where $\mathcal{C} \neq \mathcal{C}(m, f)$.

Proof. Since $G^{(\alpha)}_{m+1}(1,x)=1$ we have

$$\begin{aligned}
&\|[f-G^{*(\alpha)}_{m+1}(f)]w_\gamma\|_{L^\infty((0,x_j])} && (4.33)\\
&\le \sup_{x\in(0,x_j]}\sum_{k=1}^{m+1}\tilde{l}^2_k(x)w_\gamma(x)|f_j(x)-f_j(x_k)|\\
&= \sup_{x\in(0,x_j]}\left\{\sum_{k=1}^{j}\tilde{l}^2_k(x)w_\gamma(x)|f(x)-f(x_k)|+\sum_{k=j+1}^{m+1}\tilde{l}^2_k(x)w_\gamma(x)|f(x)|\right\}\\
&:= \sup_{x\in(0,x_j]}\left\{A_1(x)+A_2(x)\right\}. && (4.34)
\end{aligned}$$

We denote by x_d a node closest to x by assuming $x_{d-1}<x<x_d$, $d\ge 2$. We have

$$\begin{aligned}
A_1(x) &:= \sum_{k=1}^{j}\tilde{l}^2_k(x)w_\gamma(x)|f(x)-f(x_k)|\\
&= \tilde{l}^2_d(x)w_\gamma(x)|f(x)-f(x_d)|+\sum_{\substack{k=1\\k\ne d}}^{j}\tilde{l}^2_k(x)w_\gamma(x)|f(x)-f(x_k)|\\
&\le \tilde{l}^2_d(x)\frac{w_\gamma(x)}{w_\gamma(x_d)}|f(x)-f(x_d)|w_\gamma(x_d)+\sum_{\substack{k=1\\k\ne d}}^{j}\tilde{l}^2_k(x)w_\gamma(x)|f(x)-f(x_k)|\\
&\le \tilde{l}^2_d(x)\frac{w_\gamma(x)}{w_\gamma(x_d)}|f(x+(x_d-x))-f(x)|w_\gamma(x)+\sum_{\substack{k=1\\k\ne d}}^{j}\tilde{l}^2_k(x)w_\gamma(x)|f(x)-f(x_k)|\\
&\le \mathcal{C}\Omega_\varphi\left(f,\frac{1}{\sqrt{m}}\right)_{w_\gamma}+\sum_{\substack{k=1\\k\ne d}}^{j}\tilde{l}^2_k(x)w_\gamma(x)|f(x)-f(x_k)|
\end{aligned}$$

being $\tilde{l}^2_d(x)\frac{w_\gamma(x)}{w_\gamma(x_d)}\le\mathcal{C}$ (see, for instance, [14, Lemma 3.2]) and by applying (4.4). Now, in order to estimate the sum, we recall that (see, for instance, [17])

$$|p_m(w_\alpha,x)\sqrt{w_\alpha(x)}|\le\frac{\mathcal{C}}{\sqrt[4]{x}\sqrt[4]{4m-x+\frac{4m}{m^{2/3}}}},\quad x\in\left(\frac{\mathcal{C}}{m},4m\right) \tag{4.35}$$

and then by relation (4.24) and (4.25) we get that for $x\in(\frac{\mathcal{C}}{m},4m)$ the following inequality holds true

$$\tilde{l}^2_k(x)w_\gamma(x)\le\mathcal{C}\left(\frac{4m-x}{4m-x_k}\right)^2x^{\gamma-\alpha-\frac{1}{2}}\frac{\Delta x_k}{(x-x_k)^2}\;\frac{x_kw_\alpha(x_k)}{\sqrt{4m-x}}. \tag{4.36}$$

Therefore

$$\begin{aligned} &A_1(x)\\ &\leq \mathcal{C}\left[\Omega_\varphi\left(f,\frac{1}{\sqrt{m}}\right)_{w_\gamma}\right.\\ &\qquad\left.+\left\{\sum_{k=1}^{d-1}+\sum_{k=d+1}^{j}\right\}\left(\frac{4m-x}{4m-x_k}\right)^2 x^{\gamma-\alpha-\frac{1}{2}}\frac{\Delta x_k}{(x-x_k)^2}\,\frac{x_k w_\alpha(x_k)}{\sqrt{4m-x}}|f(x)-f(x_k)|\right]\\ &=\mathcal{C}\left[\Omega_\varphi\left(f,\frac{1}{\sqrt{m}}\right)_{w_\gamma}+\Sigma_1(x)+\Sigma_2(x)\right]. \end{aligned}$$

About Σ_1 we write

$$\begin{aligned} \Sigma_1(x)&=\sum_{k=1}^{d-1}\left(\frac{4m-x}{4m-x_k}\right)^2 x^{\gamma-\alpha-\frac{1}{2}}\frac{\Delta x_k}{(x-x_k)^2}\,\frac{x_k^{1+\alpha-\gamma}}{\sqrt{4m-x}}|f(x)-f(x_k)|w_\gamma(x_k)\\ &=\sum_{k=1}^{d-1}\left(\frac{4m-x}{4m-x_k}\right)^2\sqrt{\frac{x}{4m-x}}\frac{\Delta x_k}{(x-x_k)^2}\left(\frac{x_k}{x}\right)^{1+\alpha-\gamma}|f(x)-f(x_k)|w_\gamma(x_k). \end{aligned}$$

Now $\left(\frac{4m-x}{4m-x_k}\right)^2<\mathcal{C}$, $\left(\frac{x_k}{x}\right)^{1+\alpha-\gamma}<1$ being $\gamma\leq\alpha+1$ and $\sqrt{\frac{x}{4m-x}}\Delta x_k\leq\sqrt{\frac{x_d}{4m-x_d}}\Delta x_d\leq\Delta^2 x_d<(x-x_d)^2$, being $\Delta x_k\leq\Delta x_d$. Therefore

$$\Sigma_1(x)\leq\sum_{k=1}^{d-1}\left(\frac{x-x_d}{x-x_k}\right)^2|f(x)-f(x_k)|w_\gamma(x_k)$$

that is the sum appeared in (4.5). Then by (4.10) we get

$$\Sigma_1(x)\leq\mathcal{C}\left[\Omega_\varphi\left(f,\frac{1}{\sqrt{m}}\right)_{w_\gamma}+\sum_{k=[\frac{d}{2}]+1}^{d-2}\frac{1}{(d-k)^2}\Omega_\varphi\left(f,\frac{d-k}{\sqrt{m}}\right)_{w_\gamma}\right].$$

Now we consider Σ_2 and we write

$$\begin{aligned} &\Sigma_2(x)\\ &\leq\sum_{k=d+1}^{j}\left(\frac{4m-x}{4m-x_k}\right)^2 x^{\gamma-\alpha-\frac{1}{2}}\frac{\Delta x_k}{(x-x_k)^2}\,\frac{x_k^{1+\alpha-\gamma}}{\sqrt{4m-x}}|f(x)-f(x_k)|w_\gamma(x_k)\\ &=\sum_{k=d+1}^{j}\left(\frac{4m-x}{4m-x_k}\right)^{3/2}\sqrt{\frac{x_k}{4m-x_k}}\frac{\Delta x_k}{(x-x_k)^2}\left(\frac{x}{x_k}\right)^{-\frac{1}{2}-\alpha+\gamma}|f(x)-f(x_k)|w_\gamma(x_k). \end{aligned}$$

Then since $\Delta x_k \sim \sqrt{\frac{x_k}{4m-x_k}}$, $\frac{4m-x}{4m-x_k} < \mathcal{C}$, $w_\gamma(x_k) \le \mathcal{C} w_\gamma(x)$, $\left(\frac{x}{x_k}\right)^{-\frac{1}{2}-\alpha+\gamma} \le 1$ being $\gamma \ge \alpha + \frac{1}{2}$, we have by applying (2.3)

$$\Sigma_2(x) \le \mathcal{C} \sum_{k=d+1}^{j} \frac{1}{(k-d)^2} |f(x) - f(x_k)| w_\gamma(x),$$

that is the sum estimated in (4.12). Thus

$$\Sigma_2(x) \le \mathcal{C} \left[\Omega_\varphi \left(f, \frac{1}{\sqrt{m}} \right)_{w_\gamma} + \sum_{k=d+2}^{j} \frac{1}{(k-d)^2} \Omega_\varphi \left(f, \frac{k-d}{\sqrt{m}} \right)_{w_\gamma} \right].$$

Hence, by the previous estimates, we deduce

$$A_1(x) \le \mathcal{C} \sum_{i=1}^{j} \frac{1}{i^2} \Omega_\varphi \left(f, \frac{i}{\sqrt{m}} \right)_{w_\gamma} \le \mathcal{C} \sum_{i=1}^{j} \frac{1}{i^2} \omega_\varphi \left(f, \frac{i}{\sqrt{m}} \right)_{w_\gamma} \tag{4.37}$$

by using (2.8).
Now we estimate A_2. We write

$$A_2(x) = \tilde{l}^2_{m+1}(x) |(w_\gamma f)(x)| + \sum_{k=j+1}^{m} \tilde{l}^2_k(x) w_\gamma(x) |f(x)|.$$

The first term tends to zero exponentially. In fact

$$\begin{aligned} |l^2_{m+1}(x)(f w_\gamma)(x)| &= \left| \frac{w_\gamma(x)}{w_\gamma(4m)} \frac{p^2_m(w_\alpha, x)}{p^2_m(w_\alpha, 4m)} \frac{w_\gamma(4m)}{w_\gamma(x)} (f w_\gamma)(x) \right| \\ &\le \mathcal{C} e^{-2m} \|f w_\gamma\|_\infty \end{aligned}$$

being $\frac{w_\gamma(x)}{w_\gamma(4m)} \frac{p^2_m(w_\alpha, x)}{p^2_m(w_\alpha, 4m)} \le C$ (see, for instance, [14]).

It remains to estimate the sum. To this end we use relation (4.36). Hence we have

$$\begin{aligned} &\sum_{k=j+1}^{m} \tilde{l}^2_k(x) w_\gamma(x) |f(x)| \\ &\le \mathcal{C} \|f w_\gamma\|_\infty \sum_{k=j+1}^{m} \left(\frac{4m-x}{4m-x_k} \right)^2 \left(\frac{x}{x_k} \right)^{\gamma-\alpha-\frac{1}{2}} \frac{\Delta x_k}{(x_k - x)^2} \sqrt{\frac{x_k}{4m-x}} \left(\frac{x_k}{x} \right)^\gamma e^{-(x_k - x)}. \end{aligned}$$

Thus being $x < x_k < 4m$, $\gamma > \alpha + \frac{1}{2}$ and $\sqrt{\frac{x_k}{4m-x}} \le \mathcal{C}$ we have

$$\sum_{k=j+1}^{m} \tilde{l}^2_k(x) w_\gamma(x) |f(x)| \le \sum_{k=j+1}^{m} \frac{\Delta x_k}{(4m - x_k)^2} \left(\frac{x_k}{x} \right)^\gamma e^{-(x_k - x)}.$$

If $x < \frac{x_k}{2}$ then the sum tends to zero exponentially since $e^{-(x_k - x)} < e^{-\frac{x_k}{2}} < e^{-m}$. Then now we assume $\frac{x_k}{2} < x \leq x_j$ and we write

$$\sum_{k=j+1}^{m} \frac{\Delta x_k}{(4m - x_k)^2} \left(\frac{x_k}{x}\right)^{\gamma} e^{-(x_k - x)}$$
$$\leq \mathcal{C} \|f w_\gamma\|_\infty \left[\sum_{j+1 \leq k \leq [\frac{3}{2} j]} + \sum_{[\frac{3}{2} j]+1 \leq k \leq m} \right] \frac{\Delta x_k}{(4m - x_k)^2} e^{-(x_k - x)}.$$

The second sum tends again to zero exponentially. In fact $x < x_j$, $x_k > x_{[\frac{3}{2} j]} > \mathcal{C} \frac{9}{4} \frac{j^2}{m} > \mathcal{C} \frac{9}{4} x_j$ being $x_i \sim \frac{i^2}{m}$ and consequently $e^{-(x_k - x)} \leq e^{-\mathcal{C} \frac{5}{2} m}$.
About the first sum we have that it is bounded by

$$\mathcal{C} \frac{e^{-(x_{j+1} - x_j)}}{m^2} \sum_{x_{j+1} \leq x_k \leq \frac{3}{2} x} \int_{x_k}^{x_{k+1}} dt \leq \frac{\mathcal{C}}{m}.$$

In definitive

$$A_2(x) \leq \mathcal{C} \frac{\|f w_\gamma\|_\infty}{m}.$$

Thus by replacing this estimate and (4.37) in (4.33) we have the assertion. □

Proof. [Proof of Theorem 3.4] Let x_j, f_j, Ψ and Ψ_j be as in (3.9), (3.10), (4.13) and (4.14), respectively. Then being
$G^{*(\alpha)}_{m+1}(f) = G^{(\alpha)}_{m+1}(f_j)$ we have

$$\begin{aligned} \|[f - G^{*(\alpha)}_{m+1}(f)] w_\gamma\|_\infty &\leq \|[f - f_j] w_\gamma\|_\infty + \|[f_j - G^{(\alpha)}_{m+1}(f_j)] w_\gamma\|_\infty \\ &\leq \mathcal{C} \Big[\|f w_\gamma\|_{L^\infty((x_j, \infty))} + \|[f - G^{*(\alpha)}_{m+1}(f)] w_\gamma\|_{L^\infty((0, x_j])} \\ &\qquad + \|G^{*(\alpha)}_{m+1}(f) w_\gamma\|_{L^\infty((x_j, +\infty))} \Big]. \end{aligned} \tag{4.38}$$

The first term is estimated in Theorem 3.1 (see estimate (4.19)) while the second one is given by (4.32). Therefore we have

$$\|[f - G^{*(\alpha)}_{m+1}(f)] w_\gamma\|_\infty \leq \mathcal{C} \left[\sum_{i=1}^{j} \frac{1}{i^2} \omega_\varphi \left(f, \frac{i}{\sqrt{m}}\right)_{w_\gamma} + \frac{\|f w_\gamma\|_\infty}{m} \right] + \|G^{*(\alpha)}_{m+1}(f) w_\gamma\|_{L^\infty((x_j, +\infty))}$$

by using (2.8) and (2.6). Consequently we have only to bound the last term. To this end it is sufficient to observe that G^*_{m+1} is a bounded operator and $G^*_{m+1}(f)$ is a polynomial of degree $2m$. Hence by applying (4.17) we get

$$\|G^{*(\alpha)}_{m+1}(f) w_\gamma\|_{L^\infty((x_j, +\infty))} \leq \mathcal{C} e^{-Am} \|f w_\gamma\|_\infty$$

and the proof is complete. □

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